Biometrika (1996), **83**, 3, *pp.* 627–638
Printed in Great Britain

Testing for a change of the long-memory parameter

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Summary

Long-range dependence is often observed in long time series. Correlations decay approximately like $|k|^{2H-2}$, with $H \in (0{\cdot}5, 1)$, as the lag k tends to infinity. The long-term features of the data are essentially characterised by the parameter H. Small changes of H have strong implications for the long-term behaviour of the process. In particular, rates of convergence of estimators for the mean, and for many other parameters of interest, differ for different values of H. For some data sets, H appears to change with time. In this paper we consider a simple test of the null hypothesis that H is constant. The test is based on a functional central limit theorem for quadratic forms. Critical values for the test statistic are given. Simulations confirm the validity of the test. A data example illustrates its practical application.

Some key words: Change point; Fractional ARIMA; Fractional Gaussian noise; Long-range dependence; Quadratic form; Stationarity.

1. Introduction

The phenomenon of long memory, or long-range dependence, has received wide attention in the last few years. While a systematic statistical theory has been developed mainly in the last two decades, long-range dependence had been observed in many areas of application a long time before stochastic models were known; see e.g. Cox (1984), Hampel (1987), Künsch (1986) and Beran (1992a, 1994). A mathematical definition can be given as follows. Let X_t be a stationary process with autocovariances $r(k) = \text{cov}(X_t, X_{t+k})$. Then X_t is said to have long memory if, as $|k| \to \infty$,

$$r(k) \sim L_1(k)|k|^{2H-2}, \quad H \in (\tfrac{1}{2}, 1), \tag{1}$$

where $L_1(k)$ is a slowly varying function as $|k| \to \infty$, that is $L_1(ta)/L_1(t) \to 1$ as $t \to \infty$ for any $a > 0$. This property implies that the correlations are not summable, and the spectral density has a pole at zero. Under suitable conditions on $L_1(.)$, the spectral density

$$f(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r(k)e^{ikx} \sim L_2(x)|x|^{1-2H} \tag{2}$$

as $|x| \to 0$ for some $L_2(.)$ slowly varying at the origin. The best known models with (1) and (2) are fractional Gaussian noise (Mandelbrot & van Ness, 1968; Mandelbrot & Wallis, 1969) and fractional ARIMA (Granger & Joyeux, 1980; Hosking, 1981). These models are stationary with a constant long-memory parameter H. For some time series, however, the long-term dependence structure seems to change over time. This can be due to a change in the physical mechanism that generates the data. Typical examples are very long time series with long memory in telecommunication engineering (Beran et al., 1995), where changes in the data generating mechanism are likely to happen. In other situations, changes in the way observations are taken may cause H to vary with time. For instance, there could be several stretches of data, each measured by a different observer.

Even small changes of H are relevant, because they imply an essential change of the long-term behaviour of the process. In particular, the rate of convergence of confidence intervals for constants and for parameter estimates in regression with certain classes of design matrices, e.g. polynomial regression, changes when H changes; see e.g. Adenstedt (1974), Samarov & Taqqu (1988), Beran (1989, 1991) and Yajima (1988, 1991). Also, H has a strong impact on long-term forecasts and the size of forecast intervals; see e.g. Ray (1993) and Beran (1994, Ch. 1, 8). The question arises, therefore, how to decide whether H is constant over the whole observational period or not.

To illustrate this, let us look at a typical data set with long memory, namely, the yearly Nile river minima based on measurements at the Roda gauge near Cairo during the years 622–1284 (Tousson, 1925, pp. 366–85). The data are listed in the appendix of Beran

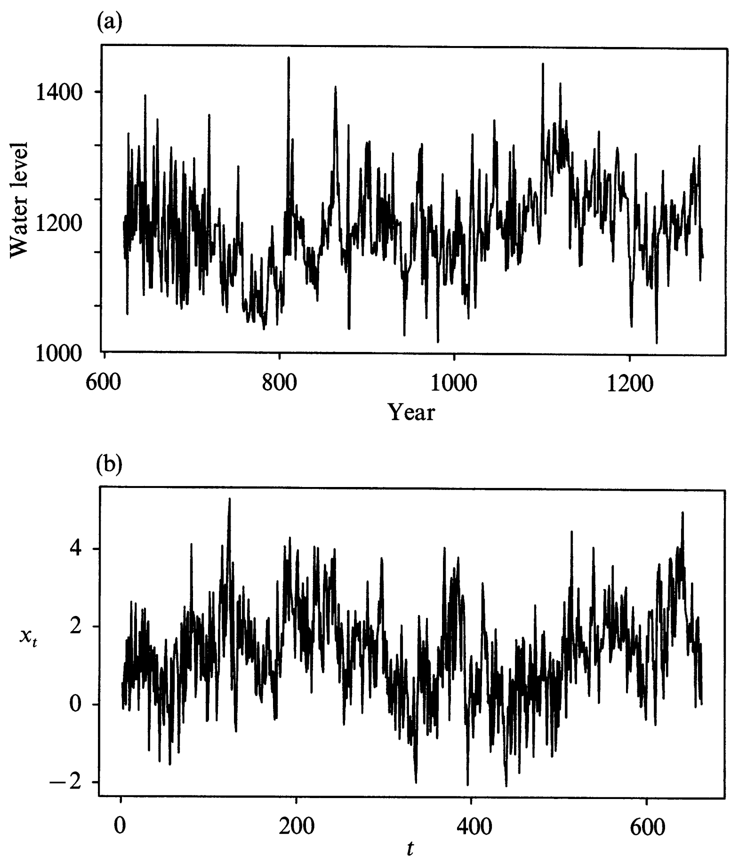

Fig. 1. Nile river data. (a) Shows yearly minimum water level at the Roda gauge for the years 622–1281 (Tousson, 1925, pp. 366–85). (b) Displays a simulated series of a fractional ARIMA$(0, d, 0)$ process with $H = 0{\cdot}90$.

(1994) and can be obtained from the first author by e-mail upon request (e-mail address `beran@stat.math.ethz.ch`). This is one of the time series that led to the discovery of the Hurst effect (Hurst, 1951) and motivated Mandelbrot and co-workers (Mandelbrot & Wallis, 1969; Mandelbrot & van Ness, 1968) to introduce fractional Gaussian noise into statistics. The series is plotted in Fig. 1(a). It exhibits some typical features of stationary long-memory processes: locally there are spurious trends and/or cycles of varying frequencies which disappear after some time, the mean seems to be changing with time but the overall mean is constant. Fractional Gaussian noise with H around 0·83 turns out to be a good model (Mandelbrot & Wallis, 1969; Beran, 1992b). Similarly, a fractional ARIMA(0, d, 0) model with $H = d + \frac{1}{2} = 0{\cdot}90$ fits well. In particular, the correlation structure of the series is well described by this one parameter H only. This can be seen by comparing the periodogram with the fitted spectral densities (Fig. 2). Yet a closer look at the data reveals an inhomogeneity: observations 1 to about 100 seem to be more independent than the subsequent observations, implying that the value of H might be lower for the first 100 observations than for the subsequent data. This is illustrated by Fig. 3. There, the periodogram, in log–log-coordinates, for the first 100 and the last 553 observations is plotted. Clearly, Fig. 3(b) plot shows a negative slope for all frequencies, whereas Fig. 3(a) looks

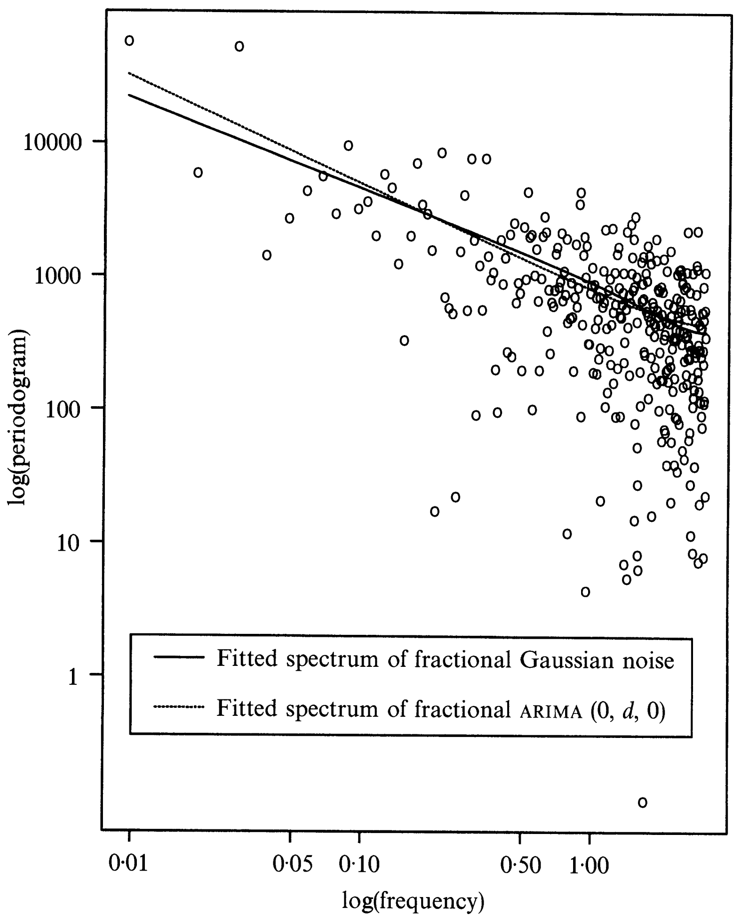

Fig. 2. Periodogram of the Nile river minima (in log–log coordinates), and fitted spectral densities of a fractional Gaussian noise process and a fractional ARIMA(0, d, 0) process.

horizontal, suggesting uncorrelated observations. The question arises: is this inhomogeneity spurious, due to randomness, or real, due to a change of the dependence structure? The answer is not obvious, because not only the raw periodogram but also optimal parametric estimates of H vary considerably when calculated for short disjoint parts of a stationary series with long memory. In order to be able to assess quantitatively how much the estimates of H can vary when estimated from different portions of the data, we need to derive the joint distribution of these estimates.

In this paper we answer this question and suggest a simple method for testing the null hypothesis H_0: 'H is constant' against H_a: 'H is not constant'. Related results in a nonparametric setting and results on testing for change points in the marginal distribution can be found in Giraitis & Leipus (1992, 1994).

The outline of the paper is as follows. In § 2 we state a functional central limit theorem for quadratic forms. The test statistic is defined in § 3. Its asymptotic distribution follows from the limit theorem in § 2. Some relevant quantiles are given. Simulations in § 4 demonstrate the validity of the test. The test is applied to the Nile river data in § 5. Concluding remarks are given in § 6. Proofs are given in the Appendix.

2. A FUNCTIONAL LIMIT THEOREM

Let $\{\xi_s: -\infty < s < \infty\}$ be a sequence of mean-zero, independent and identically distributed random variables with variance τ^2, fourth cumulant χ_4 and all moments finite. Define a moving average

$$X_t = \sum_{k=-\infty}^{\infty} a(k)\xi_{t-k} \quad (t = 0, \pm 1, \ldots), \tag{3}$$

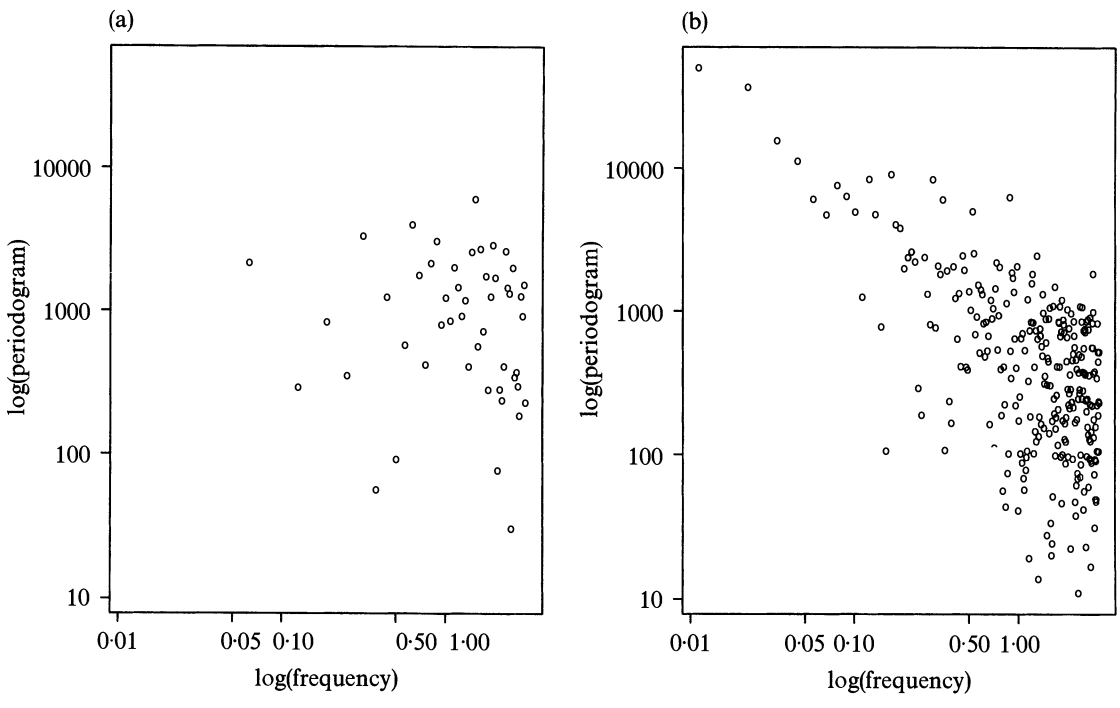

Fig. 3. Periodogram (in log–log coordinates) of (a) the first 100 observations of the Nile river minima series, and (b) the Nile river minima excluding the first 100 observations.

where

$$a(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\lambda} \tilde{a}(\lambda)\, d\lambda$$

for some complex-valued function $\tilde{a}(\lambda)$ satisfying $\tilde{a}(\lambda) = \tilde{a}(-\lambda)^*$, where z^* denotes the complex conjugate of z,

$$\int_{-\pi}^{\pi} |\tilde{a}(\lambda)|^2 \, d\lambda < \infty, \quad |\tilde{a}(\lambda)|^2 = |\lambda|^{1-2H} L(|\lambda|^{-1}),$$

where $0 < H < 1$, and L is slowly varying at infinity. The spectral density of X_t is

$$f(\lambda) = \frac{\tau^2}{2\pi} |\tilde{a}(\lambda)|^2 = \frac{\tau^2}{2\pi} |\lambda|^{1-2H} L(|\lambda|^{-1}).$$

Note that long memory in the sense of (1) and (2) occurs for $\frac{1}{2} < H < 1$. The following theorem however holds also if H is less than or equal to $\frac{1}{2}$. In this section, we therefore extend the range of H to the interval $(0, 1)$. Let

$$Q_1(t) = \sum_{j,k=1}^{\lfloor tN \rfloor} b_{j-k} \{X_j X_k - r(j-k)\}, \quad Q_2(t) = \sum_{j,k=\lfloor tN \rfloor + 1}^{N} b_{j-k} \{X_j X_k - r(j-k)\},$$

where $\lfloor . \rfloor$ denotes the integer part, $r(k) = EX_0 X_k$ and

$$b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} g(x)\, dx$$

for some bounded, real, even function $g(x)$. Suppose further that

$$\frac{\mathrm{tr}(R_N B_N)^2}{N} \to \frac{\tau^4}{2\pi} \int_{-\pi}^{\pi} \{|\tilde{a}(x)|^2 g(x)\}^2 \, dx$$

as $N \to \infty$, where $(R_N)_{t,s} = r(t-s)$ and $(B_N)_{t,s} = b(t-s)$. For a discussion of this condition, see Beran & Terrin (1994, p. 271). Let

$$\sigma^2 = \frac{\tau^4}{\pi} \int_{-\pi}^{\pi} \{|\tilde{a}(x)|^2 g(x)\}^2 \, dx + \chi_4 \frac{\tau^4}{4\pi^2} \left\{ \int_{-\pi}^{\pi} |\tilde{a}(\lambda)|^2 g(x) \, dx \right\}^2,$$
$$Z_N(t) = N^{-\frac{1}{2}} \sigma^{-1} \{t(1-t)\}^{\frac{1}{2}} \left\{ \frac{1}{t} Q_1(t) - \frac{1}{1-t} Q_2(t) \right\}. \tag{4}$$

Denote by $D[0, 1]$ the space that consists of those functions on the interval $[0, 1]$ that are right-continuous and have left-hand limits. The Skorohod topology on $D[0, 1]$ is an extension of the uniform topology on the space $C[0, 1]$ of continuous functions. In the uniform topology, functions are close if they differ by only a short distance in the vertical scale. In the Skorohod topology, functions are also considered close if they differ only slightly in the horizontal scale. For a precise definition, see, for example, Billingsley (1968, p. 111).

THEOREM 1. *Under the above assumptions, $Z_N(t)$ converges in the Skorohod topology on $D[0, 1]$ to the Gaussian process*

$$Z(t) = \{t(1-t)\}^{\frac{1}{2}} \left\{ \frac{1}{t} B_1(t) - \frac{1}{1-t} B_2(1-t) \right\}, \tag{5}$$

where B_1 and B_2 are two independent standard Brownian motions.

For a proof see the Appendix.

3. THE TEST STATISTIC

Assume that X_t is given by (3), that (2) holds and the spectral density is characterised by a finite dimensional parameter vector $\theta = (\tau, \eta) = (\tau, H, \eta_2, \ldots, \eta_m)$ such that

$$f(x; \theta) = \tau f\{x; (1, \eta)\}, \quad \int_{-\pi}^{\pi} \log f\{x; (1, \eta)\}\, dx = 0.$$

By definition, τ is the expected mean squared error of the best linear prediction of X_t given $\{X_s, s \leqslant t-1\}$. The long-memory behaviour is characterised by H, the additional parameters $\eta_2, \ldots, \eta_m$ allow for flexible modelling of short-term features. Define

$$\alpha_k(\eta) = \int_{-\pi}^{\pi} e^{ikx} f^{-1}\{x; (1, \eta)\}\, dx.$$

Given $X_1, X_2, \ldots, X_N$, let $\hat{\eta}$ be the value of η that minimises

$$Q(\eta) = \sum_{i,j=1}^{N} \alpha_{i-j}(\eta)(X_i - \bar{X})(X_j - \bar{X}).$$

The vector $\hat{\eta}$ is called the Whittle estimator (Whittle, 1951). Note that the integral $\int e^{ikx} f^{-1}\, dx$ may be replaced by a Riemann sum; see e.g. Beran (1994). Giraitis & Surgailis (1990) show that, under some additional regularity conditions on f, $N^{\frac{1}{2}}(\hat{\eta} - \eta)$ is asymptotically normal; see also Fox & Taqqu (1986), Dahlhaus (1989) and Yajima (1985). For Gaussian processes, the asymptotic covariance matrix is the same as for the exact maximum likelihood estimator. Efficiency in the Gaussian case is shown in Dahlhaus (1989).

The coefficients α_k, as well as their partial derivatives with respect to η, fulfill the conditions given in § 2. The results in § 2 then imply the following functional central limit theorem for Whittle's estimator of H. For a proof see the Appendix.

COROLLARY 1. *Define, for $0 < t < 1$,*

$$Q_1(t; \eta) = \sum_{i,j=1}^{\lfloor Nt \rfloor} \alpha_{i-j}(\eta)(X_i - \bar{X})(X_j - \bar{X}), \quad Q_2(t; \eta) = \sum_{i,j=\lfloor Nt \rfloor + 1}^{N} \alpha_{i-j}(\eta)(X_i - \bar{X})(X_j - \bar{X}).$$

Let $\hat{\eta}^{(1)}(t)$ and $\hat{\eta}^{(2)}(t)$ $(j = 1, 2)$ be defined by

$$\hat{\eta}^{(j)}(t) = \operatorname{argmin}\{Q_j(t; \hat{\eta}^{(j)})\} \quad (j = 1, 2),$$

and denote by $\hat{H}^{(j)}(t) = \hat{\eta}_1^{(j)}(t)$ the corresponding estimates of H. Moreover, define $\kappa^2 = 2D_{11}^{-1}(\eta)$, where D is the $k \times k$ matrix with elements

$$D_{ij}(\eta) = (2\pi)^{-1} \int_{-\pi}^{\pi} \frac{\partial}{\partial \eta_i} \log f\{x; (1, \eta)\} \frac{\partial}{\partial \eta_j} \log f\{x; (1, \eta)\}\, dx.$$

Then the process

$$\tilde{Z}_N(t) := N^{\frac{1}{2}} \kappa^{-1} \{t(1-t)\}^{\frac{1}{2}} \{\hat{H}^{(1)}(t) - \hat{H}^{(2)}(t)\}$$

converges in the Skorohod topology on $D[0, 1]$ *to the Gaussian process* $Z(t)$ *defined by* (5).

More generally, Theorem 1 implies a multivariate functional central limit theorem for $\hat{\eta}$. However, here the focus is on testing for constancy of H so that Corollary 1 is sufficient. A natural way of testing H_0: 'H = constant' against the alternative that H changes somewhere in the observed time period is to compare estimated values of H for many different subseries. We therefore suggest the test statistic

$$T_N = \sup_{\delta < t < 1-\delta} |\tilde{Z}_N(t)| \tag{6}$$

for some $0 < \delta < 1$. Corollary 1 implies $T_N \to Y$ in distribution, where

$$Y = \sup_{\delta < t < 1-\delta} |Z(t)| \tag{7}$$

and $Z(t)$ is defined by (5). Note that, due to the standardisation by $\{t(1-t)\}^{\frac{1}{2}}$, $Z(t)$ is a standard normal random variable for each fixed t.

Quantiles of Y can be obtained for instance by simulation. For example, the upper 90%-, 95%- and 99%-quantiles of Y obtained from 10 000 simulations turned out to be 2·65, 2·93 and 3·54 respectively. Depending on the model, κ may have to be replaced by a consistent estimate. However, simulations of the asymptotic distribution have to be done only once, since the distribution of Y does not depend on any parameters. For some model classes, even κ, and thus T_N, does not depend on any parameters. For example, for a fractional ARIMA(0, d, 0) model, κ^2 is equal to $6/\pi^2$. For computational reasons one may want to calculate $\hat{H}^{(j)}(t)$ ($j = 1, 2$) only for a subset of all possible cut points $t \in [\delta, 1-\delta]$. Thus, one may consider

$$T_{N,k} = \sup_{\delta < t = jk/N < 1-\delta} |\tilde{Z}_N(t)| \tag{8}$$

for a fixed integer k. The supremum is taken over all integers j for which the two inequalities hold. Keeping k fixed and letting N tend to infinity, the asymptotic distribution of $T_{N,k}$ is the same as for T_N.

Finally note that, in practice, δ will have to be chosen such that the resulting shortest series can still be used for estimating the long-memory parameter. For instance, for $N = 1000$ and a fractional ARIMA(0, d, 0) model, we may choose $\delta = 0{\cdot}1$. The shortest series is then of length $1000\delta = 100$.

4. Simulations

We simulated 400 series of a fractional ARIMA(0, d, 0) process with $d = H - \frac{1}{2} = 0{\cdot}2$ and 0·4 respectively and lengths $N = 500$ and 1000. For each series, the test statistic $T_{N,k}$ with $k = 20$ and $\delta = 0{\cdot}1$ was calculated. Table 1 gives, for each value of d and N, the relative frequency of simulations where $T_{N,k}$ was above 10%, 5% and 1% critical limits 2·65, 2·93 and 3·54 respectively, which were obtained from the asymptotic distribution of Corollary 1. The results show satisfactory agreement with the nominal rejection levels.

It should be noted that the method assumes that the fitted parametric model is the correct model. In practice, it is usually not known a priori which model is correct. Thus, one may have to choose an appropriate parametric model, for instance by applying data

Table 1. *Simulated rejection probabilities for $H_0: H = constant$, based on $T_{N,k}$ and asymptotic* 10%-, 5%- *and* 1%-*quantiles obtained from Corollary* 1

	$d=0\cdot2$			$d=0\cdot4$		
N	$\alpha=0\cdot1$	$\alpha=0\cdot05$	$\alpha=0\cdot01$	$\alpha=0\cdot1$	$\alpha=0\cdot05$	$\alpha=0\cdot01$
500	0·1125	0·0800	0·0175	0·1325	0·0850	0·0275
1000	0·1275	0·0675	0·0175	0·1125	0·0625	0·0300

The simulated process is fractional ARIMA(0, d, 0).

driven model selection criteria such as Akaike's information criterion or consistent versions of it; see e.g. Akaike (1970) and Schwarz (1978). It is known that using the wrong parametric model can lead to considerable bias in the estimation of H; see e.g. Cheung (1993), Agiakloglou, Newbold & Wohar (1993) and Agiakloglou & Newbold (1994). The possible effect of model misspecification on the proposed test is illustrated in Table 2. There, the true model is a fractional ARIMA(1, d, 0) process with positive autoregressive coefficients $\phi_1 = 0\cdot1$ and $0\cdot4$ respectively, however an ARIMA(0, d, 0) model is fitted. Thus, X_t is generated by

$$\sum_{k=0}^{\infty} (-1)^k \binom{d}{k} B^k(X_t - \phi_1 X_{t-1}) = \varepsilon_t$$

with independent identically distributed $N(0, \sigma_\varepsilon^2)$ variables ε_t and $B^k X_t = X_{t-k}$. Instead of using this model and estimating d and ϕ_1 from the data, d is estimated under the assumption that ϕ_1 is equal to zero. Table 2 is based on 400 simulations. The results show that, under mild misspecification of the model ($\phi_1 = 0\cdot1$), the actual level of the test is practically the same as assumed. In contrast, for the case with $\phi_1 = 0\cdot4$, assuming $\phi_1 = 0$ leads to a gross misspecification of the model and hence to a considerable deviation from the nominal significance levels. In the case considered here, the wrong model leads to an unduly conservative test. The effect is most dramatic for $\hat{\phi} = 0\cdot4$ and $d = 0\cdot4$. The reason is that assuming $\phi_1 = 0$ leads to a large positive bias in $\hat{d}$, while d is restricted to the stationary range $-\frac{1}{2} < d < \frac{1}{2}$. With high probability $\hat{d}$ is then approximately equal to $\frac{1}{2}$ for all subseries and hence $T_{N,k} \simeq 0$.

Table 2. *Effect of model misspecification on simulated rejection probabilities using $k = 20$ and asymptotic* 10%-, 5%- *and* 1%-*quantiles obtained from Corollary* 1

		$d=0\cdot2$			$d=0\cdot4$		
ϕ_1	N	$\alpha=0\cdot1$	$\alpha=0\cdot05$	$\alpha=0\cdot01$	$\alpha=0\cdot1$	$\alpha=0\cdot05$	$\alpha=0\cdot01$
0·1	500	0·0678	0·0352	0·0075	0·0503	0·0327	0·0126
	1000	0·0980	0·0553	0·0101	0·0653	0·0327	0·0050
0·4	500	0·0151	0·0126	0·0050	0	0	0
	1000	0·0201	0·0126	0·0025	0	0	0

The simulated process is a fractional ARIMA(1, d, 0) process with autoregressive parameter $\phi_1 = 0\cdot1$ and $0\cdot4$ respectively. A fractional ARIMA(0, d, 0) process is used to estimate d.

Table 3. *Simulated rejection probabilities for the alternative X_t $(t=1,\ldots,N/2)$ are independent identically distributed standard normal random variables, that is $d=d^{(1)}=0$, and X_t $(t=N/2+1,\ldots,N)$ is an independent fractional* ARIMA$(0,d,0)$ *process with $d=d^{(2)}=0{\cdot}25$ and $0{\cdot}4$ respectively.*

	$d^{(2)}=0{\cdot}25$			$d^{(2)}=0{\cdot}4$		
N	$\alpha=0{\cdot}1$	$\alpha=0{\cdot}05$	$\alpha=0{\cdot}01$	$\alpha=0{\cdot}1$	$\alpha=0{\cdot}05$	$\alpha=0{\cdot}01$
200	0·4150	0·2950	0·1275	0·6700	0·5630	0·3025
500	0·5975	0·4475	0·1725	0·9725	0·9300	0·7125
1000	0·9250	0·8400	0·5500	1	1	0·9975

The test statistic $T_{N,k}$ with $k=20$ is used together with asymptotic 10%-, 5%- and 1%-quantiles obtained from Corollary 1.

Table 3 gives some simulated powers for the alternative that half of the data consist of independent standard normal observations, that is $d=0$, and the other half are generated by a fractional ARIMA$(0,d,0)$ process, independent of the first half, with $d=H-\frac{1}{2}=0{\cdot}25$ and 0·4 respectively. The results are based on 400 simulations. Note that, owing to the relatively high variability of $\hat{H}$, $n=200$ is about the smallest sample size where one still may be able to detect a change in H.

5. Application to the Nile river data

For the Nile river data, an omnibus goodness-of-fit test supports the model of fractional Gaussian noise with H around 0·83 (Beran, 1992b) with a P-value of about 0·7. The same is true for a fractional ARIMA$(0,d,0)$ process with $H=0{\cdot}9$. Naturally, overall goodness-of-fit tests tend to have bad power against specific alternatives. As we saw in § 1, a closer look at the data indeed discloses a possible change of the dependence structure after about the first 100 observations. A preliminary analysis yields estimates of H for the subseries $X_{1+100(j-1)},\ldots,X_{100j}$ $(j=1,\ldots,6)$ of 0·5433, 0·8531, 0·8652, 0·8281, 0·8435 and 0·9354. The biggest jump occurs between $j=1$ to $j=2$. Even otherwise the variability of $\hat{H}$ seems high. However, a relatively high variability of $\hat{H}$ between disjoint stretches of a series is to be expected, as can be illustrated by looking at a typical realisation of a fractional ARIMA$(0,d,0)$ process with $H=0{\cdot}9$; see Fig. 1(b). Here, the estimates for the six disjoint subseries of length 100, defined as above, are equal to 0·6648, 0·9401, 0·8404, 0·9146, 0·7627 and 0·7549 respectively. It is therefore not necessarily easy to decide 'by eye' whether for the Nile river data H is constant or the differences between the estimates are merely due to random error. The method of § 3 answers this question in a formal way. Consider, for instance, $\delta=0{\cdot}1$ and $k=20$. Figure 4 displays $\tilde{Z}_N(t)$ as a function of $t=jk/N$ $(k=20)$ together with horizontal lines corresponding to the 5% and 1% critical limits for T_N. Figure 4 shows that there is statistical evidence, at the 1% level of significance, that H is not constant. The value of $\tilde{Z}_N(t)$ is largest around $jk=100$, with $T_{N,k}$ equal to 4·4 which is clearly above the 1% critical limit of 3·54. This confirms the visual impression in Fig. 1(a) and Fig. 3.

6. Final remarks

In this paper, we considered testing whether the long-memory parameter H remains constant. In situations where the long-term behaviour is the main focus, the parameter H

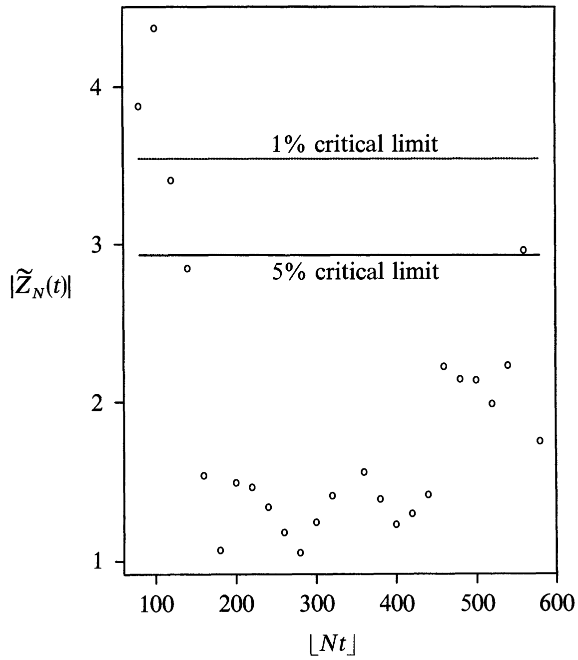

Fig. 4. $|\tilde{Z}_N(t)|$ versus Nt for the Nile river minima and critical lines for the 5% and 1% level of significance. The null hypothesis is rejected at $\alpha = 0{\cdot}05$ or $0{\cdot}01$, respectively, if $|\tilde{Z}_N(t)|$ exceeds the corresponding critical line at least once.

is most important. This is for instance the case when long-term forecasts are required or when one needs to calculate confidence intervals for location or regression estimates; see e.g. Beran (1989) and Yajima (1988, 1991). In other situations, the other components of η may be of interest as well. A test for constancy of η or certain components of η can be developed by analogous arguments from Theorem 1.

In the proposed method, the long-memory parameter was assumed to be estimated by exact or approximate maximum likelihood. Thus, in particular, $\hat{H}$ is based on the periodogram at all frequencies. In some situations, it is preferable to base the estimate of H on only a certain number of low frequencies, since it may be difficult or unnecessary to model the whole spectral shape and a misspecified model may lead to biased estimates of H: see the remarks and simulations in § 4. Estimates which rely mainly on low periodogram ordinates are discussed, for example, in Geweke & Porter-Hudak (1983), Graf, Hampel & Tacier (1984), Robinson (1994), Hurvich & Beltrao (1994) and in an as yet unpublished report by N. Terrin and C. M. Hurvich. A generalisation of our test to the situation where H is estimated by one of these methods should be straightforward.

ACKNOWLEDGEMENT

We would like to thank the referee and the associate editor for their constructive suggestions which improved the quality of the paper.

Appendix

Proofs

Proof of Theorem 1. Convergence of the finite dimensional distributions of $Z_N(t)$ follows from Theorem 1 of Beran & Terrin (1994). The relative compactness of $N^{-\frac{1}{2}}Q_1(t)$ is demonstrated in the proof of Theorem 3.1 of Giraitis & Leipus (1992). Therefore, $Z_N(t)$ is tight, and the theorem is proved. □

Proof of Corollary 1. Let η be the true parameter vector, $Q_i'(t, \eta)$ ($i = 1$ or 2) the vector of all partial derivatives

$$Q_i'(t, \eta) \equiv \left(\frac{\partial}{\partial \eta_j} Q_i(t, \eta)\right)_{j=1,\ldots,m}$$

and $Q_i''(t, \eta)$ the matrix of all second partial derivatives

$$Q_i''(t, \eta) \equiv \left(\frac{\partial^2}{\partial \eta_j \eta_k} Q_i(t, \eta)\right)_{j,k=1,\ldots,m}.$$

Then $\hat{\eta}^{(i)}(t)$ is the solution of $Q_i'(t, \hat{\eta}) = 0$. Applying a Taylor expansion at η, we obtain

$$\hat{\eta}^{(i)}(t) - \eta = \{Q_i''(t, \eta)\}^{-1} Q_i'(t, \eta) + o(N^{-\frac{1}{2}}).$$

The result then directly follows from Theorem 1. □

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[*Received February* 1995. *Revised September* 1995]