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AN LM TEST FOR A UNIT ROOT IN THE PRESENCE OF A STRUCTURAL CHANGE

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In this paper, we examine a suitably modified version of the unit root test proposed by Schmidt and Phillips (1992). A one-time structural break in the intercept does not affect its asymptotic distribution under the null hypothesis, and this is true whether the break is allowed for in the model or not. This implies that the asymptotic validity of this test statistic under the null is not affected by the incorrect placement of the structural break, by the allowance for a break when there is no break, or by no allowance for a break when there is a break.

1. INTRODUCTION

In this paper, we show that the asymptotic null distributions of the Schmidt and Phillips (1992) (henceforth, SP) unit root test statistics are not affected by a one-time structural break in the intercept. This invariance result holds whether the break is allowed for in the model or not. Our result contrasts with that of Perron's (1989) or Perron and Vogelsang's (1993) tests wherein the structural break affects the asymptotic distributions of the test statistics even under the null hypothesis of a unit root. Their tests are based on the Dickey and Fuller (1979, henceforth, DF) parameterization. The exogeneity of the structural break has been questioned by Banerjee, Lumsdaine, and Stock (1992), Christiano (1992), Perron (1990), and Zivot and Andrews (1992) in which the break is allowed to occur at an unknown time so that its placement is decided by the data. However, the asymptotic validity of the SP test statistics under the null is not affected by the incorrect placement of the structural break, by the allowance for a break when there is no break, or by no allowance for a break when there is a break.

In addition, we show that both the SP and DF tests are not affected by a one-time structural break when the null hypothesis is true. It is also shown that the SP tests are biased toward accepting the null when the alternative hypothesis of trend-stationarity around the trend containing a one-time break is true. Thus, the contribution made by Perron (1989), who suggested that

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the usual DF tests are biased toward accepting the null hypothesis when the alternative is true, is still valid in the SP framework. The empirical application to the Nelson and Plosser (1982) data indicates that we often fail to reject the null of a unit root.

The plan of this paper is as follows. In Section 2, we provide the asymptotic results for the SP-type tests in the presence of a possible structural break. In Section 3, we briefly provide the size and power properties of our tests and demonstrate the empirical evidence for the Nelson–Plosser data. Section 4 gives our concluding remarks. Throughout this paper, $\rightarrow$ indicates weak convergence as $T \rightarrow \infty$.

2. INVARIANCE RESULTS

We consider the data generating process (DGP), which is based on the unobserved components representation:

$$y_t = \delta' Z_t + X_t, \qquad X_t = \rho X_{t-1} + \varepsilon_t, \tag{1}$$

where Z_t contains exogenous variables. The unit root null hypothesis is $\rho = 1$. This DGP is consistent with Perron's models. We consider the crash model with $Z_t = [1, t, DU_t]'$, where $DU_t = 1$ for $t \geq T_B + 1$ and 0 otherwise and where T_B stands for the time period when the structural change occurs. We denote $\delta = (\delta_1, \delta_2, \delta_3)'$. It should be noted that our main results are for this crash model only and would not hold for the changing growth model or the model of sudden change in the level and growth path. Also, Perron (1989) considered two versions of the crash model, the additive outlier (AO) and the innovative outlier (IO) versions, where the former allows the economy to react instantaneously to a shock to its level and the latter allows a gradual reaction. In this paper, we consider only the AO version of Perron's model. Depending on whether ρ is equal to 1 or not, (1) implies

$$\text{null:} \quad y_t = \mu_0 + d \cdot D(TB)_t + y_{t-1} + v_t \tag{2a}$$

$$\text{alternative:} \quad y_t = \mu_1 + \beta \cdot t + (\mu_2 - \mu_1) \cdot DU_t + v_t, \tag{2b}$$

where v_t is stationary, and $D(TB)_t = 1$ for $t = T_B + 1$ and 0 elsewhere. We have $\mu_0 = \delta_2$, $d = \delta_3$, $y_0 = \delta_1 + X_0$, and $v_t = \varepsilon_t$ under the null and $\mu_1 = \delta_1$, $\beta = \delta_2$, $(\mu_2 - \mu_1) = \delta_3$, and $v_t = X_t$ under the alternative.

The SP-type test statistics allowing for structural change are obtained from the following regression according to the LM (score) principle as in SP.

$$\Delta y_t = \delta' \Delta Z_t + \phi \tilde{S}_{t-1} + e_t, \tag{3}$$

where $\tilde{S}_t = y_t - \tilde{\psi}_x - Z_t \tilde{\delta}$, $t = 2, \ldots, T$, $\tilde{\delta}$ are the coefficients in the regression of Δy_t on ΔZ_t, and $\tilde{\psi}_x$ is the restricted maximum likelihood estimate (MLE) of ψ_x ($\equiv \psi + X_0$) given by $y_1 - Z_1 \tilde{\delta}$. The unit root hypothesis is $\phi = 0$

(see Schmidt and Phillips, 1992, for more details). Then, the test statistics are defined by

$$\tilde{\rho} = T \cdot \tilde{\phi}, \tag{4a}$$

$$\tilde{\tau} = t - \text{statistic for the hypothesis } \phi = 0. \tag{4b}$$

To establish the asymptotic distributions of the preceding SP-type tests, we need the following assumption, which allows for serially correlated as well as heterogeneously distributed innovations. We define the two nuisance parameters σ^2 and σ_ε^2 as in (19) and (20) of SP.

Assumption 1.

(i) The data are generated according to (1) with $Z_t = (1, t, DU_t)'$.
(ii) The innovations ε_t satisfy the regularity conditions of Phillips and Perron (1988, p. 336).
(iii) $T_B/T \to \lambda$ as $T \to \infty$.

The DGP in the preceding considers crash models (2a) and (2b), where a one-time structural break occurs in the intercept.

THEOREM 1. *Suppose that Assumption* 1 *holds and* $\rho = 1$. *Let* $\tilde{\rho}$ *and* $\tilde{\tau}$ *be defined as in* (4). *Then, the asymptotic distributions of these statistics are the same as in equations* (21) *and* (22) *of* SP.

Theorem 1 says that allowing for a structural break at a single known time does not affect the asymptotic distributions of the SP-type test statistics under the null hypothesis of a unit root. This is so whether there is a break ($\delta_3 \neq 0$) or not ($\delta_3 = 0$), and in particular the asymptotic distributions of the statistics are unaffected by the allowance for a break when there is no break. The intuitive reason is that the SP-type tests are based on a regression in differences, and $\Delta DU_t = D(TB)_t$ equals 1 at only one point, so that its inclusion has no effect asymptotically. This result differs from that of the DF-type tests, as suggested in Perron's Theorem 2 (1989, p. 1373), which shows that the asymptotic distributions of his DF-type unit root test statistics depend on $\lambda = T_B/T$ even under the null. Perron's DF-type tests are based on a regression that includes DU_t in levels, and DU_t equals 1 for a constant fraction $(1 - \lambda)$ of the sample even asymptotically.

Next, we consider what will be the effect on the asymptotic distributions of the usual DF and SP tests under the null hypothesis, if there occurs a one-time structural break, but it is ignored.

THEOREM 2. *Suppose that Assumption* 1 *holds and* $\rho = 1$. *Suppose that there occurs a structural break at time* T_B. *Then, the asymptotic distributions of the usual* DF *and* SP *tests are unaffected by the structural break.*

Theorem 2 indicates that the asymptotic null distributions of both the usual DF and SP tests are not affected by the presence of a one-time structural break. Thus, the asymptotic null distributions of the DF tests do not depend on the break if one does not allow for it even when it exists. However, as suggested by Perron (1989), they depend on it if one allows for it in the model. This result may imply that the change in the null distributions of the DF extension tests arises from the way in which they are formulated to allow for a break. Theorems 1 and 2 combined indicate that while the asymptotics of the SP-type tests under the null are unaffected either by the presence of a break or by allowance for such a break, the asymptotics of Perron's tests under the null are affected by the presence of a break if one allows for it in the model but are unaffected if no allowance is made for the break when it exists.

Incorrect placement of a structural break is a combination of allowing for a break where none occurs and failing to allow for a break where it does occur. Combining Theorems 1 and 2, the asymptotic null distributions of the SP statistics are unaffected by the incorrect placement of a structural break. Thus, their asymptotic null distributions are invariant to allowance for a break, whether or not it occurs and whether or not it is correctly placed, and also to the failure to allow for a break that does occur.

This raises an obvious question: Why should we allow for a break in performing the SP-type tests? The answer is that we do so to increase the power of the tests. Perron shows in his Theorem 1 that the usual DF tests will be biased toward accepting the null, when the alternative is true and the break occurs. This fact is also true for the SP tests. The following theorem formally shows that the equivalent of Perron's Theorem 1 also holds for the usual SP tests.

THEOREM 3. *Suppose that Assumption* 1 *holds with* $\rho < 1$. *Suppose that a structural break occurs at time* T_B. *Define*

$$\sigma_x^2 = \lim_{T\to\infty} \sum_{t=1}^{T} \mathrm{Var}(X_t),$$

$$\gamma_1 = \lim_{T\to\infty} T^{-1} \sum_{t=1}^{T} E(X_t X_{t-1}),$$

$$\rho_1 = \gamma_1/\sigma_x^2.$$

Let X_∞ *represent the weak limit of* X_t *as* $t \to \infty$. *Then, if* $\tilde{\phi}$ *is the coefficient of* $\tilde{S}_{t-1}$ *in the regression of* Δy_t *on* $[1, \tilde{S}_{t-1}]$,

$$\tilde{\phi} \to \frac{\sigma_x^2(\rho_1 - 1)}{\sigma_x^2 + \frac{1}{3}(X_\infty^2 + X_1 X_\infty + X_1^2) + (\lambda^2 - \lambda + \frac{1}{3})\delta_3^2 + (X_\infty - X_1)(\lambda^2 - \frac{1}{3})\delta_3 + X_1\delta_3}.$$

Ignoring the break if one exists reduces the power of the tests under the alternative. Theorem 3 shows that when the alternative is true ($\rho < 1$), the asymptotic distribution of the estimated coefficient in the usual SP tests depends on the magnitude of the structural change. In particular, $\tilde{\phi} \to 0$ as $\delta_3 \to \infty$, which corresponds to a potential bias toward accepting the null, as the magnitude of the structural change increases.

3. FURTHER RESULTS

In this section, we provide results on the size and power of the test statistics by Monte Carlo simulation and demonstrate the empirical evidence. The FORTRAN subroutine program GASDEV/RAN3 of Press, Flannery, Teukolsky, and Vetterling (1986) has been used for generating the pseudo-i.i.d. $N(0,1)$ random numbers.

To allow for autocorrelated errors, we employ the augmented t-test as in Said and Dickey (1984). The DGP for the crash model is (1) with the moving average error structure: $\varepsilon_t = u_t + \theta u_{t-1}$. For this model, which corresponds to the AO version of Perron's crash model, Perron and Vogelsang (1993) corrected the procedure in Perron (1989) to eliminate the dependency of Perron's statistic on nuisance parameters by introducing the term $D(TB)_t$ in the regression:

$$\tilde{y}_t = \sum_{j=0}^{k} \omega_j D(TB)_{t-j} + \rho \tilde{y}_{t-1} + \sum_{j=1}^{k} c_j \Delta \tilde{y}_{t-1} + \text{error}, \tag{5}$$

where $\tilde{y}_t$ is the residual from the regression of y_t on $(1, t, DU_t)$. We denote the t-test statistic on ρ from this regression as $\hat{\tau}_{PV}$. The SP statistic $\tilde{\tau}$ comes from regression (3) including additional augmented terms of $\Delta\tilde{S}_t$.

When the errors are autocorrelated, there is a question as to whether equation (3) should be augmented only with lags of $\Delta\tilde{S}_{t-1}$ or lags of ΔZ_t should also be included. As a matter of notation, we let $\tilde{\tau}^*$ denote the $\tilde{\tau}$ test augmented with lags of both $\Delta\tilde{S}_t$ and $D(TB)_t$. We address this question by following Schmidt and Lee (1991) and rewrite equation (3) as

$$\Delta\tilde{S}_t = \gamma' \Delta Z_t + \phi \tilde{S}_{t-1} + e_t. \tag{6}$$

Because $\Delta\tilde{S}_t = \Delta y_t - \tilde{\delta}_2 - \tilde{\delta}_3 D(TB)_t$, (3) and (6) are the same except that the coefficients of ΔZ_t differ. Thus, least squares applied to (3) and (6) yield the same statistics $\tilde{\tau}$ and $\tilde{\rho}$. We note that $\gamma = 0$ in the population, and the regressors ΔZ_t are accordingly redundant in (6) in this case. There does not appear to be any compelling reason to include lags of ΔZ_t in the augmented tests. Adding lags of $D(TB)_t$ to the regression amounts to "dummying-out" a number of observations immediately after the break point and can be criticized as overfitting, as discussed in Phillips and Loretan (1991). Alternatively, if we view the error e_t in (3) or (6) as having the AR representation $A(L)e_t = w_t$, we can write $\Delta\tilde{S}_t = A(L)\delta'\Delta Z_t + (1 - A(L))\Delta\tilde{S}_t + A(L)\phi\tilde{S}_{t-1} + w_t$. This introduces lags of ΔZ_t but not in an unrestricted fashion. The $\tilde{\tau}^*$ test ignores the restrictions that $A(L)$ determines the lag coefficients of both

ΔZ_t and $\Delta \tilde{S}_t$. Again, ignoring these restrictions leads one to fit too many parameters and might not be expected to lead to a test with desirable finite-sample performance. In our case, this problem is not serious as we see in the following simulations, because $D(TB)_t$ is negligible both asymptotically and in finite samples of reasonable size.

Table 1 contains size and power results for three test statistics we consider; $\tilde{\tau}$, $\tilde{\tau}^*$, and $\hat{\tau}_{PV}$ for the sample size $T = 100$ and with $k = 4$ and $k = 12$, where k is the number of augmentations. The appropriate critical values for the $\tilde{\tau}$ and $\tilde{\tau}^*$ test are the same as the usual SP statistics, as suggested in Theorem 1. They are -3.63, -3.32, and -3.06 for $T = 100$ at the 1, 2.5, and 5% levels. The parameter λ for the break point is set at .5 for the simulation consisting of 10,000 replications.

Experiment A in Table 1 considers the size of the tests under the unit root null in the presence of MA errors ($\theta = -.8, -.5, 0, .5, .8$). When the structural change occurs ($\delta_3 = 5$), all three tests are similar, and they display no significant size distortions when enough augmentations are taken. When different magnitudes of the structural change are considered (the results are not reported here), the sizes of these tests do not change much, which indicates that they successfully eliminate the effect of the structural change. The small difference between the $\tilde{\tau}$ and $\tilde{\tau}^*$ tests reflects the fact that, because the lagged terms of $D(TB)_t$ are asymptotically negligible in the SP-type test procedure, adding lags of $D(TB)_t$ has minimal effect on the performance of the tests.

TABLE 1. Size comparisons, 5% rejection

					$\tilde{\tau}$		$\tilde{\tau}^*$		$\hat{\tau}_{PV}$	
Experiment	T	ρ	δ_3	θ	$(k = 4)$	$(k = 12)$	$(k = 4)$	$(k = 12)$	$(k = 4)$	$(k = 12)$
A	100	1	5	−.8	.244	.054	.232	.056	.298	.026
				−.5	.062	.049	.063	.055	.042	.020
				.0	.053	.056	.050	.058	.039	.029
				.5	.045	.058	.043	.058	.053	.048
				.8	.029	.056	.027	.055	.047	.061
					$(k = 0)$		$(k = 0)$		$(k = 0)$	
B	100	.9	5	0	.261		.261		.159	
		.8			.728		.728		.529	
		.7			.951		.951		.907	
		.5			.998		.998		1.00	

A slight size distortion of all tests under i.i.d. errors is due to using more than enough augmentations.

Experiment B considers the power of the tests under i.i.d. errors (where virtually no size distortions are found) for different values of δ_3 and ρ. The initial value, $y_0 = X_0 = 0$, is chosen (see Schmidt and Phillips, 1992, for details on the role of the initial value in power comparison). The results show that the SP-type tests, $\tilde{\tau}$ and $\tilde{\tau}^*$, are generally not less powerful than the $\hat{\tau}_{PV}$ test, and sometimes more powerful. When there are large divergences from a unit root, for instance, $\rho = .5$ or lower, the DF-type tests are powerful relative to the SP-type tests, but in this case the power is close to 1.0. The overall results indicate that the correct size of the SP extension tests is not obtained under the sacrifice of power.

Table 2 shows the results of applying the $\tilde{\tau}$, $\tilde{\tau}^*$, and $\hat{\tau}_{PV}$ tests to the 11 Nelson and Plosser (1982) data series considered in Perron (1989). These statistics are applicable to the AO versions of the crash model, and their applicability would be potentially limited by the finding in Perron (1989) that the IO model is more appropriate for these series. All series are in logs except the "interest rate" series. The number of augmentations, k, is determined by the same procedure in Perron. We reject the null in favor of the alternative

TABLE 2. Empirical test results[a]

			$\tilde{\tau}$		$\tilde{\tau}^*$		$\hat{\tau}_{PV}$	
Series	T	T_B	k	Statistic	k	Statistic	k	Statistic
Real GNP	62	21	1	−3.15*	2	−3.35*	7	−3.49
Nominal GNP	62	21	1	−2.58	2	−2.56	6	−2.74
Real per capita GNP	62	21	1	−2.96	2	−1.47	7	−3.24
Industrial production	111	70	5	−2.64	2	−4.22***	7	−3.50
Employment	81	40	6	−2.96	3	−3.01	6	−2.81
GNP deflator	82	41	1	−2.25	1	−2.28	5	−2.99
Consumer Price Index	111	70	5	−2.73	5	−2.83	5	−2.31
Nominal wage	71	30	7	−3.04	2	−2.42	7	−3.34
Money stock	82	41	1	−3.18*	2	−2.76	1	−3.65
Velocity	102	61	0	−1.71	0	−1.72	0	−1.73
Interest rate	71	30	2	−1.12	0	−0.838	2	−1.66

[a]Critical values for $\tilde{\tau}$ and $\tilde{\tau}^*$ at the 1, 2.5, and 5% significance levels are −3.67, −3.36, and −3.09 for T = 62 and T_B = 21, and −3.64, −3.32, and −3.06 for T = 111 and T_B = 70. Critical values for other sample sizes can be calculated via interpolation. Asymptotic critical values for $\hat{\tau}_{PV}$ at the same significance levels are −4.39, −4.03, and −3.76 for $\lambda = .3$.

*Significant at the 5% level. **Significant at the 2.5% level. ***Significant at the 1% level.

of trend stationarity for the "real GNP" series with both tests at the 5% level, the "money stock" series with the $\tilde{\tau}$ test at the same level, and the "industrial production" series with the $\tilde{\tau}^*$ test at the 1% level of significance. Most of the other series are found to have a unit root. We note that the results for the $\hat{\tau}_{PV}$ test are also supportive of a unit root for these series.

4. CONCLUDING REMARKS

We find that allowing or not allowing for a one-time structural change does not affect the asymptotic distribution theory for SP statistics under the null hypothesis. As a result, the asymptotic validity of the SP tests is not affected by possibly incorrect placement of the structural break or by allowance for a break when there is not a break (or vice versa).

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APPENDIX

Proof of Theorem 1. We note that $\Delta\tilde{S}_t = \varepsilon_t - (\tilde{\delta}' - \delta')Z_t$ and $\tilde{S}_t = \sum_{j=2}^{t} \varepsilon_j - (\tilde{\delta}' - \delta')(Z_t - Z_1)$. Let $S_t = \sum_{j=2}^{t} \varepsilon_j$ and $[rT]$ be the integer part of rT, for $r \in [0,1]$. Then, we get

$$T^{-1/2}\tilde{S}_{[rT]} = T^{-1/2}S_{[rT]} - ([rT] - 1)T^{-1}T^{1/2}(\tilde{\delta}_2 - \delta_2)$$
$$- T^{-1}(DU_{[rT]} - DU_1)\cdot T^{1/2}(\tilde{\delta}_3 - \delta_3). \quad \textbf{(A.1)}$$

The first term on the right-hand side of (A.1) follows: $T^{-1/2}S_{[rT]} \to \sigma W(r)$, which is a standard result. For the second term, we note that $(\tilde{\delta}_2 - \delta_2)$ is asymptotically equivalent to $\bar{\varepsilon}$, because $D(TB)_t$ equals 1 at only one point. Thus, $([rT] - 1)T^{-1} \times T^{1/2}(\tilde{\delta}_2 - \delta_2) \to \sigma r W(1)$. In the third term, $DU_{[rT]} - DU_1 \to 0$ if $r < \lambda$, and 1 if $r \geq \lambda$. We note $(\tilde{\delta}_2 - \delta_2) = [D(TB)'MD(TB)]^{-1}D(TB)'M\varepsilon$, where $M = I - i(i'i)^{-1}i' = I - T^{-1}ii'$, and i is a vector of ones. Then, because $D(TB)'M\varepsilon = \varepsilon_{TB+1} - \bar{\varepsilon}$, and $T^{1/2}(\tilde{\delta}_3 - \delta_3) = [1 - T^{-1}]^{-1}T^{1/2}(\varepsilon_{TB+1} - \bar{\varepsilon}) \to -\sigma W(1)$, the third term is $o_p(1)$. Thus, (A.1) follows:

$$T^{-1/2}\tilde{S}_{[rT]} \to \sigma[W(r) - rW(1)] = \sigma V(r), \quad \textbf{(A.2)}$$

where $V(r)$ is a Brownian bridge. This is the same expression as that we obtain from the usual SP tests ignoring the break; see the equation just before (A3.1) in Schmidt and Phillips (1992, p. 24). Therefore, $\tilde{\rho}$ and $\tilde{\tau}$ have the same asymptotic distributions as the original SP tests. ■

Proof of Theorem 2. The DGP is (1) with $\rho = 1$. When the SP test statistics (ignoring the break) are calculated from data following DGP, we have $\Delta y_t = \delta_2 + \delta_3 D(TB)_3 + \varepsilon_t$, $\tilde{\delta}_2 = \delta_2 + T^{-1}\delta_3 + \bar{\varepsilon}$. We can express $\tilde{S}_t$ as

$$\tilde{S}_t = (y_t - y_1) - \tilde{\delta}_2(t - 1) = (X_t - X_1) + \delta_3 DU_t - (\tilde{\delta}_2 - \delta_2)(t - 1)$$
$$= \sum_{j=2}^{t} \varepsilon_j - (t - 1)\bar{\varepsilon} + \delta_3 DU_t - (t - 1)\delta_3/T. \quad \textbf{(A.3)}$$

From this we obtain $T^{-1/2}\tilde{S}_{[rT]} = T^{-1/2}\sum_{j=2}^{rT} \varepsilon_j - ([rT] - 1)T^{-1}\cdot T^{1/2}\bar{\varepsilon} + T^{-1/2}\delta_3 \times DU_t - ([rT] - 1)T^{-3/2}\delta_3$. Because the last two terms vanish as $T \to \infty$, we obtain $T^{-1/2}\tilde{S}_{[rT]} \to \sigma[W(r) - rW(1)]$. This result is the same as (A.2) and also the same as that we obtain from the usual SP tests ignoring the break. Therefore, ignoring the break in the usual SP tests does not matter asymptotically under the null.

Next, we want to show that this result also holds for the DF tests. The DGP is again (1). Define $W_t \equiv [1,t]$. Because the DF tests are based on the regression $y_t = \gamma_1 + \gamma_2 t + \rho y_{t-1} + \varepsilon_t$, we can express $\hat{S}_t$ as the residual from regression of y_t on W_t. Thus, we obtain

$$T^{-1/2}\hat{S}_t = T^{-1/2}\delta_3\cdot\widetilde{DU}_t + T^{-1/2}\tilde{X}_t, \quad \textbf{(A.4)}$$

where $\widetilde{DU}_t$ and $\tilde{X}_t$ are the residuals from the regression of DU_t and X_t on W_t, respectively. The properties of the DF tests will depend on the properties of this projection residual, because the DF tests are obtained by a regression of $\hat{S}_t$ on $\hat{S}_{t-1}$. Notice that the second term on the right-hand side in (A.4) converges to a demeaned and detrended Wiener process, as shown in equation (24) in Kwiatkowski, Phillips, Schmidt, and Shin (1992, p. 168) and in equation (16) in Park and Phillips (1988, p. 474). This is exactly the same asymptotic result as we encounter in considering the usual DF test statistics that do not allow for the break. Therefore, to complete the proof, we only need to show that the first term vanishes asymptotically. To do so, define the convergence rate matrix: $R = \text{diagonal}[T^{-1/2}, T^{-3/2}]$. Then, the first term on the right-hand side in (A.4) follows:

$$T^{-1/2}\delta_3 DU_t - T^{-1/2}\delta_3 W_t R(RW'WR)^{-1}RW'DU$$
$$= \delta_3 T^{-1/2} DU_t - \delta_3 (T^{-1/2}, tT^{-3/2})(RW'WR)^{-1}(T^{-1}\Sigma DU_t, T^{-2}\Sigma tDU_t)' \quad \textbf{(A.5)}$$

Here, notice that the following limits exist:

$$(RW'WR) \to \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}, \quad T^{-1}\Sigma DU_t \to \lambda, \quad \text{and} \quad T^{-2}\Sigma tDU_t \to (1-\lambda^2)/2.$$

But because $T^{-1/2}DU_t \to 0$ and $(T^{-1/2}, tT^{-3/2}) \to 0$, the whole terms in (A.5) vanish asymptotically, which finishes the proof. ■

Proof of Theorem 3. The DGP is (1) with $\rho < 1$ under the alternative hypothesis. In this case, we may express $\tilde{S}_t$ in (A.3) for the usual SP test statistics as $\tilde{S}_t = (X_t - X_1) - (t-1)\overline{\Delta X} + \delta_3[DU_t - (t-1)/T]$. Note that $\overline{\Delta X} = O_p(T^{-1})$, because $T\overline{\Delta X} = X_t - X_1 = O_p(1)$. Thus, $\overline{\Delta X}$ and $\sqrt{T}\overline{\Delta X}$ vanish asymptotically. Now, consider the following expression for the denominator of $\tilde{\phi}$:

$$T^{-1}\Sigma\tilde{S}_t^2 = T^{-1}\Sigma[(X_t - X_1) - (t-1)\overline{\Delta X}]^2 + T^{-1}\delta_3^2\Sigma[DU_t - (t-1)/T)]^2$$
$$+ 2T^{-1}\delta_3\Sigma[(X_t - X_1) - (t-1)\overline{\Delta X}][DU_t - (t-1)/T].$$

By expanding each term of the preceding expression, we obtain the asymptotic distribution of the denominator as given in the theorem. For the numerator of $\tilde{\delta}$, we consider

$$\begin{aligned} T^{-1}\Sigma\tilde{S}_t\Delta\tilde{S}_t &= T^{-1}\Sigma[(X_{t-1} - X_1) - (t-2)\overline{\Delta X} + \delta_3(DU_{t-1} - (t-2)/T)] \\ &\quad \cdot [(\Delta X_t - \overline{\Delta X}) + \delta_3(D(TB)_t - T^{-1})] \\ &= T^{-1}\Sigma(\Delta X_t - \overline{\Delta X})\{(X_{t-1} - X_1) - (t-2)\overline{\Delta X}\} \\ &\quad + T^{-1}\delta_3\Sigma(\Delta X_t - \overline{\Delta X})\{DU_{t-1} - (t-2)/T)\} \\ &\quad + T^{-1}\delta_3[D(TB)_t - T^{-1}]\cdot[(X_{t-1} - X_1) - (t-2)\overline{\Delta X}] \\ &\quad + T^{-1}\delta_3^2\Sigma[D(TB)_t - T^{-1}]\cdot[DU_{t-1} - (t-2)/T]. \qquad \textbf{(A.6)} \end{aligned}$$

We can show that the first term follows as $\sigma_x^2(\rho_1 - 1)$ and that the other terms vanish asymptotically. This completes the proof. From the theorem, we notice that $\tilde{\delta}$ of the usual SP test ignoring break is not a consistent estimator of $(\rho_1 - 1)$. Interestingly, this is so even when there is no break ($\delta_3 = 0$). A nonzero value of δ_3 will tend to move the distribution of $\tilde{\delta}$ toward zero because $(\lambda^2 - \lambda + \frac{1}{3}) > 0$, and this is certainly so for large enough $|\delta_3|$. ■

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