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Intervention analysis and multiple time series

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SUMMARY

A general model is introduced to encapsulate interventions in a multiple time series. The estimation of this model is discussed, and a bivariate economic example is presented to illustrate the methods.

Some key words: Autoregressive-moving average vector process; Estimation; Intervention; Time series.

1. INTRODUCTION

Quite often decisions are taken or policy changes are effected in different sectors expecting some form of change in a set of response variables occurring sometimes in the form of a multiple time series. For example, in October 1975 the Federal Government of Canada created an Anti-inflation Board hoping to influence a variety of economic variables. Such actions, or in general intrusions, to a series, referred to as interventions (Box & Tiao, 1975), are expected to affect the time series in some way and it is worthwhile investigating whether or not the expected change has materialized.

An intervention can abruptly change the level of a series or change the level after a short delay; it could deflect a series going downward causing it to drift upward or it could cause some other form of change. A single intervention can also have different effects on different time series. For example, the creation of the Anti-inflation Board in Canada may have different effects on the rate of change of the Consumer Price Index, CPI, and on the rate of change of wages and salaries; the introduction of an 'electroshock and tranquillizer' on schizophrenic patients may have one effect on a psychological variable and another on a biochemical variable. The approach adopted here is quite similar to that of Box & Tiao (1975) where interventions are considered in a univariate time series. Here we modestly extend the results to the multiple time series case where we build a multiple time series model which includes the possibilities of the changes of the form expected.

2. A MULTIPLE TIME SERIES MODEL WITH INTERVENTIONS

2·1. *General set-up*

Suppose that, after relevant transformations, the data Z_t $(t = \ldots -1, 0, 1, \ldots)$ are available as a vector series observed at equispaced time intervals. We then consider the model

$$Z_t = F(\omega, X_t, t) + N_t, \tag{2·1}$$

where $Z'_t = (z_{1t}, \ldots, z_{mt})$, $F' = (f_1, \ldots, f_m)$, $f_i(\omega, X_t, t)$ is a function of the parameters ω, exogenous variables X_t and time t, and $N'_t = (n_{1t}, \ldots, n_{mt})$ stands for noise.

2·2. *Intervention model*

The function f_i in (2·1) can allow for the effects of interventions by taking some or all of the exogenous variables to be indicator variables as described below.

Suppose that k known interventions occurred in the vector series Z_t at $t = T_1, \ldots, T_k$ $(T_1 < \ldots < T_k)$ and there are no other exogenous variables present. Then the function F can be written as $F(\omega, X_t, t) = R(B) I_t$, where $R(B)$ is an $m \times k$ matrix of rational functions of B with elements $R_{ij}(B) = \beta_{ij}(B)/\alpha_{ij}(B)$ $(i = 1, \ldots, m;\ j = 1, \ldots, k)$, B is a backward shift operator such that

$$Bz_t = z_{t-1}, \quad \beta_{ij}(B) = \beta_{ij}(0) - \beta_{ij}(1) B - \ldots - \beta_{ij}(s_{ij}) B^{s_{ij}},$$
$$\alpha_{ij}(B) = 1 - \alpha_{ij}(1) B - \ldots - \alpha_{ij}(r_{ij}) B^{r_{ij}}.$$

It is also supposed that $\beta_{ij}(B)$ has roots outside and $\alpha_{ij}(B)$ on or outside the unit circle. Further I_t is a vector of indicator variables $I'_t = (I_t(T_1), \ldots, I_t(T_k))$, where $I_t(T_j)$ could take the form of 'step' or 'pulse' inputs as described in (i) and (ii) respectively,

$$\text{(i)} \quad I_t(T_j) = \begin{cases} 0 & (t < T_j), \\ 1 & (t \geqslant T_j), \end{cases} \qquad \text{(ii)} \quad I_t(T_j) = \begin{cases} 0 & (t \neq T_j), \\ 1 & (t = T_j). \end{cases}$$

The step input takes the values zero and one to denote the nonoccurrence and occurrence of interventions while the pulse input takes the value one at the time of intervention and zeros elsewhere.

2·3. *Noise model*

Now we suppose that $N_t = Z_t - R(B) I_t$ can be modelled by a mixed autoregressive-moving average vector process,

$$\Phi(B) N_t = \Theta(B) a_t, \tag{2·2}$$

where $\Phi(B) = I_m - \Phi_1 B - \Phi_2 B^2 - \ldots \Phi_p B^p$, $\Theta(B) = I_m - \Theta_1 B - \ldots - \Theta_q B^q$, I_m is the unit $m \times m$ matrix,

$$\Phi_i = ((\phi_{ijk})), \quad \Theta_l = ((\theta_{ljk})) \quad (i = 1, \ldots, p;\ l = 1, \ldots, p;\ j, k = 1, \ldots, m)$$

are $m \times m$ parameter matrices, and $a'_t = (a_{1t}, \ldots, a_{mt})$ for $t = \ldots -1, 0, 1, \ldots$ are independent and identically distributed vector normal random variables with mean vector zero and nonsingular dispersion matrix Σ. We require that the parameter space is such that $|\Phi(B)| \neq 0$, and $|\Theta(B)| \neq 0$ for $|B| \leqslant 1$. The former condition which is also the stationarity condition ensures the expansion of the rational matrix function

$$\Phi^{-1}(B)\,\Theta(B) = \psi(B) = I_m + \sum_{j=1}^{\infty} \psi_j B^j$$

for all $|B| \leqslant 1$ and hence

$$N_t = a_t + \sum_{j=1}^{\infty} \psi_j a_{t-j}.$$

The second condition, which is the invertibility condition, ensures that $\Theta^{-1}(B)\,\Phi(B) = \pi(B) = I_m - \Sigma \pi_i B^i$ for all $|B| \leqslant 1$ and hence $a_t = N_t - \Sigma \pi_i N_{t-i}$. The process in (2·2) will be referred to as an ARMAV(p, q) model. These can be extended to include multiple seasonal time series also. Further discussions of them are skipped here to avoid complicated notations.

3. ESTIMATION

3·1. *Maximum likelihood*

Suppose that we have the observational vectors $Z_1, \ldots, Z_n$, possibly after differencing to get rid of some form of trend. Then the likelihood may be obtained in terms of an $m \times n$

matrix M whose tth column is given by $N_t = \Phi^{-1}(B)\,\Theta(B)\,a_t$ which is stationary. Now suppose that γ is the vector of unknown parameters in $\alpha(B)$, $\beta(B)$, Φ and Θ. Since the stationarity and invertibility conditions are satisfied the likelihood of the parameters γ and Σ is proportional to the joint density of $(a_1, \ldots, a_n)$. The logarithm of the likelihood may be given as

$$l(\gamma, \Sigma) \simeq \tfrac{1}{2} n \log|\Sigma^{-1}| - \tfrac{1}{2}\operatorname{tr}\{\Sigma^{-1} C(\gamma)\}, \tag{3·1}$$

where $C(\gamma) = \Sigma\, a_{it} a_{jt}$ is the $m \times m$ matrix of conditional residual sum of squares and products conditional on the starting values of N_t and a_t, and Σ^{-1} is the inverse of Σ. Following Rao (1973, pp. 446–9) and Pandit & Wu (1977), approximate maximum likelihood estimates of γ can be obtained by minimizing $|C(\gamma)|$. However, Hillmer & Tiao (1979) have given an algorithm to obtain the exact maximum likelihood estimates of the parameters of a multiple time series model without the intervention model part in it. This algorithm can be adapted to suit the present case without much difficulty. The resulting estimates will be denoted by $\hat{\gamma}$ and for large n the variance–covariance matrix of $\hat{\gamma}$ can be given as $V(\hat{\gamma}) = ((l_{ij}))^{-1}$, where $l_{ij} = \partial^2 l(\gamma, \Sigma)/\partial\gamma_i\,\partial\gamma_j$.

3·2. *Two-stage estimation*

We have seen before that approximate or exact maximum likelihood estimates can be obtained by minimizing or maximizing a nonlinear function of the parameters, which can be very difficult if m, the number of time series, or k, the number of interventions, is large. An alternative approach to ease the computations to some extent is outlined below.

We propose to build a time series model $\Phi(B)\,N_t = \Theta(B)\,a_t$ using the first $T-1$ observations, where T is the time at which the first intervention starts. This model, which does not contain the intervention parameters, can be estimated by the methods described previously and the minimization or maximization problem here is simpler because the number of parameters is smaller. The general model can now be rewritten as

$$Z_t = R(B)\,I_t + \hat{\Phi}^{-1}(B)\,\hat{\Theta}(B)\,a_t, \tag{3·2}$$

where the circumflexes imply that the corresponding operators are already estimated. We can rewrite (3·2) as

$$Y_t = Q(B)\,I_t + a_t, \tag{3·3}$$

where $Y_t = \hat{\Theta}^{-1}(B)\,\hat{\Phi}(B) Z_t$, $Q(B) = \hat{\Theta}^{-1}(B)\,\hat{\Phi}(B)\,R(B)$, and $Q(B)$ contains the unknown intervention parameters. The situation now is quite similar to a multivariate nonlinear regression estimation which can be handled.

For illustration, consider a bivariate time series model with one intervention, and

$$R(B) = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

Now taking $\hat{\pi}(B) = \hat{\Theta}^{-1}(B)\,\hat{\Phi}(B)$, we can write

$$Y_t = \hat{\pi}(B) Z_t = \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix}, \quad \hat{\pi}(B)\,I_t = \begin{bmatrix} \hat{\pi}_{11}(B) & \hat{\pi}_{12}(B) \\ \hat{\pi}_{21}(B) & \hat{\pi}_{22}(B) \end{bmatrix} I_t = \begin{bmatrix} x_{1t} & x_{2t} \\ x_{3t} & x_{4t} \end{bmatrix}.$$

When there are n observations $(t = 1, \ldots, n)$ we can write the model corresponding to (3·3) as

$$Y = X\beta + a, \tag{3·4}$$

where

$$Y' = [y_{11}, \ldots, y_{1n}, y_{21}, \ldots, y_{2n}], \quad X' = \begin{bmatrix} x_{11} \ldots x_{1n} & x_{31} \ldots x_{3n} \\ x_{21} \ldots x_{2n} & x_{41} \ldots x_{4n} \end{bmatrix},$$

$$\beta' = (\beta_1, \beta_2), \quad a' = [a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}].$$

If the estimated parameter matrices Φ, Θ and Σ are treated as known, then the model (3·4) satisfies the conditions of the generalized least squares with the variance–covariance matrix of a being $V(a) = \Omega = (\hat{\Sigma} \otimes I)$ where $\otimes$ denotes the Kronecker product. Hence conditionally on Φ, Θ and Σ

$$\hat{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}Y, \quad V(\hat{\beta}) = (X'\Omega^{-1}X)^{-1}.$$

The situation is very similar to two-stage least squares in econometrics (Theil, 1971, pp. 497–500), and it does not seem to be worthwhile repeating the properties of these estimators.

4. EXAMPLE

The data shown in Fig. 1(a) and (b) are respectively the percentage changes from month to month of wages and salaries, z_{1t}, and of the Consumer Price Index, z_{2t}, from February 1967 to December 1977 for the whole of Canada. We consider the following two events as interventions to the bivariate series $Z_t = (z_{1t}, z_{2t})'$:

$I_t(T_1)$, the price boost in the oil in May 1973 by the Orgnaization of Petroleum Exporting Countries, OPEC,

$I_t(T_2)$, creation in October 1975 by the Federal Government of Canada of the Anti-inflation Board, AIB, to control inflation.

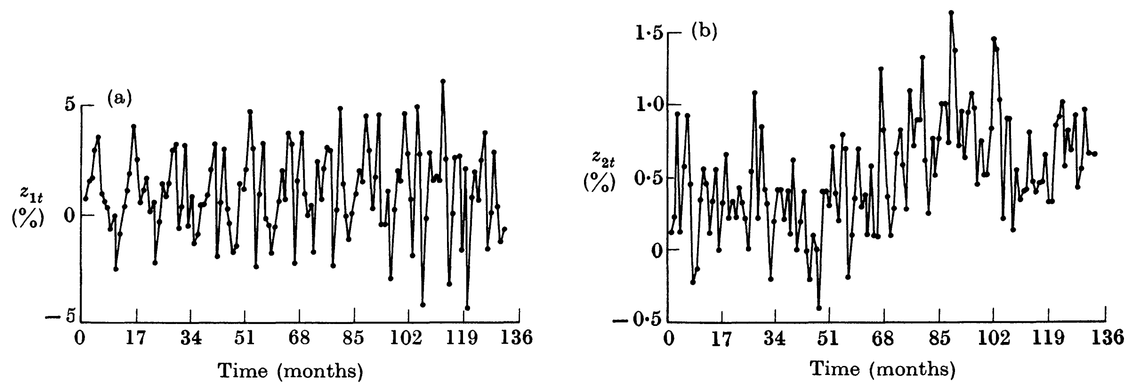

Fig. 1. Percentage change from month to month, February 1967–December 1977, in (a) Canadian wages and salaries, and (b) Canadian Consumer Price Index.

Considering the data prior to the first intervention, examining the series z_{it}, $(1-B^{12})z_{it}$ $(i = 1, 2)$, their autocorrelations and cross-correlations, we arrive at a model

$$(1-B^{12})Z_t = (I-\theta B)(I-\Theta B^{12})a_t, \tag{4·1}$$

where

$$\theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}, \quad \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}, \quad a'_t = (a_{1t}, a_{2t}).$$

The estimation of the parameters in this model yields

$$\hat{\theta} = \begin{bmatrix} 0\cdot16 \pm 0\cdot10 & -0\cdot53 \pm 0\cdot29 \\ -0\cdot01 \pm 0\cdot03 & -0\cdot30 \pm 0\cdot10 \end{bmatrix}, \quad \hat{\Theta} = \begin{bmatrix} 0\cdot45 \pm 0\cdot11 & -0\cdot26 \pm 0\cdot29 \\ 0\cdot03 \pm 0\cdot04 & 0\cdot46 \pm 0\cdot10 \end{bmatrix},$$

where the corresponding standard errors are included. The diagnostic checks on the residuals revealed no model inadequacy, except that the ninth lag autocorrelation of the residuals from the CPI series was found to be larger than twice the approximate standard error. This was attributed to chance because no reasonable explanation could be found. Note that $\hat{\theta}_{21}$, $\hat{\Theta}_{12}$ and $\hat{\Theta}_{21}$ are not significantly different from zero and this will be used in the choice of the final model.

One would expect $I_t(T_1)$ to have no direct effect on z_{1t}; however, it might cause a gradual increase in the level of z_{2t}. Also, $I_t(T_2)$ could be expected to lower the levels of both the series. Hence we tentatively entertain the model

$$Z_t = R(B)\, I_t + \{(I - \theta B)\,(I - \Theta B^{12})/(1 - B^{12})\}\, a_t, \tag{4·2}$$

where

$$R(B) = \begin{bmatrix} 0 & \beta_{21} B \\ \beta_{12} B (1 - \alpha_{12} B)^{-1} & \beta_{22} B \end{bmatrix}, \quad I_t = \begin{bmatrix} I_t(T_1) \\ I_t(T_2) \end{bmatrix},$$

$$I_1(T_1) = \begin{cases} 0 & (t < \text{May } 1973), \\ 1 & (t \geqslant \text{May } 1973), \end{cases} \quad I_t(T_2) = \begin{cases} 0 & (t < \text{November } 1975), \\ 1 & (t \geqslant \text{November } 1975). \end{cases}$$

Maximum likelihood estimation as discussed in § 3 yields

$$\hat{\beta}_{21} = -1\cdot14 \pm 0\cdot45, \quad \hat{\beta}_{12} = 0\cdot18 \pm 0\cdot16, \quad \hat{\alpha}_{12} = 0\cdot51 \pm 0\cdot46, \quad \hat{\beta}_{22} = -0\cdot36 \pm 0\cdot01,$$

$$\hat{\theta} = \begin{bmatrix} 0\cdot18 \pm 0\cdot10 & -0\cdot50 \pm 0\cdot29 \\ 0 & -0\cdot31 \pm 0\cdot10 \end{bmatrix}, \quad \hat{\Theta} = \begin{bmatrix} 0\cdot46 \pm 0\cdot11 & 0 \\ 0 & 0\cdot52 \pm 0\cdot10 \end{bmatrix},$$

$$\hat{\Sigma} = \begin{bmatrix} 0\cdot97 & 0\cdot03 \\ 0\cdot03 & 0\cdot12 \end{bmatrix}.$$

Under this model OPEC intervention led to a gradual increase in CPI rate; however, the AIB intervention lowered the level of CPI rate slightly although it did not seem to repair the damage caused by the first intervention. The AIB seems to have reduced the level of the percentage changes in wages and salaries more significantly.

The estimated model (4·1) indicates that there is only a one-way relation contrary to the usual notion of a two-way relation between 'wages' and 'prices'. Here the CPI seems to be leading the wages and salaries but not vice versa. This suggests putting $\theta_{21} = \Theta_{12} = \Theta_{21} = 0$ and writing the model (4·1) as

$$w_{1t} = -\{\theta_{12} B/(1 - \theta_{22} B)\}\, w_{2t} + (1 - \theta_{11} B)\,(1 - \Theta_{11} B^{12})\, a_{1t}, \quad w_{2t} = (1 - \theta_{22} B)\,(1 - \Theta_{22} B^{12})\, a_{2t}, \tag{4·3}$$

where

$$w_{it} = (1 - B^{12})\, z_{it} \quad (i = 1, 2).$$

One could consider interventions in (4·3) and it was found that this leads to very nearly the same estimates.

Univariate analyses of z_{1t} and z_{2t} were carried out with the same intervention model to contrast that with the multivariate analysis done. As expected the CPI series led to the same estimates. However, the univariate model for wages and salaries is

$$z_{1t} = -0{\cdot}26 \pm 0{\cdot}21 B I_t(T_2) + \{(1 - 0{\cdot}24 \pm 0{\cdot}09B)(1 - 0{\cdot}72 \pm 0{\cdot}06B^{12})/(1 - B^{12})\} a_{1t}.$$

We find here that the impact of the AIB intervention is less significant in the univariate analysis than it was in the multivariate analysis.

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