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Journal of Statistical Planning and Inference 60 (1997) 291–310

journal of statistical planning and inference

Effect of dependence on statistics for determination of change [1]

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Received 25 July 1995; revised 15 April 1996

Abstract

Quite a number of test statistics and estimators for detection of a change in the mean of a series of independent observations were proposed and studied. The purpose of this paper is to examine the behaviour of these statistics if the observations are dependent, particularly, if they form a linear process.

AMS classification: 62M10, 62F05, 62F12

Keywords: Change-point problem; Linear process; Location model; AR(1)

1. Introduction

Consider the following time series model:

$$\begin{aligned} X_t &= \mu + e_t, \qquad t = 1,2,\ldots,[n\theta], \\ &= \mu + \delta + e_t, \qquad t = [n\theta]+1,\ldots,n, \end{aligned} \tag{1.1}$$

where $[s]$ denotes the integer part of s, μ, $\delta \neq 0$ and $\theta \in (0,1]$ are unknown parameters and $\{e_t\}_{t=1}^{\infty}$ is a linear process satisfying

$$e_t = \sum_{j=0}^{\infty} w_j \varepsilon_{t-j}, \quad t = 1,2,\ldots \tag{1.2}$$

where $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ are i.i.d. random variables with $E\varepsilon_t = 0$, $\operatorname{var}\varepsilon_t = \sigma^2 > 0$ and $E|\varepsilon_t|^{2+\Delta} < \infty$ for some $\Delta > 0$ and the weights $\{w_j\}_{j=0}^{\infty}$ satisfy

$$\sum_{j=0}^{\infty} j|w_j| < \infty, \qquad \sum_{j=0}^{\infty} w_j \neq 0. \tag{1.3}$$

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[1] Partially supported by grant GAČR–201/94/0472.

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PII S0378-3758(96)00138-3

We wish to test the null hypothesis

$$\mathrm{H}_0:\ \theta = 1$$

against the alternative

$$\mathrm{H}_1:\ \theta \in (0,1)$$

and if $0<\theta<1$ to estimate θ.

In case $\{e_t\}_{t=1}^{\infty}$ are i.i.d. random variables with mean zero and var $e_t = \sigma^2 > 0$ one often uses procedures based on the partial sums

$$S_k = \sum_{i=1}^{k}(X_i - \overline{X}_n), \quad k = 1,...,n \tag{1.4}$$

where $\overline{X}_n = n^{-1}\sum_{i=1}^{n} X_i$.

Typically, the test statistics are of the form

$$T_{n1}(q) = \max_{1\leqslant k\leqslant n}\left\{\frac{1}{\sqrt{n}\,q(k/n)}\,|S_k|\right\}, \tag{1.5}$$

$$T_{n2} = \max_{1\leqslant k<n}\left\{\sqrt{\frac{n}{k(n-k)}}\,|S_k|\right\}, \tag{1.6}$$

$$T_{n3}(G) = \max_{G<k\leqslant n}\left\{\frac{1}{\sqrt{G}}\,|S_k - S_{k-G}|\right\}, \tag{1.7}$$

$$T_{n4}(r) = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{r(k/n)}\,S_k^2 \tag{1.8}$$

where $q(\cdot)$ and $r(\cdot)$ are weight functions defined on $(0,1)$ and $G < n$.

We shall also consider the following estimators of the change point $m = [n\theta]$:

$$\hat{m}_1 = \min\{\arg\max\{|S_k|;\ k = 1,...,n\}\}, \tag{1.9}$$

$$\hat{m}_2 = \min\left\{\arg\max\left\{\sqrt{\frac{n}{k(n-k)}}\,|S_k|;\ k = 1,...,n-1\right\}\right\}. \tag{1.10}$$

The behaviour of the test statistics (1.5)–(1.8) and of the estimators (1.9) and (1.10) was widely studied for the case when $\{e_t\}$ are i.i.d. random variables with zero mean and finite variance. The information on these results can be found in Csörgő and Horváth (1988), Antoch and Hušková (1992, 1994), Antoch et al. (1995), Jandhyala and MacNeill (1989, 1991) among others.

The purpose of this paper is to examine the behaviour of the test statistics (1.5)–(1.8) under H_0 and of the estimators (1.9) and (1.10) under local alternatives in the set-up (1.1)–(1.3). In other words, we are interested in the effect of dependency of $\{e_t\}$ described by (1.2) and (1.3) on the behaviour of the procedures developed for detection and identification of the change in the mean of series of independent observations.

Situations with dependent error terms occur quite often in econometrics, hydrology as well as in other applications.

Tang and MacNeill (1993) investigated the influence of the serial correlation of the error term on the Bayesian test statistics for the detection change in regression models. Their test statistics are related to $T_{n4}(r)$. They also surveyed the results till 1993.

Nagaraj and Reddy (1993) dealt with Bayesian-type test statistics related to $T_{n4}(r)$. Brodsky and Darkhovsky (1993) studied statistic $T_{n1}(q)$ with $q(u) = 1$, $u \in (0, 1)$, and the consistency of the estimator related to $\hat{m}_1$ when $\{e_t\}$ form a ψ-mixing sequence.

Bai (1994) treated the estimator $\hat{m}_2$.

Horváth and Shao (1994) and Horváth and Kokoszka (1995) treated the case when $\{e_t\}$ are Gaussian random variables with long-range dependence.

There is also a number of papers concerning the partial sums of residuals related to the changes in stationary processes, see e.g. Kulperger (1985), Picard (1985), Bai (1993), Horváth (1993), Johnson and Bagshaw (1974).

The main results formulated in Section 2 can be summarized as follows: *The limit distribution of the statistics* (1.5)–(1.8) *under* H_0 *and of the estimators* (1.9) *and* (1.10) *are up to a multiplicative constant the same as in the case of independent observations.*

Results of simulation study are demonstrated in Section 3 while the proofs are presented in Section 4. The crucial step in proofs is Lemma 4.3, which says that the standardized partial sums S_k are close to standardized partial sums of i.i.d. random variables and as a consequence one receives a generalization of the Darling–Erdős and the Deheuvels–Revész theorems.

2. Main results

We assume that the weight function q belongs to the class

$$Q_{0,1} = \{\, q; q \text{ is nondecreasing in a neighbourhood of zero, nonincreasing in a neighbourhood of one and } \inf_{\eta \leqslant t \leqslant 1-\eta} q(t) > 0 \text{ for all } 0 < \eta < \tfrac{1}{2} \}.$$

Put

$$a(x) = \sqrt{2 \log x}, \quad b(x) = 2 \log x + \tfrac{1}{2} \log \log x - \tfrac{1}{2} \log \pi, \tag{2.1}$$

$$I^*(q, c) = \int_0^1 \frac{1}{t(1-t)} \exp\left\{ \frac{-cq^2(t)}{t(1-t)} \right\} \mathrm{d}t \tag{2.2}$$

and

$$\sigma^{*2} = \sigma^2 \left(\sum_{j=0}^{\infty} w_j \right)^2. \tag{2.3}$$

The main results on the limit behaviour of the test statistics $T_{n1}(q)$, T_{n2}, $T_{n3}(G)$ and $T_{n4}(r)$ say the following:

Theorem 2.1. *Assume that* (1.1)–(1.3) *and* H_0 *hold.*

(i) *If* $q \in Q_{0,1}$ *and* $I^*(q,c) < \infty$ *for some* $c > 0$, *then*

$$\lim_{n\to\infty} P\Big(T_{n1}(q) \leqslant x\sigma^*,\ a(\log n)\, T_{n2} \leqslant \sigma^*\big(b(\log n) + y\big)\Big)$$
$$= P\left(\sup_{0<t<1} \frac{|B(t)|}{q(t)} \leqslant x\right) \cdot \exp\{-2\exp\{-y\}\},$$
$$x \in \mathbb{R}^+, \quad y \in \mathbb{R} \tag{2.4}$$

where $B(t)$ *is the Brownian bridge.*

(ii) *If, as* $n \to \infty$, $G = G_n \to \infty$ *and* $G^{-1} n^{2/(2+\Delta)} \log n \to 0$, *then*

$$\lim_{n\to\infty} P\big(a(n/G)\, T_{n3}(G) \leqslant \sigma^*\big(b(n/G) + y\big)\big) = \exp\{-2\exp\{-y\}\}, \quad y \in \mathbb{R}. \tag{2.5}$$

(iii) *If* $\inf_t r(t) > 0$ *and* $\int_0^1 (t(1-t)/r(t))\,\mathrm{d}t < \infty$, *then*

$$\lim_{n\to\infty} P\big(T_{n4}(r) \leqslant x\sigma^*\big) = P\left(\int_0^1 \frac{B^2(t)}{r(t)}\,\mathrm{d}t \leqslant x\right), \quad x \in \mathbb{R}. \tag{2.6}$$

Remark 1. Notice that if $w_0 = 1$ and $w_j = 0$, $j = 1,2,\ldots$, the assertion reduces to the case of the i.i.d. error terms and $\sigma^{*2} = \sigma^2$, see Csörgő and Horváth (1988), Csörgő, et al. (1993), Deheuvels and Révesz (1987) among others.

Theorem 2.2. *Assume that* (1.1)–(1.3) *with* $\theta \in (0,1)$ *are satisfied and, as* $n \to \infty$,

$$\delta = \delta_n \to 0 \quad \textit{and} \quad \frac{|\delta|\sqrt{n}}{\sqrt{\log\log n}} \to \infty. \tag{2.7}$$

Then, as $n \to \infty$,

$$\frac{\delta^2(\hat{m}_1 - [n\theta])}{\sigma^{*2}} \xrightarrow{\mathscr{D}} \min\Big\{\arg\max$$
$$\times\Big\{W(t) - |t|\big((1-\theta)I\{t<0\} + \theta I\{t \geqslant 0\}\big);\quad t \in \mathbb{R}\Big\}\Big\} \tag{2.8}$$

and

$$\frac{\delta^2(\hat{m}_2 - [n\theta])}{\sigma^{*2}} \xrightarrow{\mathscr{D}} \min\{\arg\max\ \{W(t) - |t|/2;\ t \in \mathbb{R}\}\} \tag{2.9}$$

where $\{W(t), t \in \mathbb{R}\}$ *is the two-sided Wiener process and* $I\{A\}$ *is the indicator of the set* A.

Remark 2. Again, if $w_o = 1$ and $w_j = 0$, $j = 1,2,\ldots$, the assertions reduce to the known i.i.d. error terms case.

These two theorems exhibit the effect of the dependency on the behaviour of several test statistics and estimators developed for a change in the mean in a series of independent observations. The results can be extended to other test statistics and estimators

for a change in the mean in the location model and further to the case of a change in linear models.

Under (1.1)–(1.3) and H_0, $\{X_t\}$ is the second-order stationary with spectral density $f(\cdot)$. Hence the asymptotic variance of $\sqrt{n}\overline{X}_n$ is $2\pi f(0) = \sigma^{*2}$, see Brockwell and Davis (1991, Ch. 7.2). As usual, σ^{*2} is unknown and should be estimated. Here we use the estimator

$$\sigma_n^{*2}(L) = \hat{R}(0) + 2\sum_{k=1}^{L}\left(1 - \frac{k}{L}\right)\hat{R}(k), \quad L < n \tag{2.10}$$

where

$$\hat{R}(k) = \frac{1}{n}\sum_{t=1}^{n-k}\left(X_t - \overline{X}_n\right)\left(X_{t+k} - \overline{X}_n\right), \quad k \geqslant 0 \tag{2.11}$$

which has the following consistency properties.

Theorem 2.3. *Assume that* (1.1)–(1.3) *hold and*

$$L = L_n \to \infty, \quad \frac{L^2 \log L}{n} \to 0 \quad \text{as} \quad n \to \infty. \tag{2.12}$$

Then, under H_0

$$\sigma_n^{*2}(L) - \sigma^{*2} = O_p\left(\frac{1}{L} + \sqrt{\frac{L^2 \log L}{n}}\right) \quad \text{as} \quad n \to \infty \tag{2.13}$$

and, if moreover (2.7) *holds, then*

$$\sigma_n^{*2}(L) - \sigma^{*2} = o_p(1) \quad \text{as} \quad n \to \infty. \tag{2.14}$$

Remark 3. If we choose L such that $L^2/\log n \to \infty$ and $n^{-1}L^2 \log L = o(\log n)$, then the assertions in Theorems 2.1 and 2.2 remain true if σ^* is replaced by $\sigma_n^*(L)$. For practical purposes one should preferably use (in accordance with the i.i.d. case)

$$\tilde{\sigma}_n^{*2}(L) = \hat{R}^*(0) + 2\sum_{k=1}^{L}\left(1 - \frac{k}{L}\right)\hat{R}^*(k) \tag{2.15}$$

where

$$\hat{R}^*(k) = \frac{1}{n}\sum_{t=1}^{\hat{m}-k}\left(X_t - \overline{X}_{\hat{m}}\right)\left(X_{t+k} - \overline{X}_{\hat{m}}\right) + \sum_{t=\hat{m}+1}^{n-k}\left(X_t - \overline{X}^*_{\hat{m}}\right)\left(X_{t+k} - \overline{X}^*_{\hat{m}}\right),$$

$$\overline{X}^*_{\hat{m}} = \frac{1}{n - \hat{m}}\sum_{i=\hat{m}+1}^{n} X_i$$

and $\hat{m}$ is any of the estimators $\hat{m}_1$ or $\hat{m}_2$.

Remark 4. The estimator (2.10) corresponds to $2\pi\hat{f}(0)$, where $\hat{f}(0)$ is the Bartlett lag window estimator of the spectral density at frequency zero with the observations X_t centered around the sample mean, see Priestley (1981, Ch. 6).

In case $\{X_t\}$ is a stationary series with zero mean, the asymptotic variance of $\hat{f}(0)$ is of order $O(L/n)$, $L = L_n \to \infty$ as $n \to \infty$. Moreover, the choice $L = O(n^{1/3})$ makes the asymptotic order of the squared bias of Bartlett estimator equal to the order of asymptotic variance and thus minimizes the mean square error, see Parzen (1957, 1958). There exists a variety of sufficient conditions that guarantee this consistency result but all they need at least the fourth moment of $\{X_t\}$, see Priestley (1981, Ch. 6.2). The rate of consistency provided by (2.13) is not optimal in the above sense. On the other hand, we use the minimal moment conditions on $\{X_t\}$ only.

3. Simulations

The purpose of the simulations is to illustrate how the described procedures work. We concentrated on the test statistics $T_{n1}(q)$ and T_{n2} given by (1.5) and (1.6), resp. on the estimators $\hat{m}_1$ and $\hat{m}_2$ given by (1.9) and (1.10). Aside from that, we were interested in the behaviour of the statistics $\sigma_n^*(L)$ and $\tilde{\sigma}_n^*(L)$ given by (2.10) and (2.15) and the influence of the choice of L.

For the simulation study we took the model (1.1), where $\{e_t\}$ forms AR(1) sequence with autoregressive coefficient $\varrho \in \{-0.5, -0.3 : 0.1 : 0.3, 0.5, 0.7\}$ and $\{\varepsilon_j\}$ are i.i.d. N(0,1), hence $\sigma^* = (1-\varrho)^{-1}$. Sample sizes n used were 80, 120, 160 and 200, the change points under the alternative were set equal either to $n/4$, $n/2$ or $3n/4$, $L \in \{[n/4], [n/5], [n/7], [n/10]\}$ and $\delta \in \{1, 1.5, 2\}$. The algorithms itself were coded in MATLAB and for each situation we run 10 000 repetitions. Due to the similarity of some results and limitation of space we present only a small part of results in Tables 1–6.

Table 1(a)
Simulated critical values for the statistics $T_{n1}(q)$

	$n = 80$			$n = 120$			$n = 200$		
ϱ	10%	5%	2.5%	10%	5%	2.5%	10%	5%	2.5%
0.0	1.15	1.31	1.41	1.17	1.31	1.43	1.16	1.28	1.45
0.1	1.27	1.44	1.55	1.29	1.45	1.59	1.27	1.42	1.60
0.2	1.41	1.61	1.73	1.44	1.62	1.77	1.42	1.59	1.80
0.3	1.59	1.82	1.96	1.62	1.84	2.01	1.60	1.81	2.05
0.5	2.16	2.44	2.66	2.23	2.49	2.77	2.19	2.50	2.82
0.7	3.38	3.89	4.17	3.53	4.01	4.45	3.53	4.01	4.52
−0.1	1.06	1.20	1.29	1.08	1.20	1.31	1.06	1.17	1.33
−0.2	0.98	1.11	1.20	1.00	1.11	1.20	0.98	1.08	1.22
−0.3	0.93	1.03	1.12	0.93	1.03	1.13	0.90	1.01	1.14
−0.5	0.83	0.92	0.99	0.83	0.91	1.01	0.80	0.90	1.01

Table 1(b)
Theoretical 5% critical values (5% tcv) for the statistics $T_{n1}(q)$

ϱ	0.0	0.1	0.2	0.3	0.5	0.7	−0.1	−0.2	−0.3	−0.5
5% tcv	1.36	1.51	1.70	1.94	2.72	4.53	1.23	1.13	1.04	0.91

Table 2(a)
Simulated critical values for the statistics T_{n2}

	$n = 80$			$n = 120$			$n = 200$		
ϱ	10%	5%	2.5%	10%	5%	2.5%	10%	5%	2.5%
0.0	2.74	3.00	3.21	2.85	3.08	3.32	2.85	3.10	3.25
0.1	2.96	3.22	3.46	3.08	3.34	3.67	3.10	3.39	3.56
0.2	3.27	3.55	3.81	3.39	3.69	3.98	3.41	3.73	3.93
0.3	3.65	3.98	4.23	3.80	4.10	4.49	3.82	4.18	4.42
0.5	4.87	5.33	5.71	5.04	5.58	6.14	5.20	5.60	6.02
0.7	7.45	8.31	8.92	7.91	8.74	9.51	8.15	8.86	9.67
−0.1	2.57	2.81	3.01	2.66	2.87	3.11	2.65	2.89	3.06
−0.2	2.44	2.69	2.90	2.54	2.78	2.94	2.53	2.72	2.92
−0.3	2.35	2.57	2.87	2.44	2.69	2.87	2.42	2.62	2.79
−0.5	2.37	2.65	2.91	2.48	2.67	2.97	2.37	2.60	2.86

Table 2(b)
Theoretical 5% critical values (5% tcv) for the statistics T_{n2}

ϱ	0.0	0.1	0.2	0.3	0.5	0.7	−0.1	−0.2	−0.3	−0.5
5% tcv	3.64	4.04	4.55	5.20	7.28	12.13	3.31	3.03	2.80	2.43

Table 3(a)
Behavior of the estimator $\hat{m}_1$ under alternative $\theta = \frac{1}{4}$ and $\delta = 1.5$

	$n = 80$			$n = 120$			$n = 200$		
ϱ	$\bar{\hat{m}}_1$	$s_{\hat{m}_1}$	MAD	$\bar{\hat{m}}_1$	$s_{\hat{m}_1}$	MAD	$\bar{\hat{m}}_1$	$s_{\hat{m}_1}$	MAD
0.0	22.36	4.27	2.88	32.52	4.60	3.01	52.90	5.68	3.59
0.1	22.79	4.98	3.41	32.76	5.03	3.31	53.47	6.68	4.31
0.2	23.10	5.59	3.86	33.32	6.19	4.09	54.21	8.08	5.25
0.3	23.68	6.72	4.66	34.27	7.72	5.21	54.97	9.01	6.09
0.5	26.38	10.14	7.61	36.98	11.67	8.42	58.12	14.18	9.76
0.7	31.55	14.40	11.83	43.81	19.28	15.58	67.06	26.48	19.88
−0.1	22.04	3.64	2.45	32.24	4.13	2.65	52.25	4.20	2.78
−0.2	21.79	3.28	2.17	31.99	3.70	2.34	51.98	3.83	2.45
−0.3	21.58	2.73	1.89	31.68	3.15	2.01	51.81	3.53	2.25
−0.5	21.38	2.47	1.65	32.32	2.47	1.59	51.41	2.75	1.77

The (simulated) 50–99% quantiles of the test statistics were calculated with step 1%. However, in Tables 1(a) and 2(a) we present for statistics $T_{n1}(q)$, $q(t) = 1$, $t \in (0, 1)$ and $q(t) = 0$ otherwise, and for T_{n2}, only 10%, 5% and 2.5% simulated critical values.

Table 3(b)
Behavior of the estimator $\hat{m}_1$ under alternative $\theta = \frac{1}{2}$ and $\delta = 1.5$

	$n = 80$			$n = 120$			$n = 200$		
ϱ	$\bar{\hat{m}}_1$	$s_{\hat{m}_1}$	MAD	$\bar{\hat{m}}_1$	$s_{\hat{m}_1}$	MAD	$\bar{\hat{m}}_1$	$s_{\hat{m}_1}$	MAD
0.0	40.01	1.89	1.05	59.85	2.08	1.19	99.90	1.98	1.11
0.1	40.05	2.26	1.26	59.85	2.55	1.40	99.90	2.57	1.37
0.2	40.13	2.69	1.56	59.82	3.04	1.70	99.86	3.07	1.69
0.3	40.12	3.44	1.95	59.67	3.77	2.16	99.89	3.76	2.13
0.5	40.13	5.89	3.49	59.77	6.51	3.85	99.95	7.06	4.07
0.7	40.16	10.28	7.11	59.53	12.33	7.97	99.91	15.52	9.57
−0.1	39.99	1.65	0.90	59.87	1.70	1.00	99.91	1.67	0.95
−0.2	39.97	1.39	0.81	59.89	1.58	0.92	99.92	1.37	0.80
−0.3	39.96	1.27	0.74	59.89	1.45	0.86	99.94	1.26	0.74
−0.5	40.00	1.07	0.63	59.89	1.23	0.76	99.94	1.10	0.66

Table 4(a)
Behavior of the estimator $\hat{m}_2$ under alternative $\theta = \frac{1}{4}$ and $\delta = 1.5$

	$n = 80$			$n = 120$			$n = 200$		
ϱ	$\bar{\hat{m}}_2$	$s_{\hat{m}_2}$	MAD	$\bar{\hat{m}}_2$	$s_{\hat{m}_2}$	MAD	$\bar{\hat{m}}_2$	$s_{\hat{m}_2}$	MAD
0.0	20.19	2.93	1.54	30.18	2.76	1.38	50.10	2.52	1.35
0.1	20.29	3.84	1.97	30.23	3.30	1.67	50.08	2.95	1.60
0.2	20.56	4.84	2.50	30.45	5.32	2.40	50.28	3.97	2.11
0.3	21.03	6.99	3.57	30.65	6.76	3.29	50.23	5.23	2.74
0.5	23.15	12.04	7.44	32.08	12.79	6.96	51.82	14.32	7.12
0.7	29.66	18.68	15.17	40.49	25.26	19.22	59.46	34.55	22.51
−0.1	20.11	2.40	1.28	30.09	2.01	1.13	50.04	2.16	1.14
−0.2	20.06	2.12	1.11	30.05	1.67	0.94	49.96	1.82	1.01
−0.3	20.07	1.79	0.97	30.02	1.52	0.84	49.97	1.57	0.90
−0.5	20.04	1.26	0.83	30.01	1.17	0.68	49.99	1.26	0.71

Table 4(b)
Behavior of the estimator $\hat{m}_2$ under alternative $\theta = \frac{1}{2}$ and $\delta = 1.5$

	$n = 80$			$n = 120$			$n = 200$		
ϱ	$\bar{\hat{m}}_2$	$s_{\hat{m}_2}$	MAD	$\bar{\hat{m}}_2$	$s_{\hat{m}_2}$	MAD	$\bar{\hat{m}}_2$	$s_{\hat{m}_2}$	MAD
0.0	40.05	2.58	1.31	59.81	2.62	1.38	99.90	2.44	1.26
0.1	40.06	3.21	1.66	59.76	3.16	1.68	99.89	2.76	1.46
0.2	40.04	4.27	2.13	59.76	4.05	2.12	99.86	3.48	1.88
0.3	40.08	5.57	2.83	59.75	5.17	2.75	99.91	4.80	2.49
0.5	40.23	9.72	5.80	59.97	10.81	5.87	99.96	10.58	5.62
0.7	39.88	15.34	11.07	59.23	19.81	13.21	99.56	28.24	17.61
−0.1	40.01	2.12	1.09	59.83	2.24	1.20	99.86	2.01	1.07
−0.2	39.98	1.80	0.94	59.87	1.86	1.01	99.90	1.59	0.89
−0.3	39.98	1.52	0.84	59.90	1.54	0.90	99.92	1.36	0.78
−0.5	39.99	1.19	0.69	59.91	1.29	0.79	99.94	1.14	0.69

Table 5(a)
Values of $\sigma_n^*(L)$ and ratios of $\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$ under hypothesis, i.e. $\delta = 0$, estimator $\hat{m}_1$ used

		$n = 80$							
		$L = 8$		$L = 11$		$L = 16$		$L = 20$	
ϱ	σ^*	$\sigma_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$
0.0	1.00	0.93	1.12	0.91	1.16	0.87	1.21	0.85	1.25
0.1	1.11	1.02	1.13	1.00	1.17	0.96	1.23	0.93	1.27
0.2	1.25	1.12	1.15	1.11	1.19	1.07	1.25	1.04	1.29
0.3	1.43	1.26	1.16	1.25	1.20	1.21	1.27	1.17	1.32
0.5	2.00	1.66	1.20	1.67	1.25	1.63	1.32	1.59	1.38
0.7	3.33	2.44	1.26	2.52	1.32	2.53	1.42	2.49	1.50
−0.1	0.91	0.86	1.11	0.84	1.15	0.80	1.20	0.78	1.23
−0.2	0.83	0.80	1.10	0.78	1.13	0.74	1.18	0.72	1.22
−0.3	0.77	0.75	1.09	0.73	1.12	0.69	1.16	0.67	1.20
−0.5	0.67	0.68	1.06	0.66	1.09	0.62	1.13	0.60	1.15

Table 5(b)
Values of $\sigma_n^*(L)$ and ratios of $\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$ under hypothesis, i.e. $\delta = 0$, estimator $\hat{m}_1$ used

		$n = 120$							
		$L = 12$		$L = 17$		$L = 24$		$L = 30$	
ϱ	σ^*	$\sigma_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$
0.0	1.00	0.93	1.11	0.91	1.15	0.87	1.19	0.85	1.22
0.1	1.11	1.03	1.12	1.00	1.16	0.97	1.20	0.93	1.24
0.2	1.25	1.14	1.13	1.12	1.17	1.08	1.22	1.04	1.25
0.3	1.43	1.29	1.14	1.26	1.18	1.22	1.23	1.19	1.27
0.5	2.00	1.73	1.17	1.72	1.21	1.67	1.27	1.63	1.31
0.7	3.33	2.66	1.22	2.69	1.28	2.66	1.35	2.60	1.41
−0.1	0.91	0.86	1.10	0.83	1.14	0.80	1.18	0.77	1.21
−0.2	0.83	0.79	1.10	0.77	1.13	0.74	1.17	0.71	1.19
−0.3	0.77	0.74	1.09	0.72	1.12	0.69	1.15	0.66	1.18
−0.5	0.67	0.66	1.07	0.64	1.09	0.61	1.12	0.59	1.14

Aside from that, for the comparison limiting (theoretical) 5% critical values are given in tables 1(b) and 2(b).

Tables 3 and 4 present the behaviour of the change point estimators $\hat{m}_1$ and $\hat{m}_2$ when a change occured. Tables contain mean values of $\bar{\hat{m}}_i$, standard deviations $s_{\hat{m}_i}$, $i = 1, 2$, as well as MAD (median absolute deviation) values of $\hat{m}_i$'s for $\theta = \frac{1}{4}$ and $\frac{1}{2}$ and $\delta = 1.5$. It has to be said that the median values of $\bar{\hat{m}}_i$ do not differ practically from the mean values so that we do not present them.

Tables 5 and 6 provide information about the behaviour of estimates $\sigma_n^*(L)$ and $\tilde{\sigma}_n^*(L)$ of σ^* both under the hypothesis and the alternative $\delta = 1.5$ and $\theta = \frac{1}{2}$. The influence of the choice of the constant L is also demonstrated.

Table 5(c)
Values of $\sigma_n^*(L)$ and ratios of $\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$ under hypothesis, i.e. $\delta = 0$, estimator $\hat{m}_1$ used

		$n = 200$							
		$L = 20$		$L = 28$		$L = 40$		$L = 50$	
ϱ	σ^*	$\sigma_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$
0.0	1.00	0.93	1.10	0.91	1.13	0.88	1.18	0.85	1.21
0.1	1.11	1.03	1.11	1.01	1.14	0.97	1.18	1.94	1.21
0.2	1.25	1.15	1.11	1.13	1.15	1.08	1.19	1.05	1.23
0.3	1.43	1.31	1.12	1.28	1.16	1.23	1.20	1.20	1.24
0.5	2.00	1.79	1.14	1.76	1.18	1.70	1.23	1.65	1.27
0.7	3.33	2.84	1.17	2.82	1.22	2.76	1.28	2.69	1.33
−0.1	0.91	0.85	1.09	0.83	1.13	0.80	1.17	0.77	1.20
−0.2	0.83	0.79	1.09	0.77	1.12	0.74	1.16	0.71	1.19
−0.3	0.77	0.73	1.08	0.71	1.11	0.68	1.15	0.66	1.17
−0.5	0.67	0.65	1.07	0.63	1.09	0.60	1.13	0.58	1.14

Table 6(a)
Values of $\tilde{\sigma}_n^*(L)$ and ratios of $\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$ under alternative $\delta = 1.5$ and $\theta = 1/2$, estimator $\hat{m}_1$ used

		$n = 80$							
		$L = 8$		$L = 11$		$L = 16$		$L = 20$	
ϱ	σ^*	$\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$
0.0	1.00	0.86	2.60	0.82	3.08	0.76	3.86	0.71	4.47
0.1	1.11	0.94	2.43	0.90	2.87	0.83	3.57	0.77	4.13
0.2	1.25	1.03	2.26	0.99	2.65	0.91	3.28	0.86	3.78
0.3	1.43	1.14	2.10	1.10	2.43	1.02	2.99	0.95	3.43
0.5	2.00	1.46	1.79	1.42	2.03	1.33	2.44	1.24	2.76
0.7	3.33	2.03	1.56	2.02	1.71	1.91	1.98	1.80	2.20
−0.1	0.91	0.80	2.77	0.76	3.30	0.70	4.14	0.66	4.80
−0.2	0.83	0.75	2.92	0.71	3.50	0.66	4.41	0.61	5.12
−0.3	0.77	0.71	3.07	0.67	3.69	0.62	4.67	0.58	5.42
−0.5	0.67	0.65	3.30	0.61	4.00	0.56	5.09	0.52	5.93

3.1. Conclusions

From Tables 1 and 2 we can conclude that the simulated critical values of $T_{n1}(q)$ are in a good agreement with the asymptotic theory. The situation is a bit worse in the case of T_{n2}. Of course, the theoretical critical values are slightly larger than the simulated ones because they correspond to the maximum over a larger set. It is worth to notice that the simulated critical values divided by $\sigma*$ do not practically depend on ϱ which is in accordance with the theoretical results.

Tables 3 and 4 clearly demonstrate, according to our expectation, that the estimator $\hat{m}_1$ is preferable in the case when the change point is in the middle of the sample while $\hat{m}_2$ in the case when the change point is either at the beginning or the end of

Table 6(b)
Values of $\tilde{\sigma}_n^*(L)$ and ratios of $\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$ under alternative $\delta = 1.5$ and $\theta = 1/2$, estimator $\hat{m}_1$ used

		$n = 120$							
		$L = 12$		$L = 17$		$L = 24$		$L = 30$	
ϱ	σ^*	$\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$
0.0	1.00	0.87	3.09	0.83	3.75	0.77	4.63	0.73	5.36
0.1	1.11	0.96	2.86	0.91	3.46	0.85	4.26	0.79	4.93
0.2	1.25	1.06	2.63	1.01	3.16	0.94	3.88	0.88	4.47
0.3	1.43	1.18	2.40	1.13	2.86	1.05	3.50	0.99	4.03
0.5	2.00	1.56	1.97	1.50	2.29	1.40	2.76	1.31	3.15
0.7	3.33	2.29	1.61	2.23	1.81	2.10	2.10	1.97	2.35
−0.1	0.91	0.81	3.31	0.76	4.05	0.71	5.01	0.67	5.81
−0.2	0.83	0.75	3.53	0.71	4.32	0.66	5.37	0.62	6.22
−0.3	0.77	0.70	3.74	0.67	4.59	0.62	5.72	0.58	6.64
−0.5	0.67	0.64	4.10	0.60	5.07	0.55	6.34	0.52	7.37

Table 6(c)
Values of $\tilde{\sigma}_n^*(L)$ and ratios of $\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$ under alternative $\delta = 1.5$ and $\theta = 1/2$, estimator $\hat{m}_1$ used

		$n = 200$							
		$L = 20$		$L = 28$		$L = 40$		$L = 50$	
ϱ	σ^*	$\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$	$\tilde{\sigma}_n^*(L)$	$\sigma_n^*(L)/\tilde{\sigma}_n^*(L)$
0.0	1.00	0.89	3.86	0.84	4.70	0.78	5.87	0.74	6.78
0.1	1.11	0.97	3.54	0.93	4.29	0.86	5.34	0.81	6.17
0.2	1.25	1.08	3.21	1.03	3.88	0.96	4.82	0.91	5.56
0.3	1.43	1.22	2.90	1.17	3.49	1.07	4.32	1.02	4.97
0.5	2.00	1.65	2.28	1.57	2.70	1.46	3.31	1.38	3.79
0.7	3.33	2.52	1.73	2.43	1.97	2.27	2.34	2.14	2.64
−0.1	0.91	0.81	4.18	0.77	5.11	0.72	6.39	0.68	7.38
−0.2	0.83	0.75	4.50	0.71	5.50	0.66	6.89	0.62	7.96
−0.3	0.77	0.70	4.81	0.66	5.89	0.62	7.39	0.58	8.55
−0.5	0.67	0.62	5.37	0.59	6.61	0.54	8.31	0.51	9.64

the sample. The corresponding sample standard deviation $s_{\hat{m}}$ and the median absolute deviation MAD increase as the autoregressive coefficient ϱ increases.

Tables 5 and 6 show that $\sigma_n^*(L)$ and $\tilde{\sigma}_n^*(L)$ are quite different ($\tilde{\sigma}_n^*(L) < \sigma_n^*(L)$) and that their ratio is influenced also by the choice of L, n and ϱ (the smallest value of the ratio is attained for $n = 200$ and $L = [n/10]$). While $\sigma_n^*(L)$ is clearly preferable in the case when there is no change (cf. Table 5) the contents of Table 6 show (again according to our expectation) that $\tilde{\sigma}_n^*(L)$ is much better if the change occurred.

Table 6 concerns the estimators of σ^* when the change occurred at $\theta = \frac{1}{2}$. However, the results for $\theta = \frac{1}{4}$ were practically the same and we can expect similar behaviour for all θ that are not close to 0 or 1.

Generally, the results of simulations are in a good agreement with the theoretical results for $|\rho| \leqslant 0.3$, while the agreement for $0.3 < |\rho| < 1$ is much poorer.

4. Proofs

We start with two auxiliary lemmas proved elsewhere.

Lemma 4.1. *Let $Y_1, ..., Y_n$ be i. i. d. random variables with zero mean, unit variance and $E|Y_1|^{2+\Delta} < \infty$ for some $\Delta > 0$. Then*

$$\lim_{n\to\infty} P\left(\max_{\gamma_1(n)\leqslant k\leqslant \gamma_2(n)} \left\{ \sqrt{\frac{n}{k(n-k)}} \left| \sum_{i=1}^{k} (Y_i - \overline{Y}_n) \right| \right\} a\left(\log \frac{\gamma_2(n)}{\gamma_1(n)}\right) \right.$$
$$\left. \leqslant y + b\left(\log \frac{\gamma_2(n)}{\gamma_1(n)}\right) \right) = \exp\{-2\exp\{-y\}\}, \quad y \in \mathbb{R} \tag{4.1}$$

and

$$\lim_{n\to\infty} P\left(\max_{G<k\leqslant n} \max_{0\leqslant s<G} \left\{ \frac{1}{\sqrt{G}} \left| \sum_{i=k-G+1}^{k-s} Y_i \right| \right\} a(n/G) \leqslant y + b(n/G) \right)$$
$$= \exp\{-2\exp\{-y\}\}, \quad y \in \mathbb{R} \tag{4.2}$$

for $1 \leqslant \gamma_1(n) < \gamma_2(n) \leqslant n$, $\gamma_2(n)/\gamma_1(n) \to \infty$, $G/n \to 0$, $G^{-1}n^{2/(2+\Delta)} \log n \to \infty$, where $a(\cdot)$ and $b(\cdot)$ are defined in (2.1).

Proof. Assertion (4.1) follows from Csörgő and Horváth (1988) while (4.2) is a modification of Deheuvels and Révész (1987). □

Lemma 4.2. *Let the assumptions of Lemma* 4.1 *be satisfied.*
(i) *If $q \in Q_{01}$ and if $I^*(q,c) < \infty$, then*

$$\lim_{n\to\infty} P\left(\max_{1\leqslant k\leqslant n} \left\{ \frac{1}{\sqrt{n}\, q(k/n)} \left| \sum_{i=1}^{k} (Y_i - \overline{Y}_n) \right| \right\} \leqslant x \right)$$
$$= P\left(\sup_{0<t<1} \frac{|B(t)|}{q(t)} \leqslant x \right), \quad x \in \mathbb{R} \tag{4.3}$$

where $\{B(t), t \in (0,1)\}$ is the Brownian bridge.
(ii) *If $\inf_{0<t<1} r(t) > 0$ and $\int_0^1 t(1-t)/r(t)\,\mathrm{d}t < \infty$, then*

$$\lim_{n\to\infty} P\left(\frac{1}{n} \sum_{k=1}^{n} \left(\sum_{i=1}^{k} (Y_i - \overline{Y}_n) \right)^2 \frac{1}{r(k/n)} \leqslant x \right)$$
$$= P\left(\int_0^1 \frac{B^2(t)}{r(t)}\,\mathrm{d}t \leqslant x \right), \quad x \in \mathbb{R} \tag{4.4}$$

Proof. Assertion (i) follows from Theorem 2.1 in Szyszkowicz (1992), while (ii) is a consequence of the results in Csörgő et al. (1993). □

To prove Theorem 2.1 it suffices to show that Lemmas 4.1 and 4.2 remain true if the i. i. d. random variables are replaced by a stationary sequence of random variables $\{e_t\}$ fulfilling (1.2) and (1.3). Toward this the following two lemmas play the crucial role.

Lemma 4.3. *Assume that* (1.1)–(1.3) *and* H_0 *hold. Then, as* $n \to \infty$,

$$\max_{1 \leqslant k \leqslant \log n} \left\{ \frac{1}{\sqrt{k}} \left| \sum_{i=1}^{k} (e_i - \bar{e}_n) \right| \right\} = \mathrm{O_p}\left(\sqrt{\log\log\log n}\right), \tag{4.5}$$

$$\max_{n-\log n \leqslant k < n} \left\{ \frac{1}{\sqrt{n-k}} \left| \sum_{i=1}^{k} (e_i - \bar{e}_n) \right| \right\} = \mathrm{O_p}\left(\sqrt{\log\log\log n}\right), \tag{4.6}$$

$$\begin{aligned} &\max_{\log n \leqslant k \leqslant n} \left\{ \frac{1}{\sqrt{k}} \left| \sum_{i=1}^{k} \left(e_i - \varepsilon_i \sum_{j=0}^{\infty} w_j \right) \right| \right\} \\ &\quad + \max_{1 \leqslant k \leqslant n-\log n} \left\{ \frac{1}{\sqrt{n-k}} \left| \sum_{i=k+1}^{n} \left(e_i - \varepsilon_i \sum_{j=0}^{\infty} w_j \right) \right| \right\} \\ &= \mathrm{o_p}\left(\frac{1}{\sqrt{\log\log n}} \right). \end{aligned} \tag{4.7}$$

Proof. According to Bai (1994, p. 470)

$$e_i = \varepsilon_i \sum_{j=0}^{\infty} w_j - e_i^* + e_{i-1}^*, \quad i = 1, \ldots \tag{4.8}$$

where $\{e_i^*\}$ is a second-order stationary process with

$$E|e_i^*|^{2+\Delta} < \infty. \tag{4.9}$$

Thus,

$$\frac{1}{\sqrt{k}} \sum_{i=1}^{k} \left(e_i - \varepsilon_i \sum_{j=0}^{\infty} w_j \right) = \frac{1}{\sqrt{k}} e_0^* - \frac{1}{\sqrt{k}} e_k^*, \quad k = 1, \ldots, n \tag{4.10}$$

and

$$\bar{e}_n - \bar{\varepsilon}_n \sum_{j=0}^{\infty} w_j = -\frac{1}{n}(e_0^* - e_n^*) = \mathrm{O_p}(n^{-1}). \tag{4.11}$$

Since $\bar{\varepsilon}_n = \mathrm{O_p}(n^{-1/2})$, we also have

$$\bar{e}_n = \mathrm{O_p}(n^{-1/2}). \tag{4.12}$$

It can be easily shown that

$$\max_{1 \leqslant k \leqslant \log n} \left\{ \frac{1}{\sqrt{k}} |e_0^*| \right\} = \mathrm{O_p}(1) \tag{4.13}$$

and

$$P\left(\max_{1 \leqslant k \leqslant \log n} \left\{ \frac{1}{k^{1/2}} |e_k^*| \right\} > A \right) \leqslant \sum_{k=1}^{[\log n]} k^{-(2+\Delta)/2} E|e_k^*|^{2+\Delta} A^{-(2+\Delta)} \tag{4.14}$$

for any $A > 0$. Hence,

$$\max_{1 \leqslant k \leqslant \log n} \left\{ \frac{1}{k^{1/2}} |e_k^*| \right\} = \mathrm{O_p}(1) \tag{4.15}$$

which in combination with (4.8), (4.11)–(4.13) and Lemma 4.1 implies (4.5). The relation (4.6) can be derived in the same way and therefore, the proof is omitted.

Towards (4.7) we notice that

$$\max_{\log n \leqslant k \leqslant n} \left\{ \frac{1}{\sqrt{k}} |e_0^*| \right\} \leqslant \frac{1}{\sqrt{\log n}} |e_0^*| = \mathrm{O_p}((\log n)^{-1/2})$$

and

$$P\left(\max_{\log n \leqslant k \leqslant n} \mathrm{big} \left\{ \frac{1}{k^{1/2-\eta}} |e_k^*| \right\} > 1 \right) \leqslant \sum_{k=[\log n]}^{\infty} \frac{1}{k^{(1/2-\eta)(2+\Delta)}} E|e_k^*|^{2+\Delta} \to 0 \tag{4.16}$$

for η fulfilling $(1-2\eta)(2+\Delta)/2 > 1$. The last relations together with (4.8) imply that

$$\max_{\log n \leqslant k < n} \left\{ \frac{1}{\sqrt{k}} \left| \sum_{i=1}^{k} \left(e_i - \varepsilon_i \sum_{j=0}^{\infty} w_j \right) \right| \right\} = \mathrm{O_p}((\log n)^{-\eta}). \tag{4.17}$$

Similarly, we have

$$\begin{aligned} &\max_{1 \leqslant k \leqslant n - \log n} \left\{ \frac{1}{\sqrt{n-k}} \left| \sum_{i=1}^{k} \left(e_i - \varepsilon_i \sum_{j=0}^{\infty} w_j \right) \right| \right\} \\ &= \mathrm{O_p}\left(\max_{1 \leqslant k \leqslant n - \log n} \left\{ \frac{1}{\sqrt{n-k}} (|e_n^*| + |e_k|) \right\} \right) = \mathrm{O_p}((\log n)^{-\eta}). \end{aligned} \tag{4.18}$$

The assertion follows from (4.17) and (4.18). □

Lemma 4.4. *Let assumptions of Lemma 4.3 be satisfied and let for $G = G_n$, as $n \to \infty$,*

$$\frac{G}{n} \to 0 \quad \textit{and} \quad G^{-1} n^{2/(2+\Delta)} \log n \to 0. \tag{4.19}$$

Then, as $n \to \infty$,

$$\max_{G<k\leqslant n}\left\{\frac{1}{\sqrt{G}}\left|\sum_{i=k-G+1}^{k}\left(e_i-\varepsilon_i\sum_{j=0}^{\infty}w_j\right)\right|\right\}=\mathrm{o_p}\left(\frac{1}{\sqrt{\log(n/G)}}\right). \tag{4.20}$$

Proof. Notice that by (4.8)

$$\max_{G<k\leqslant n}\left\{\left|\sum_{i=k-G+1}^{k}\left(e_i-\varepsilon_i\sum_{j=0}^{\infty}w_j\right)\right|\right\}=\max_{G<k\leqslant n}\{(|e_k^*|+|e_{k-G}^*|)\}.$$

Further, standard tools and assumption (4.19) give

$$\begin{aligned}
&P\left(\max_{1\leqslant k\leqslant n}\left\{\frac{1}{\sqrt{G}}|e_k^*|\right\}>1\right)\\
&\quad=\mathrm{O_p}\left(G^{-(2+\Delta)/2}\sum_{i=1}^{n}E|e_k^*|^{2+\Delta}\right)=\mathrm{O_p}\left(G^{-(2+\Delta)/2}n\right)\\
&\quad=\mathrm{O_p}\left((\log n)^{-(2+\Delta)/2}\right)
\end{aligned}$$

which immediately implies the desired assertion. □

Proof of Theorem 2.2. The proof is quite close to that in the case of i.i.d. error terms that is treated in detail in Antoch et al. (1993); therefore we give only the sketch of the proof of (2.9).

The estimator $\hat{m}_2$ can be equivalently defined as

$$\hat{m}_2=\min\left\{\arg\max\left\{S_k^2\frac{n}{k(n-k)}-S_m^2\frac{n}{m(n-m)},\ k=1,\ldots,n-1\right\}\right\}$$

where $m=[n\theta]$. Elementary calculations give for $k\leqslant m$

$$S_k^2\frac{n}{k(n-k)}-S_m^2\frac{n}{m(n-m)}=N_{k1}+N_{k2}+2N_{k3}\delta+N_{k4}\delta^2 \tag{4.21}$$

where

$$\begin{aligned}
N_{k1}&=\frac{n(mn-m^2-kn+k^2)}{(n-k)m(n-m)}\left(\sum_{i=1}^{k}(e_i-\bar{e}_n)\right)^2,\\
N_{k2}&=-\frac{n}{m(n-m)}\sum_{i=k+1}^{m}(e_i-\bar{e}_n)\left(\sum_{i=1}^{k}(e_i-\bar{e}_n)\right)\left(\sum_{i=1}^{m}(e_i-\bar{e}_n)\right),\\
N_{k3}&=-\sum_{i=1}^{k}(e_i-\bar{e}_n)\frac{n-m}{n-k}+\sum_{i=1}^{m}(e_i-\bar{e}_n),\\
N_{k4}&=\frac{n-m}{n-k}(k-m).
\end{aligned}$$

By Lemma 4.1 for $z \in (0,1)$ we have

$$\max_{1 \leqslant k \leqslant nz} \{|N_{k1}|\} = \mathrm{O_p}\left(\sqrt{\log\log n}\right), \tag{4.22}$$

$$\max_{nz < k \leqslant m} \left\{\frac{|N_{k1}|}{m-k}\right\} = \mathrm{O_p}\left(n^{-1}\right). \tag{4.23}$$

Next, for $A > 0$

$$\max_{1 \leqslant k \leqslant m - A/\delta^2} \{|N_{k2}|\} = \mathrm{O_p}(1), \tag{4.24}$$

$$\max_{m - A/\delta^2 \leqslant k \leqslant m} \{|N_{k2}|\} = \mathrm{o_p}\left(|\delta|/\sqrt{n}\right) = \mathrm{o_p}(1). \tag{4.25}$$

Concerning N_{k3}, Lemma 4.1 and the fact that

$$N_{k3} = \sum_{i=k+1}^{m} e_i - m\bar{e}_n + \sum_{i=1}^{k} \left(e_i - \bar{e}_n\right) \frac{m-k}{n-k} \tag{4.26}$$

give

$$\max_{1 \leqslant k \leqslant m} \left\{\frac{1}{m-k}\left|N_{k3} - \sum_{i=k+1}^{m} e_i\right|\right\} = \mathrm{O_p}\left(n^{-1/2}\right). \tag{4.27}$$

Finally,

$$\max_{1 \leqslant k \leqslant m} \left\{\left|\frac{N_{k4} + m - k}{k - m}\right|\right\} = \mathrm{O}\left(n^{-1}\right). \tag{4.28}$$

Proceeding similarly as in Antoch et al. (1995), we find that (4.21)–(4.28) imply

$$\delta^2(\hat{m}_2 - m) = \mathrm{O_p}(1)$$

and that the limit distribution of $\hat{m}_2$ is the same as that of

$$\begin{aligned}\tilde{m}_2 &= \min\left\{t \in R;\ \max\left\{2\delta \sum_{i=[m+s/\delta^2]}^{m} e_i I\{s<0\} - \sum_{i=m}^{[m+s/\delta^2]} e_i I\{s>0\} - |s|, s \in \mathbb{R}\right\}\right. \\ &= \left. 2\delta \sum_{i=[m+t/\delta^2]}^{m} e_i I\{t<0\} - \sum_{i=m}^{[m+t/\delta^2]} e_i I\{t>0\} - |t|\right\}.\end{aligned}$$

Hence it remains to investigate the limit distribution of the process

$$V(s) = \begin{cases} \displaystyle\sum_{i=[m+s/\delta^2]}^{m} e_i, & s < 0, \\ \displaystyle -\sum_{i=[m-s/\delta^2]}^{m} e_i, & s > 0. \end{cases}$$

A slight modification of Lemma 4.1 yields for $A > 0$

$$\max_{0<s\leqslant A}\left\{\delta\left|\sum_{i=m+1}^{[m+s/\delta^2]}\left(e_i-\varepsilon_i\sum_{j=0}^{\infty}w_j\right)\right|\right\}=\mathrm{o_p}(1)$$

and

$$\max_{-A\leqslant s<0}\left\{\delta\left|\sum_{i=[m+s/\delta^2]}^{m}\left(e_i-\varepsilon_i\sum_{j=0}^{\infty}w_j\right)\right|\right\}=\mathrm{o_p}(1).$$

The proof can be finished in a standard way.

Proof of Theorem 2.3 If $\delta = 0$ then $\hat{\sigma}_n^2$ can be decomposed as

$$\hat{\sigma}_n^2 = D_{n1} + D_{n2} + D_{n3} \tag{4.29}$$

where

$$D_{n1} = \frac{1}{n}\sum_{i=1}^{n} e_i^2,$$

$$D_{n2} = \frac{2}{n}\sum_{k=1}^{L}\left(1-\frac{k}{L}\right)\sum_{t=1}^{n-k} e_t e_{t+k},$$

$$D_{n3} = \bar{e}_n\left(\bar{e}_n + \frac{2}{n}\sum_{k=1}^{L}\left(1-\frac{k}{L}\right)\sum_{t=1}^{n-k}\left(\bar{e}_n - e_t - e_{t+k}\right)\right).$$

Utilizing (4.8) we receive after simple calculation

$$D_{n3} = \mathrm{O_p}\left(L/n\right) \tag{4.30}$$

Next, we split the term D_{n1} into four summands

$$D_{n1} = D_{n11} + D_{n12} + D_{n13} + D_{n14} \tag{4.31}$$

where

$$D_{n11} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=0}^{L} w_j^2\varepsilon_{i-j}^2,$$

$$D_{n12} = \frac{2}{n}\sum_{i=1}^{n}\sum_{j=0}^{L-1}\sum_{v=j+1}^{L} w_j w_v \varepsilon_{i-j}\varepsilon_{i-v},$$

$$D_{n13} = \frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=L+1}^{\infty} w_j\varepsilon_{i-j}\right)^2,$$

$$D_{n14} = \frac{2}{n}\sum_{i=1}^{n}\sum_{j=0}^{L}\sum_{v=L+1}^{\infty} w_j w_v \varepsilon_{i-j}\varepsilon_{i-v}.$$

Straightforward calculations give

$$ED_{n12}^2 = \mathrm{O}(n^{-1}), \tag{4.32}$$

$$ED_{n13} = \mathrm{O}(L^{-1}), \tag{4.33}$$

$$ED_{n14}^2 = \mathrm{O}(L^{-2} + n^{-1}). \tag{4.34}$$

The term D_{n11} is the leading term in D_{n1} and it can be checked that

$$D_{n11} - \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2 \sum_{j=0}^{L} w_j^2 = \mathrm{O_p}(L/n). \tag{4.35}$$

It remains to investigate the term D_{n2}. We decompose it into several summand and treat them separately. We put

$$D_{n2} = D_{n21} + D_{n22} + D_{n23} \tag{4.36}$$

where

$$D_{n21} = \frac{2}{n}\sum_{k=1}^{L}\sum_{j=0}^{L}\sum_{t=1}^{n-k}\left(1 - \frac{k}{L}\right) w_j w_{j+k} \varepsilon_{t-j}^2,$$

$$D_{n22} = \frac{2}{n}\sum_{k=1}^{L}\sum_{j=L+1}^{\infty}\sum_{t=1}^{n-k}\left(1 - \frac{k}{L}\right) w_j w_{j+k} \varepsilon_{t-j}^2,$$

$$D_{n23} = \frac{2}{n}\sum_{k=1}^{L}\sum_{j=0}^{\infty}\sum_{q=0}^{\infty}\sum_{t=1}^{n-k}\left(1 - \frac{k}{L}\right) w_j w_q \varepsilon_{t-j}\varepsilon_{t+k-q} I\{t - j \neq t + k - q\}.$$

After tedious but straightforward calculations we get

$$E\,|D_{n22}| = \mathrm{O}(L^{-1}), \tag{4.37}$$

$$ED_{n23}^2 = \mathrm{O}(L^2 n^{-1} \log L), \tag{4.38}$$

$$ED_{n23} - \frac{2}{n}\sum_{k=1}^{L}\sum_{j=0}^{L}\sum_{t=1}^{n-k}\left(1 - \frac{k}{L}\right) w_j w_{j+k} \varepsilon_t^2 = \mathrm{O_p}(L n^{-1} \log n). \tag{4.39}$$

Combining (4.29)–(4.39) we observe that

$$\hat{\sigma}_n^2 - \sigma^{*2} = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^2 \sum_{j=0}^{L} w_j^2 + \frac{2}{n}\sum_{k=1}^{L}\sum_{j=0}^{L}\sum_{t=1}^{n-k}\left(1 - \frac{k}{L}\right) w_j w_{j+k} \varepsilon_t^2$$
$$-\sigma^2\left(\sum_{j=0}^{\infty} w_j^2 + 2\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} w_j w_{j+k}\right) + \mathrm{O_p}\left(L^{-1} + L\sqrt{\frac{\log L}{n}}\right).$$

Then applying the Marcienkiewicz–Zygmund Theorem, see Theorem 5.2.2 in Chow–Teicher (1988), we get the assertion (2.12). The assertion (2.13) can be proved along the same line. □

Acknowledgements

The authors wishes express their gratitude to the referees for the valuable comments that considerably improved the paper. Particularly, the remark concerning relation (4.8) shortened the proofs of Lemmas 4.3 and 4.4.

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