Nonparametric Statistics
Vol. 16(1–2), February–April 2004, pp. 197–216

Taylor & Francis
Taylor & Francis Group

APPLICATIONS OF PERMUTATIONS TO THE SIMULATIONS OF CRITICAL VALUES

ISTVÁN BERKES[a], LAJOS HORVÁTH[b], MARIE HUŠKOVA[c,*] and JOSEF STEINEBACH[d]

[a]*A. Rényi Institute of Mathematics, Hungarian Academy of Sciences, P.O. Box 127, H-1364 Budapest, Hungary;* [b]*Department of Mathematics, University of Utah, 155 South 1440 East, Salt Lake City, UT 84112-0090, USA;* [c]*Department of Statistics, Charles University and ÚTIA CAS, Sokolovská 83, CZ-18600 Praha, Czech Republic;* [d]*Mathematisches Institut, Universität zu Köln, Weyertal 86-90, D-50931 Köln, Germany*

(Received 9 October 2002; Revised 18 June 2003; In final form 22 June 2003)

Permuting the observations can provide an approximation for the distribution function of our test statistic. We show that invariance principles provide rates of convergence for the simulation. Bounds for the rate of convergence of cumulative sum (CUSUM), moving sum (MOSUM) and maximally selected (weighted) CUSUM statistics are examples for our method.

Keywords: Permutations; Invariance; Partial sums; CUSUM; MOSUM

2000 AMS Subject Classification: Primary: 62G09; Secondary: 62G20

1 INTRODUCTION

Let $X_1, X_2, \ldots, X_n$ be independent random variables with distribution functions (d.f.'s) $F_{(1)}, F_{(2)}, \ldots, F_{(n)}$. Let $T_n = T_n(X_1, X_2, \ldots, X_n)$ be a test statistic for the testing problem

$$H_0\colon F_{(1)}(x) = \cdots = F_{(n)}(x) \quad \text{for all } x$$

against

$$H_1\colon F_{(1)}(x) = \cdots = F_{(m)}(x), \quad F_{(m+1)}(x) = \cdots = F_{(n)}(x) \quad \text{for all } x$$
$$\text{and } F_{(m)}(x_0) \neq F_{(m+1)}(x_0) \quad \text{with some } x_0 \quad \text{and} \quad m < n.$$

Typically, the null hypothesis H_0 is rejected for (say) large values of T_n. The basic problem is then to obtain critical values for T_n, *i.e.* to determine the (upper) $(1-\alpha)$-quantiles of the d.f. of T_n, where α is the level of the test. However, the distribution function of T_n is usually unknown and asymptotic results even if they are available, may not be suitable in practice due to dependence on unknown parameters and slow rates of convergence. To overcome these

* Corresponding author. E-mail: marie.huskova@karlin.mff.cuni.cz

ISSN 1048-5252 print; ISSN 1029-0311 online
DOI: 10.1080/10485250310001624747

difficulties, resampling procedures have become a very popular tool for simulating the d.f. of T_n. Antoch and Hušková (2001) advocated the application of permutations as an alternative method to approximate the d.f. of T_n. Let $\mathbf{R} = (R_1, R_2, \ldots, R_n)$ be a random permutation of $(1, 2, \ldots, n)$, independent of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$. We define the statistic $T_n^* = T_n^*(\mathbf{R}) = T_n(X_{R_1}, \ldots, X_{R_n})$, in which we just shuffle the original observations in the computation of the test statistic. Using N independent permutations $\mathbf{R}_1, \ldots, \mathbf{R}_N$, independent of each other and also of $\mathbf{X}$, we obtain the permuted statistics $T_n^*(\mathbf{R}_1), \ldots, T_n^*(\mathbf{R}_N)$. The function

$$H_{n,N}(x) = \frac{1}{N} \sum_{1 \le i \le N} I\{T_n^*(\mathbf{R}_i) \le x\}$$

is used to approximate

$$H_n(x) = P_{H_0}\{T_n \le x\}.$$

In this paper, we are interested in the magnitude of the difference $|H_{n,N}(x) - H_n(x)|$. Proving limit theorems for T_n, a typical result says that there are random variables $\xi_1, \xi_2, \ldots$ such that

$$P_{H_0}\{|T_n - \xi_n| \ge q_n\} \le q_n, \tag{1}$$

i.e. the Prokhorov–Lévy distance between T_n and ξ_n is not greater than q_n. Similar results can be obtained for T_n^*. Let $P_\mathbf{X}$, $E_\mathbf{X}$, and $\mathrm{var}_\mathbf{X}$ denote conditional probability, expected value, and variance with respect to $\mathbf{X}$. There are random variables $\xi_1^*, \xi_2^*, \ldots$ such that

$$P_\mathbf{X}\{|T_n^* - \xi_n^*| \ge z_n\} \le z_n, \quad z_n = z_n(\mathbf{X}) \tag{2}$$

and

$$P_\mathbf{X}\{\xi_n^* \le t\} = P_{H_0}\{\xi_n \le t\} \quad \text{for almost all realizations of } \mathbf{X}. \tag{3}$$

The rate of convergence of $|H_{n,N}(x) - H_n(x)|$ to 0 will depend on the modulus of continuity of $F_n(x) = P_{H_0}\{\xi_n \le x\}$, that is on

$$m_n(\delta, x) = \sup_{y:\, |x-y| \le \delta} |F_n(x) - F_n(y)|.$$

We note that we do not require any continuity properties of the d.f. of T_n; we need only that T_n can be approximated with random variables having a smooth distribution function. Our examples will illustrate that it is usually easier to compute the modulus of continuity of the distribution function of ξ_n than that of T_n.

THEOREM 1.1 *If Eqs.* (1)–(3) *hold*, *then for any* $\lambda \ge 0$ *and* $N \ge ((4/3)\lambda)^2$ *we have*

$$\begin{aligned} P_\mathbf{X}\{|H_{n,N}(x) - H_n(x)| &\ge \lambda N^{-1/2} + q_n + m_n(q_n, x) + z_n + m_n(z_n, x)\} \\ &\le 2\exp(-\lambda^2) \quad \textit{for almost all realizations of } \mathbf{X}. \end{aligned} \tag{4}$$

Proof Conditionally on $\mathbf{X}, T_n^*(\mathbf{R}_1), \ldots, T_n^*(\mathbf{R}_N)$ are independent, identically distributed random variables. Using the Bernstein inequality (cf. Serfling, 1980, p. 95) we conclude that

$$P_\mathbf{X}\left\{\left|\sum_{1 \le i \le N} (I\{T_n^*(\mathbf{R}_i) \le x\} - P_\mathbf{X}\{T_n^* \le x\})\right| \ge Nt\right\} \le 2\exp\left(-\frac{N^2t^2}{N/2 + (2/3)Nt}\right),$$

since $|I\{T_n^*(\mathbf{R}_i) \leq x\} - P_{\mathbf{X}}\{T_n^* \leq x\}| \leq 1$ and $\mathrm{var}_{\mathbf{X}}\, I\{T_n^*(\mathbf{R}_i) \leq x\} \leq 1/4$. Choosing $t = \lambda/N^{1/2}$ and observing that

$$\frac{N^2 t^2}{N/2 + (2/3)Nt} \geq \lambda^2$$

we conclude

$$P_{\mathbf{X}}\left\{|H_{n,N}(x) - P_{\mathbf{X}}\{T_n^* \leq x\}| \geq \frac{\lambda}{N^{1/2}}\right\} \leq 2\exp(-\lambda^2). \tag{5}$$

By Eq. (1) we have

$$|P_{H_0}\{T_n \leq x\} - F_n(x)| = |P_{H_0}\{T_n \leq x\} - P_{H_0}\{\xi_n \leq x\}| \leq q_n + m_n(q_n, x) \tag{6}$$

and similarly Eqs. (2) and (3) imply

$$|P_{\mathbf{X}}\{T_n^* \leq x\} - F_n(x)| = |P_{\mathbf{X}}\{T_n^* \leq x\} - P_{\mathbf{X}}\{\xi_n^* \leq x\}| \leq z_n + m_n(z_n, x). \tag{7}$$

Now Eq. (4) follows from Eqs. (5)–(7).

COROLLARY 1.1 *If there is a random variable* ξ *such that*

$$T_n \xrightarrow{\mathcal{D}} \xi \quad \textit{under}\, H_0, \tag{8}$$

$$T_n^* \xrightarrow{\mathcal{D}_{\mathbf{X}}} \xi \quad \textit{for almost all realizations of } \mathbf{X}, \tag{9}$$

and

$$x \textit{ is a point of continuity of the distribution function of } \xi, \tag{10}$$

then, *as* $\min(n, N) \to \infty$,

$$|H_{n,N}(x) - H_n(x)| = o_{P_{\mathbf{X}}}(1)$$

for almost all realizations of **X**.

Proof By the Skorokhod–Dudley–Wichura representation theorem, there are random variables ξ_n and ξ_n^* such that Eqs. (1) and (2) hold with some $q_n \to 0$ and $z_n \to 0$ a.s. Hence the result follows from Theorem 1.1.

Theorem l.1 gives the following upper bound for the rate of convergence.

COROLLARY 1.2 *If Eqs.* (1)–(3) *hold*, *then*

$$|H_{n,N}(x) - H_n(x)| = \mathrm{O}_{P_{\mathbf{X}}}(N^{-1/2} + q_n + m_n(q_n, x) + z_n + m_n(z_n, x))$$

for almost all realizations of **X**.

Note that through the above permutation argument we get an approximation for the critical values corresponding to the null distribution, even if the observed data do not follow H_0. We shall see in the examples that Eqs. (2) and (3) hold under H_0 and under H_1. This means that $H_{n,N}(x)$ always approximates $H_n(x)$, the d.f. of the test statistic under the null hypothesis, even if H_1 holds.

In the next three sections, we discuss some examples for Theorem 1.1. The last two sections contain technical results on the maximum of a Gaussian process and approximations of linear rank statistics.

2 CUSUM STATISTICS

For the sake of simplicity, we consider the location model with a change after an unknown time point m, *i.e.*

$$X_i = \mu + \delta I\{i > m\} + e_i, \quad i = 1, \ldots, n, \tag{11}$$

where $1 \leq m \leq n$, μ, and $\delta = \delta_n > 0$ are unknown parameters and $I\{A\}$ denotes the indicator function of a set A. It is assumed that $|\delta| \leq D_0$ with some $D_0 > 0$. Assume, moreover, that

$$\begin{gathered} e_1, \ldots, e_n \text{ are independent, identically distributed random variables (i.i.d. r.v.'s)} \\ \text{with } Ee_i = 0, 0 < \sigma^2 = \operatorname{var} e_i, \text{ and } E|e_i|^\nu < \infty \text{ with some } \nu > 2. \end{gathered} \tag{12}$$

We are interested in testing the hypotheses

$$H_0\colon m = n \quad \text{against} \quad H_1\colon m < n.$$

Our test statistic is the (so-called) CUSUM statistic defined as

$$T_n = \max_{1 \leq k \leq n} \frac{1}{n^{1/2} \hat{\sigma}_n} \left(S(k) - \frac{k}{n} S(n) \right),$$

where, for later use, we set

$$S(x) = \sum_{1 \leq i \leq x} X_i \quad (x \geq 0),$$

the empty sum being 0, and

$$\hat{\sigma}_n^2 = \frac{1}{n} \sum_{1 \leq i \leq n} \left(X_i - \frac{1}{n} S(n) \right)^2. \tag{13}$$

On assuming Eqs. (11) and (12), we have that

$$T_n \xrightarrow{\mathcal{D}} \xi \quad \text{under } H_0, \tag{14}$$

where $\xi = \sup_{0 \leq t \leq 1} B(t)$ and $\{B(t), 0 \leq t \leq 1\}$ denotes a Brownian bridge (cf. Billingsley, 1968; Csörgő and Horváth, 1997). Analogously to Antoch and Hušková (2001), it follows from Hušková (1997) that the corresponding permutation statistic T_n^* satisfies

$$T_n^* \xrightarrow{\mathcal{D}} \xi \quad \text{for almost all realizations of } \mathbf{X}. \tag{15}$$

Since

$$P\{\xi \leq x\} = 1 - \mathrm{e}^{-2x^2}, \quad 0 \leq x < \infty, \tag{16}$$

we get that

$$|H_{n,N}(x) - H_n(x)| = o_{P_{\mathbf{X}}}(1), \quad \text{as } \min(n, N) \to \infty, \tag{17}$$

for almost all realizations of **X**.

Next we consider the rate of convergence in Eq. (17). In view of Eq. (12) we have

$$E_{H_0}|X_1|^\nu < \infty. \tag{18}$$

By the Komlós *et al*. (1975a, b; 1976) approximation (cf. also Borovkov, 1973) there are Brownian bridges $\{B_n(t), 0 \le t \le 1\}$ such that

$$\begin{aligned} P_{H_0} & \left\{ \left| \frac{1}{n^{1/2}\sigma} \max_{1 \le k \le n} \left(S(k) - \frac{k}{n} S(n) \right) - \sup_{0 \le t \le 1} B_n(t) \right| \ge C_1 n^{-(\nu-2)/(2(1+\nu))} \right\} \\ & \le C_1 n^{-(\nu-2)/(2(1+\nu))} \end{aligned} \tag{19}$$

with some constant C_1. Using the Rosenthal (cf. Petrov, 1995, p. 56) and the Markov inequalities we get for all $x > 0$ that

$$P_{H_0} \left\{ \left| \frac{1}{n} S(n) - E_{H_0} X_1 \right|^2 \ge x \right\} \le C_2 \frac{n^{\nu/2}}{(n x^{1/2})^\nu}$$

with some constant C_2 and therefore

$$P_{H_0} \left\{ \left| \frac{1}{n} S(n) - E_{H_0} X_1 \right|^2 \ge C_3 n^{-\nu/(2(1+\nu))} \right\} \le C_3 n^{-\nu/(2(1+\nu))}. \tag{20}$$

If $4 \le \nu < \infty$, then applying again the Rosenthal and Markov inequalities we obtain that

$$P_{H_0} \left\{ \left| \frac{1}{n} \sum_{1 \le i \le n} (X_i - E_{H_0} X_1)^2 - \sigma^2 \right| \ge x \right\} \le C_4 \frac{n^{\nu/4}}{(nx)^{\nu/2}}$$

and therefore

$$P_{H_0} \left\{ \left| \frac{1}{n} \sum_{1 \le i \le n} (X_i - E_{H_0} X_1)^2 - \sigma^2 \right| \ge C_5 n^{-\nu/(2(2+\nu))} \right\} \le C_5 n^{-\nu/(2(2+\nu))}. \tag{21}$$

If $2 < \nu < 4$, then replacing the Rosenthal inequality with the von Bahr–Esseen inequality (cf. Petrov, 1995, p. 82) we conclude

$$P_{H_0} \left\{ \left| \frac{1}{n} \sum_{1 \le i \le n} (X_i - E_{H_0} X_1)^2 - \sigma^2 \right| \ge x \right\} \le C_6 \frac{n}{(nx)^{\nu/2}}$$

and therefore

$$P_{H_0} \left\{ \left| \frac{1}{n} \sum_{1 \le i \le n} (X_i - E_{H_0} X_1)^2 - \sigma^2 \right| \ge C_7 n^{-(\nu-2)/(2+\nu)} \right\} \le C_7 n^{-(\nu-2)/(2+\nu)}. \tag{22}$$

It follows from Eq. (16) that

$$P_{H_0}\left\{\sup_{0\le t\le 1} B_n(t) \ge (\log n)^{1/2}\right\} \le \frac{1}{n^2}. \tag{23}$$

Putting together Eqs. (19)–(23) we obtain that

$$P_{H_0}\{|T_n - \xi_n| \ge q_n\} \le q_n, \tag{24}$$

where $\xi_n = \sup_{0\le t\le 1} B_n(t)$ and $q_n = C_8 n^{-(\nu-2)/(2(1+\nu))}$.

Next we establish an analogue of Eq. (24) for T_n^*, assuming that the $X_1, \ldots, X_n$ follow the model Eqs. (11) and (12). This means that Eq. (2) will be established for T_n^* under the null as well as under the alternative. Using Lemma 6.1 below for each n, there are $U_1^{(n)}, \ldots, U_n^{(n)}$, independent, identically distributed random variables, uniform on [0, 1] such that

$$\begin{aligned} P_{\mathbf{X}}&\left\{\max_{1\le k\le n}\left|S^*(k) - \frac{k}{n}S^*(n) - \left(\sum_{1\le i\le k}\xi_{i,n} - \frac{k}{n}\sum_{1\le i\le n}\xi_{i,n}\right)\right| \ge x\right\} \\ &\le \frac{C_9}{x^2}\max_{1\le i\le n}\left|X_i - \frac{S(n)}{n}\right|\left(\sum_{1\le i\le n}\left(X_i - \frac{S(n)}{n}\right)^2\right)^{1/2}, \end{aligned} \tag{25}$$

where

$$S^*(k) = \sum_{1\le i\le k} X_{R_i}$$

and

$$\xi_{i,n} = X_{[nU_i^{(n)}]+1}.$$

Since the permuted version of T_n is

$$T_n^* = \frac{1}{n^{1/2}\hat{\sigma}_n}\max_{1\le k\le n}\left(S^*(k) - \frac{k}{n}S^*(n)\right),$$

Eq. (25) implies that

$$P_{\mathbf{X}}\left\{\left|T_n^* - \frac{1}{n^{1/2}\hat{\sigma}_n}\max_{1\le k\le n}\left(\sum_{1\le i\le k}\xi_{i,n} - \frac{k}{n}\sum_{1\le i\le n}\xi_{i,n}\right)\right| \ge z_{n,1}\right\} \le z_{n,1}, \tag{26}$$

where

$$z_{n,1} = C_{10}\left\{\frac{1}{\hat{\sigma}_n^2 n}\max_{1\le i\le n}\left|X_i - \frac{S(n)}{n}\right|\left(\sum_{1\le i\le n}\left(X_i - \frac{S(n)}{n}\right)^2\right)^{1/2}\right\}^{1/3}.$$

By Sakhanenko (1980; 1984; 1985) there are Wiener processes $\{W_n^*(x)\colon x \geq 0\}$ such that

$$P_{\mathbf{X}}\left\{\max_{1\le k\le n}\left|\sum_{1\le i\le k}(\xi_{i,n} - E_{\mathbf{X}}\xi_{1,n}) - \hat{\sigma}_n W_n^*(k)\right| \geq x\right\}$$
$$\leq \frac{C_{11}}{x^\nu}\sum_{1\le i\le n} E_{\mathbf{X}}|\xi_{i,n} - E_{\mathbf{X}}\xi_{i,n}|^\nu$$
$$= \frac{C_{11}}{x^\nu}\sum_{1\le i\le n}\left|X_i - \frac{S(n)}{n}\right|^\nu. \tag{27}$$

Hence by Eqs. (26), (27), and Lemma 1.1.1 of Csörgő and Révész (1981) we have

$$P_{\mathbf{X}}\left\{\left|\frac{1}{n^{1/2}\hat{\sigma}_n}\max_{1\le k\le n}\left(\sum_{1\le i\le k}\xi_{i,n} - \frac{k}{n}\sum_{1\le i\le n}\xi_{i,n}\right) - \sup_{0\le t\le 1} B_n^*(t)\right| \geq z_{n,2}\right\} \leq z_{n,2}, \tag{28}$$

where

$$z_{n,2} = \frac{C_{12}}{\hat{\sigma}_n^{1/(1+\nu)}} n^{-\nu/(2(1+\nu))}\left(\sum_{1\le i\le n}\left|X_i - \frac{S(n)}{n}\right|^\nu\right)^{1/(1+\nu)} + 2n^{-1/2}(\log\, n)^{1/2},$$

and $B_n^*(t) = n^{-1/2}(W_n^*(nt) - tW_n^*(n))$ is a Brownian bridge. Putting together Eqs. (26) and (28) we conclude that

$$P_{\mathbf{X}}\{|T_n^* - \xi_n^*| \geq z_n\} \leq z_n,$$

where $z_n = z_{n,1} + z_{n,2}$ and $\xi_n^* = \sup_{0\le t\le 1} B_n^*(t)$. Hence Corollary 1.2 yields

$$|H_{n,N}(x) - H_n(x)| = \mathrm{O}_{P_{\mathbf{X}}}(N^{-1/2} + q_n + z_n) \tag{29}$$

for almost all realizations of **X**. By the assumptions and by the strong law of large numbers we have

$$\max_{1\le i\le n}\left|X_i - \frac{S(n)}{n}\right|\left(\sum_{1\le i\le n}\left(X_i - \frac{S(n)}{n}\right)^2\right)^{1/2} = \mathrm{O}(n^{(\nu+2)/(2\nu)}) \text{ a.s.} \tag{30}$$

and therefore

$$z_{n,1} = \mathrm{O}(n^{-(\nu-2)/(6\nu)}) \text{ a.s.} \tag{31}$$

Similarly,

$$z_{n,2} = \mathrm{O}(n^{-(\nu-2)/(2(1+\nu))}) \text{ a.s.}$$

We also conclude from Eq. (29) that

$$|H_{n,N}(x) - H_n(x)| = \mathrm{O}_{P_{\mathbf{X}}}(N^{-1/2} + n^{-(\nu-2)/(6\nu)}) \tag{32}$$

for almost all realizations of **X**.

Remark Checking carefully the proof of Eq. (32), one can see that this assertion remains true even if the observations $X_1, \ldots, X_n$ in Eq. (11) follow the more general model

$$X_i = \mu_i + e_i, \quad i = 1, \ldots, n, \tag{33}$$

where $|\mu_i| \leq D_1$ (with some $D_1 > 0$) are unknown parameters, and $e_1, \ldots, e_n$ are as in Eq. (12).

3 MOSUM STATISTICS

We assume models (11) and (12) again, but make now use of the (so-called) moving sum (MOSUM) statistic defined as

$$T_n(h) = \frac{1}{h^{1/2}\hat{\sigma}_n} \max_{1\leq k\leq n-h} \left(S(k+h) - S(k) - h\frac{S(n)}{n} \right),$$

where $h = h(n)$ satisfies

$$h = cn^{\kappa} \text{ with some } c > 0 \quad \text{and} \quad \frac{\nu+2}{2\nu} < \kappa < \frac{\nu+3}{2\nu+1}. \tag{34}$$

We refer to Antoch and Hušková (1989), Hušková (1994), Steinebach (1994) and Steinebach and Eastwood (1996) for further MOSUM procedures. Under H_0 by the Komlós, Major and Tusnády (1975a, b; 1976) approximation we can find a Wiener process $\{W(x): x \geq 0\}$ such that

$$P_{H_0}\left\{ h^{-1/2} \max_{0\leq x\leq n-h} \left| \frac{1}{\sigma} \left(S(x+h) - S(x) - h\frac{S(n)}{n} \right) - (W(x+h) \right.\right.$$
$$\left.\left. -W(x) - h\frac{W(n)}{n} \right) \right| > C_1 \left(\frac{n}{h^{\nu/2}} \right)^{1/(1+\nu)} \right\} \leq C_1 \left(\frac{n}{h^{\nu/2}} \right)^{1/(1+\nu)}. \tag{35}$$

Since $W(n)/n^{1/2}$ is standard normal for any n, the usual upper bound for the tail of the standard normal distribution function together with Eq. (34) yields

$$P\left\{ h^{1/2}\frac{|W(n)|}{n} \geq C_2 \left(\frac{n}{h^{\nu/2}} \right)^{1/(1+\nu)} \right\} \leq C_2 \left(\frac{n}{h^{\nu/2}} \right)^{1/(1+\nu)}. \tag{36}$$

Let $U(t) = W(t+1) - W(t)$. It is easy to see that for any n

$$h^{-1/2} \sup_{0\leq x\leq n-h} (W(x+h) - W(x)) \overset{\mathcal{D}}{=} \sup_{0\leq t\leq n/h-1} U(t).$$

Clearly, $\{U(t): t \geq 0\}$ is a stationary Gaussian process with $EU(t) = 0$ and

$$EU(t)U(s) = \begin{cases} 1 - |t-s|, & \text{if } |t-s| \leq 1, \\ 0, & \text{if } |t-s| > 1. \end{cases}$$

The Fernique (1975) inequality yields that

$$P_{H_0}\left\{ \sup_{0\leq t\leq n/h-1} U(t) \geq C_3 \left(\log\left(\frac{h^{\nu/2}}{n} \right) \right)^{1/2} \right\} \leq C_4 \left(\frac{n}{h^{\nu/2}} \right)^{1/(1+\nu)} \tag{37}$$

with some constants C_3 and C_4. Putting together Eqs. (20)–(22), and Eqs. (35)–(37) we get that

$$P_{H_0}\left\{\left|T_n(h) - h^{-1/2}\max_{0\le x\le n-h}(W(x+h) - W(x))\right| \ge C_5\left(\frac{n}{h^{\nu/2}}\right)^{1/(1+\nu)}\right\}$$
$$\le C_5\left(\frac{n}{h^{\nu/2}}\right)^{1/(1+\nu)}. \tag{38}$$

Let

$$a(n) = \left(2\log\left(\frac{n}{h} - 1\right)\right)^{1/2}$$

and

$$b(n) = 2\log\left(\frac{n}{h} - 1\right) + \frac{1}{2}\log\log\left(\frac{n}{h} - 1\right) - \frac{1}{2}\log\pi.$$

Our test statistic is

$$T_n = a(n)T_n(h) - b(n).$$

By Eq. (38) we have that

$$P_{H_0}\{|T_n - \xi_n| \ge q_n\} \le q_n \tag{39}$$

with $\xi_n = a(n)h^{-1/2}\max_{0\le x\le n/h-1}(W(x+h) - W(x))$ and

$$q_n = C_6\left(\log\frac{n}{h}\right)^{1/2}\left(\frac{n}{h^{\nu/2}}\right)^{1/(1+\nu)}.$$

The permutation counterpart of $T_n(h)$ is

$$T_n^*(h) = \frac{1}{h^{1/2}\hat{\sigma}_n}\max_{1\le k\le n-h}\left(S^*(k+h) - S^*(k) - h\frac{S^*(n)}{n}\right).$$

We assume that $\mathbf{X}$ is given. We show again that Eq. (2) holds for $T_n^*(h)$ under the null hypothesis as well as under the alternative. Using Eq. (25) we obtain that

$$P_{\mathbf{X}}\left\{h^{-1/2}\max_{1\le k\le n-k}\left|S^*(k+h) - S^*(k) - h\frac{S^*(n)}{n}\right.\right.$$
$$\left.\left. - \left(\sum_{1\le i\le k+h}\xi_{i,n} - \sum_{1\le i\le k}\xi_{i,n} - \frac{h}{n}\sum_{1\le i\le n}\xi_{i,n}\right)\right| \ge z_{n,1}\right\} \le z_{n,1}, \tag{40}$$

where

$$z_{n,1} = C_7 h^{-1/3}\left(\max_{1\le i\le n}\left|X_i - \frac{S(n)}{n}\right|^2\sum_{1\le i\le n}\left(X_i - \frac{S(n)}{n}\right)^2\right)^{1/6}$$

and by Eq. (30)

$$z_{n,1} = \mathrm{O}(h^{-1/3}(n^{-(\nu-2)/(6\nu)}). \tag{41}$$

The approximation in Eq. (27) implies that

$$P_{\mathbf{X}}\left\{h^{-1/2}\max_{1\le k\le n-h}\left|\left(\sum_{1\le i\le k+h}\xi_{i,n}-\sum_{1\le i\le k}\xi_{i,n}-\frac{h}{n}\sum_{1\le i\le n}\xi_{i,n}\right)\right.\right.$$
$$\left.\left.-\hat{\sigma}_n\left(W_n^*(k+h)-W_n^*(k)-\frac{h}{n}W_n^*(n)\right)\right|\ge z_{n,2}\right\}\le z_{n,2}, \tag{42}$$

where

$$z_{n,2}=C_8h^{-\nu/(2(1+\nu))}\left(\sum_{1\le i\le n}\left|X_i-\frac{S(n)}{n}\right|^2\right)^{1/(1+\nu)}.$$

By Eqs. (36), (40), (42), and Lemma 1.1.1 of Csörgő and Révész (1981) we have

$$P_{\mathbf{X}}\left\{\left|T_n^*(h)-h^{-1/2}\sup_{0\le x\le n-h}(W_n^*(x+h)-W_n^*(x))\right|\ge z_{n,3}\right\}\le z_{n,3}, \tag{43}$$

where

$$z_{n,3}=\max(1,\hat{\sigma}_n)(z_{n,1}+z_{n,2})+2h^{-1/2}(\log n)^{1/2}.$$

Defining

$$T_n^*=a(n)T_n^*(h)-b(n)$$

and

$$\xi_n^*=a(n)h^{-1/2}\sup_{0\le x\le n-h}(W_n^*(x+h)-W_n^*(x))-b(n),$$

we conclude that

$$P_{\mathbf{X}}\{|T_n^*-\xi_n^*|\ge z_n\}\le z_n, \tag{44}$$

where

$$z_n=a(n)z_{n,3}.$$

Then we obtain that

$$z_n\overset{\text{a.s.}}{=}\mathrm{O}\left(\left(\log\frac{n}{h}\right)^{1/2}\left(\left(\frac{n}{h^{\nu/2}}\right)^{1/(1+\nu)}+\left(\frac{n^{(\nu+2)/\nu}}{h^2}\right)^{1/6}\right)\right).$$

Lemma 5.1 below yields that

$$m_n(\eta)\le C_9\eta^{\gamma}\quad\text{for any }\ \gamma<\frac{1}{3},$$

where $m_n(\eta)$ denotes the modulus of continuity of the d.f. of ξ_n^*. So by Corollary 1.2 we have that

$$H_{n,N}(x)-H_n(x)=\mathrm{o}_{P_{\mathbf{X}}}\left(\left(\frac{n}{h^{\nu/2}}\right)^{\tau_1}+\left(\frac{n}{h^2}\right)^{\tau_2}+N^{-1/2}\right) \tag{45}$$

for any $\tau_1<1/(3(1+\nu))$, $\tau_2<1/18$ for almost all realizations $\mathbf{X}$.

4 WEIGHTED CUSUM STATISTIC

Assume model Eqs. (11) and (12) of Section 2 again. A very often used weighted version of the CUSUM statistic is

$$\tilde{T}_n = \frac{n^{1/2}}{\hat{\sigma}_n} \max_{1\le k\le n} \frac{1}{(k(n-k))^{1/2}} \left(S(k) - \frac{k}{n} S(n) \right),$$

where $\hat{\sigma}_n$ is given by Eq. (13), *i.e.* the partial sums $S(k) - kS(n)/n$ are normalized by the estimated standard deviation. For a survey of the results on the maximally selected CUSUM and its connection to the likelihood ratio test we refer to Csörgő and Horváth (1997). If Eq. (18) holds, Gombay and Horváth (2002) constructed a sequence of Brownian bridges $\{B_n(t), 0 \le t \le 1\}$ such that for any $0 < \alpha < 2/3$ and $0 < \beta < \alpha/2$

$$P_{H_0} \left\{ \sup_{(\log\log n)^{\alpha}/n \le t \le 1-(\log\log n)^{\alpha}/n} \frac{\left|1/\sigma n^{1/2}(S(nt) - tS(n)) - B_n(t)\right|}{(t(1-t))^{1/2}} \right.$$
$$\left. \ge C_1(\log\log n)^{-\beta} \right\} \le C_2(\log\log n)^{\nu(\beta-\alpha/2)}. \tag{46}$$

Using the Hájek–Rényi–Chow inequality (cf. Chow, 1960)

$$P_{H_0} \left\{ \max_{1\le k\le(\log\log n)^{\alpha}} \frac{1}{\sigma k^{1/2}} |S(k) - E_{H_0}S(k)| > (\log\log n)^{1/2} \right\}$$
$$= P_{H_0} \left\{ \max_{1\le k\le(\log\log n)^{\alpha}} \frac{1}{\sigma^{\nu} k^{\nu/2}} |S(k) - E_{H_0}S(k)|^{\nu} > (\log\log n)^{\nu/2} \right\}$$
$$\le (\sigma \log\log n)^{-\nu/2} \left\{ \sum_{1\le k\le(\log\log n)^{\alpha}-1} \{k^{-\nu/2} - (k+1)^{-\nu/2}\} E_{H_0}|S(k) - E_{H_0}S(k)|^{\nu} \right.$$
$$\left. +(\sigma \log\log n)^{-\alpha\nu/2} E_{H_0}|S((\log\log n)^{\alpha}) - E_{H_0}S((\log\log n)^{\alpha})|^{\nu} \right\}. \tag{47}$$

By Theorem 2.10 in Petrov (1995) we have that

$$E_{H_0}|S(k) - E_{H_0}S(k)|^{\nu} \le C_3 k^{\nu/2} \tag{48}$$

and therefore Eq. (47) yields

$$P_{H_0} \left\{ \max_{1\le k\le(\log\log n)^{\alpha}} \frac{1}{\sigma k^{1/2}} |S(k) - E_{H_0}S(k)| > (\log\log n)^{1/2} \right\}$$
$$\le C_4(\log\log n)^{-\nu/2} \log\log\log n. \tag{49}$$

Also, Eq. (48) implies

$$P_{H_0} \left\{ \max_{1\le k\le(\log\log n)^{\alpha}} \left| \frac{k^{1/2}}{\sigma n}(S(n) - ES(n)) \right| \ge (\log\log n)^{1/2} \right\}$$
$$\le C_5(\log\log n)^{-\nu/2} \log\log\log n. \tag{50}$$

Putting together Eqs. (49) and (50) we obtain that

$$P_{H_0}\left\{\max_{1\le k\le(\log\log n)^\alpha}\frac{n^{1/2}}{\sigma(k(n-k))^{1/2}}\left(S(k)-\frac{k}{n}S(n)\right)>(\log\log n)^{1/2}\right\}$$
$$\le C_5(\log\log n)^{-\nu/2}\log\log\log n \tag{51}$$

and by symmetry

$$P_{H_0}\left\{\max_{n-(\log\log n)^\alpha\le k<n}\frac{n^{1/2}}{\sigma(k(n-k))^{1/2}}\left(S(k)-\frac{k}{n}S(n)\right)>(\log\log n)^{1/2}\right\}$$
$$\le C_5(\log\log n)^{-\nu/2}\log\log\log n. \tag{52}$$

Similar arguments show that for any $\kappa>0$

$$P_{H_0}\left\{\sup_{1/n\le t\le(\log\log n)^\alpha/n}\frac{B_n(t)}{(t(1-t))^{1/2}}>(\log\log n)^{1/2}\right\}\le C_6(\log\log n)^{-\kappa} \tag{53}$$

and

$$P_{H_0}\left\{\sup_{1-(\log\log n)^\alpha/n\le t<1-1/n}\frac{B_n(t)}{(t(1-t))^{1/2}}>(\log\log n)^{1/2}\right\}\le C_6(\log\log n)^{-\kappa}. \tag{54}$$

Using Lemma 3.3 of Gombay and Horváth (2002) we conclude that

$$P_{H_0}\left\{\sup_{1/n\le t\le 1-1/n}\frac{B_n(t)}{(t(1-t))^{1/2}}\le\left(\left(\frac{3}{2}\right)\log\log n\right)^{1/2}\right\}\le C_7n^{-2}. \tag{55}$$

Putting together Eqs. (46) and (51)–(55) we conclude that for any $\kappa>0$

$$P_{H_0}\left\{\sup_{1/n\le t\le 1-1/n}\frac{B_n(t)}{(t(1-t))^{1/2}}\ne\sup_{(\log\log n)^\alpha\le t\le 1-(\log\log n)^\alpha/n}\frac{B_n(t)}{(t(1-t))^{1/2}}\right\}$$
$$\le C_8(\log\log n)^{-\kappa} \tag{56}$$

and

$$P_{H_0}\left\{\max_{1\le k<n}\left(\frac{n}{k(n-k)}\right)^{1/2}\left(S(k)-\frac{k}{n}S(n)\right)\right.$$
$$\left.\ne\max_{(\log\log n)^\alpha\le k<n-(\log\log n)^\alpha}\left(\frac{n}{k(n-k)}\right)^{1/2}\left(S(k)-\frac{k}{n}S(n)\right)\right\}$$
$$\le C_9((\log\log n)^{\nu(\beta-\alpha/2)}+\log\log\log n(\log\log n)^{-\nu/2}). \tag{57}$$

If

$$\tilde{\xi}_n=\sup_{1/n\le t\le 1-1/n}\frac{B_n(t)}{(t(1-t))^{1/2}},$$

then Eqs. (46), (57) and Eqs. (21), (22) yield

$$P_{H_0}\left\{|\tilde{T}_n - \tilde{\xi}_n| \geq C_{10}(\log\log n)^{-\beta}\right\}$$
$$\leq C_9((\log\log n)^{\nu(\beta-\alpha/2)} + \log\log\log n(\log\log n)^{-\nu/2}) \tag{58}$$

for any $0 < \alpha < 2\beta$. Let

$$T_n = a(n)\tilde{T}_n - b(n),$$

and

$$\xi_n = a(n)\tilde{\xi}_n - b(n),$$

where the normalizing sequences $\{a(n)\}$ and $\{b(n)\}$ are from Section 3. By Eq. (58) we get that

$$P_{H_0}\{|T_n - \xi_n| > q_n\} \leq q_n \tag{59}$$

with $q_n = C_{12}(\log n)^{-\nu/2}$.

Now we shall establish Eq. (59) for T_n^* under the null as well as under the alternative hypothesis. We condition with respect to $\mathbf{X}$. Lemma 6.2 and Eq. (30) yield that

$$P_{\mathbf{X}}\left\{\max_{1\leq k<n-1}\frac{n^{1/2}}{(k(n-k))^{1/2}}\left|S_k^* - \frac{k}{n}S_n^* - \left(\sum_{1\leq i\leq k}\xi_{i,n} - \frac{k}{n}\sum_{1\leq i\leq n}\xi_{i,n}\right)\right|\right.$$
$$\left.\geq C_{13}(\log\log n)^{-\kappa}\right\} \overset{\text{a.s.}}{=} \mathrm{O}((\log\log n)^{-\kappa})$$

for any $\kappa > 0$. By Sakhanenko (1980; 1984; 1985) for any n there are independent Wiener processes $\{W_{n,1}^*(x), 0 \leq x < \infty\}$, $\{W_{n,2}^*(x), 0 \leq x < \infty\}$ and random variables τ_n such that

$$P_{\mathbf{X}}\left\{\sup_{k-1\leq y<k}\left|\sum_{1\leq i\leq k}(\xi_{i,n} - E_{\mathbf{X}}\xi_{i,n}) - \hat{\sigma}_n W_{n,1}^*(y)\right| \geq x\right\} \leq \tau_n\frac{k}{x^\nu}, \quad 1 \leq k \leq \frac{n}{2}, \tag{60}$$

and

$$P_{\mathbf{X}}\left\{\sup_{k-1\leq y<k}\left|\sum_{k<i\leq n}(\xi_{i,n} - E_{\mathbf{X}}\xi_{i,n}) - \hat{\sigma}_n W_{n,2}^*(y)\right| \geq x\right\} \leq \tau_n\frac{k}{x^\nu}, \quad \frac{n}{2} \leq k \leq n, \tag{61}$$

with

$$\tau_n = \mathrm{O}(1) \text{ a.s.} \tag{62}$$

Following the arguments in the proof of Lemma 3.1 of Gombay and Horváth (2002) we conclude that by Eqs. (60)–(62) we can construct Brownian bridges $\{B_n^*(t), 0 \leq t \leq 1\}$ such that

$$P_{\mathbf{X}}\left\{\sup_{(\log\log n)^\alpha/n\leq t\leq 1-(\log\log n)^\alpha/n}\frac{\left|1/\hat{\sigma}_n n^{1/2}\left(\sum_{1\leq i\leq n_t}\xi_{i,n} - t\sum_{1\leq i\leq n}\xi_{i,n}\right) - B_n^*(t)\right|}{(t(1-t))^{1/2}}\right.$$
$$\left.\geq C_{14}(\log\log n)^{-\beta}\right\} = \mathrm{O}((\log\log n)^{\nu(\beta-\alpha/2)}) \text{ a.s.} \tag{63}$$

for any $0 < \beta < 2\alpha$. It is easy to see that

$$E_{\mathbf{X}}\left|\sum_{1\le i\le k}(\xi_{i,n}-E_{\mathbf{X}}\xi_{i,n})\right|^{\nu}\le \tau_n^* k^{\nu/2},\quad 1\le k\le n$$

and

$$\tau_n^* = \mathrm{O}(1) \text{ a.s.}$$

Thus we can follow the proofs of Eqs. (49)–(59) resulting in

$$P_{\mathbf{X}}\{|T_n^* - \xi_n^*| > z_n\} \le z_n$$

with

$$z_n \overset{\text{a.s.}}{=} \mathrm{O}((\log\log n)^{-\nu/2}),$$

where

$$T_n^* = a(n)\frac{n^{1/2}}{\hat{\sigma}_n}\max_{1\le k<n}\frac{S(k)-(k/n)S(n)}{(k(n-k))^{1/2}}$$

and

$$\xi_n^* = \sup_{1/n\le t\le 1-1/n}\frac{B_n^*(t)}{(t(1-t))^{1/2}}.$$

Next we observe that (cf. Csörgő and Horváth, 1997, p. 366)

$$\sup_{1/n\le t\le 1-1/n}\frac{B_n(t)}{(t(1-t))^{1/2}} \overset{\mathcal{D}}{=} \sup_{0\le t\le 2\log(n-1)} U^*(t)$$

and

$$\sup_{1/n\le t\le 1-1/n}\frac{B_n^*(t)}{(t(1-t))^{1/2}} \overset{\mathcal{D}_{\mathbf{X}}}{=} \sup_{0\le t\le 2\log(n-1)} U^*(t),$$

where $U^*(t)$ is an Ornstein-Uhlenbeck process. We say that $U^*(t)$ is an Ornstein-Uhlenbeck process if $U^*(t)$ is Gaussian with $EU^*(t)=0$ and $EU^*(t)U^*(s)=\exp(-|t-s|/2)$. So Corollary 1.2 and Lemma 5.1 yield that

$$H_{n,N}(x) - H_n(x) = \mathrm{o}_{P_{\mathbf{X}}}((\log\log n)^{-\nu/2} + N^{-1/2}) \tag{64}$$

for almost all realizations of $\mathbf{X}$. It is known that T_n goes in distribution to a random variable with distribution function $\exp(-\exp x)$. Gombay and Horváth (2002) indicate that the rate of convergence to the double exponential distribution is essentially $\mathrm{o}((\log\log n)^{-1/2})$, so the permutation method gives a faster convergence.

5 CONTINUITY OF THE DISTRIBUTION FUNCTION OF THE MAXIMA OF GAUSSIAN PROCESSES

Let $\{X(t), 0\le t<\infty\}$ be a stationary Gaussian process with $EX(t)=0$ and let

$$r(h) = EX(t)X(t+h).$$

We assume that there are $C > 0$ and $0 < \alpha \leq 2$ such that

$$r(h) = 1 - C|h|^\alpha + \mathrm{o}(|h|^\alpha) \quad \text{as } h \longrightarrow 0 \tag{65}$$

and

$$r(t)\log t \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{66}$$

Let

$$a(T) = (2\log T)^{1/2} \tag{67}$$

and

$$b(T) = 2\log T + \frac{2-\alpha}{2\alpha}\log\log T + \log\left(\frac{C^{1/\alpha}H_\alpha 2^{(2-\alpha)/(2\alpha)}}{(2\pi)^{1/2}}\right), \tag{68}$$

where H_α is a certain strictly positive constant ($H_1 = 1$, $H_2 = \pi^{-1/2}$). Next we define

$$F_T(x) = P\left\{a(T)\sup_{0\leq t\leq T} X(t) - b(T) \leq x\right\}.$$

It is well known (cf. Leadbetter *et al.*, 1983) that

$$\lim_{T\to\infty} F_T(x) = \exp(-e^{-x}) \qquad \text{for all } x.$$

In this section we show that $F_T(x)$ is Lipschitz continuous on $[0, \infty)$ of order $\delta < \alpha/(2+\alpha)$.

LEMMA 5.1 *If Eqs.* (65) *and* (66) *hold, then there is* $L > 0$ *such that for any* $x_0, y_0 \geq 0$

$$|F_T(x_0) - F_T(y_0)| \leq L|x_0 - y_0|^\eta \quad \textit{for all } |x_0 - y_0| \leq 1 \tag{69}$$

with any $\eta < \alpha/(2+\alpha)$.

Proof We can assume without loss of generality that $T = n$ is an integer. For any $x_0 < y_0$ let

$$x_n = \frac{x_0 + b(n)}{a(n)} \quad \text{and} \quad y_n = \frac{y_0 + b(n)}{a(n)}.$$

By the stationarity of $X(t)$ we have

$$F_n(y_0) - F_n(x_0) \leq nP\left\{x_n < \sup_{0\leq t\leq 1} X(t) \leq y_n\right\}. \tag{70}$$

For any $q > 0$ to be specified below and integer j we write

$$\begin{aligned} P\left\{x_n < \sup_{0\leq t\leq 1} X(t) \leq y_n\right\} &= P\left\{x_n < \max_{0\leq jq\leq 1} X(jq) \leq y_n\right\} \\ &\quad + \left(P\left\{\sup_{0\leq t\leq 1} X(t) > x_n\right\} - P\left\{\max_{0\leq jq\leq 1} X(jq) > x_n\right\}\right) \\ &\quad - \left(P\left\{\sup_{0\leq t\leq 1} X(t) > y_n\right\} - P\left\{\max_{0\leq jq\leq 1} X(jq) > y_n\right\}\right). \end{aligned} \tag{71}$$

By the Bonferroni inequality and Feller's (1968, p. 175) estimates for the standard normal distribution function we get

$$P\left\{x_n < \max_{0\le jq\le 1} X(jq) \le y_n\right\} \le \left(\left[\frac{1}{q}\right]+1\right) P\{x_n < X(0) \le y_n\}$$
$$\le \left(\left[\frac{1}{q}\right]+1\right) \frac{\varphi(x_n)}{x_n}\left(1-\left(\frac{x_n}{y_n}-\frac{x_n}{y_n^3}\right)\frac{\varphi(y_n)}{\varphi(x_n)}\right), \quad (72)$$

where φ denotes the standard normal density function. We note that

$$\left|\frac{\varphi(y_n)}{\varphi(x_n)}-1\right| = \left|\exp\left(\frac{y_n^2-x_n^2}{2}\right)-1\right| \le \frac{1}{2}(y_n^2-x_n^2) \le y_n(y_n-x_n) = \mathrm{O}(1)x_n(y_n-x_n)$$

and

$$\left|\frac{x_n}{y_n}-\frac{x_n}{y_n^3}-1\right| \le \frac{|y_n-x_n|}{y_n}+\frac{1}{y_n^2}+\frac{|y_n-x_n|}{y_n^3} = \mathrm{O}(1)\frac{y_n-x_n}{x_n},$$

where the O(1)-term holds uniformly in x_0, $y_0 \ge 0$.

On choosing $q = Ax_n^{-2/\alpha}$ with $A = (y_0-x_0)^\gamma$, $0<\gamma<1$, we get from Eq. (72) that

$$P\{x_n < \max_{0\le jq\le 1} X(jq) \le y_n\} = \mathrm{O}\left(\frac{x_n^{2/\alpha}}{A}\frac{\varphi(x_n)}{x_n}x_n(y_n-x_n)\right)$$
$$= \mathrm{O}\left(x_n^{2/\alpha}\frac{\varphi(x_n)}{a(n)}\right)(y_0-x_0)^{1-\gamma}. \quad (73)$$

Next we note that for any $B>0$

$$0 \le P\left\{\sup_{0\le t\le 1} X(t) > x_n\right\} - P\left\{\max_{0\le jq\le 1} X(jq) > x_n\right\}$$
$$\le P\left\{x_n - \frac{B}{x_n} \le \max_{0\le jq\le 1} X(jq) \le x_n\right\}$$
$$+ P\left\{\max_{0\le jq\le 1} X(jq) \le x_n - \frac{B}{x_n}, \sup_{0\le t\le 1} X(t) > x_n\right\}. \quad (74)$$

Choosing $B = A^\beta$ with some $\beta>0$, Lemma 12.2.6 of Leadbetter *et al.* (1983) yields that

$$\lim_{n\to\infty} \frac{P\{x_n-(B/x_n) < \max_{0\le jq\le 1} X(jq) \le x_n\}}{x_n^{2/\alpha}\varphi(x_n)/x_n} = (e^\beta-1)C^{1/\alpha}H_\alpha(A). \quad (75)$$

Hence

$$P\left\{x_n - \frac{B}{x_n} < \max_{0\le jq\le 1} X(jq) \le x_n\right\} = \mathrm{O}\left(x_n^{2\alpha}\frac{\varphi(x_n)}{x_n}\right)(y_0-x_0)^{\beta\gamma}.$$

Following the proof of Lemma 12.2.5 of Leadbetter *et al*. (1983, p. 229) with $\beta < \alpha/2$ we get that

$$\begin{aligned}\frac{P\left\{\max_{0\le jq\le 1} X(jq) \le x_n - (B/x_n) \sup_{0\le t\le 1} X(t) \ge x_n\right\}}{x_n^{2/\alpha}\varphi(x_n)/x_n} &\le \frac{C_1}{qx_n^{2/\alpha}}\int_{-\infty}^{-B} \exp(-C_2A^{-\alpha}z^2)\,\mathrm{d}z \\ &= \mathrm{O}(A^{\alpha/2-1}\Phi(-C_2BA^{-\alpha})) \\ &= \mathrm{O}(A^{\alpha-1-\beta}\exp(-C_2A^{2\beta-\alpha})) \\ &= \mathrm{O}((y_0-x_0)^{\mu}) \quad \text{for any } \mu > 0,\end{aligned}$$

due to the Feller's (1968) bound for the normal distribution. Thus we have

$$\begin{aligned}0 &\le P\left\{\sup_{0\le t\le 1} X(t) > x\right\} - P\left\{\max_{0\le jq\le 1} X(jq) > x\right\} \\ &= \mathrm{O}\left(x_n^{2/\alpha}\frac{\varphi(x_n)}{x_n}\right)(y_0-x_0)^{\beta\gamma}. \end{aligned} \tag{76}$$

Correspondingly, since $x_n/y_n \to 1$, as $n \to \infty$, we get

$$\begin{aligned}0 &\le P\left\{\sup_{0\le t\le 1} X(t) > y_n\right\} - P\left\{\max_{0\le jq\le 1} X(jq) > y_n\right\} \\ &= \mathrm{O}\left(x_n^{2/\alpha}\frac{\varphi(x_n)}{x_n}\right)(y_0-x_0)^{\beta\gamma}. \end{aligned} \tag{77}$$

We choose $\gamma = 1/(1+\beta)$, so by Eqs. (71), (73), (76) and (77) we have that

$$P\left\{x_n < \sup_{0\le t\le 1} X(t) \le y_n\right\} = \left\{\mathrm{O}\left(x_n^{2/\alpha}\frac{\varphi(x_n)}{x_n}\right) + \mathrm{O}\left(x_n^{2/\alpha}\frac{\varphi(x_n)}{a(n)}\right)\right\}(y_0-x_0)^{\beta/(1+\beta)}$$

for any $\beta < \alpha/2$. Since $x^{2/\alpha}\varphi(x)$ decreases for $x^2 > 2/\alpha$ and $x_n \ge C_3b(n)/a(n) \ge C_4a(n)$ we conclude that

$$x_n^{2/\alpha}\varphi(x_n)\left(\frac{1}{a(n)} + \frac{1}{x_n}\right) = \mathrm{O}\left(a(n)^{2/\alpha-1}\exp\left\{-\frac{(b(n)/a(n))^2}{2}\right\}\right).$$

Using the definitions of $a(n)$ and $b(n)$ one can verify that

$$\begin{aligned}\frac{1}{2}\left(\frac{b(n)}{a(n)}\right)^2 &= \frac{1}{2}\left(a(n) + \frac{2-\alpha}{2\alpha}\frac{\log\log n}{a(n)} + \mathrm{O}\left(\frac{1}{a(n)}\right)\right)^2 \\ &= \log n + \log(\log n)^{(2-\alpha)/(2\alpha)} + \mathrm{O}(1)\end{aligned}$$

and therefore

$$a_n^{-1+2/\alpha}\exp\left\{-\frac{(b(n)/a(n))^2}{2}\right\} = \mathrm{O}\left(\frac{1}{n}\right),$$

completing the proof of the lemma.

6 APPROXIMATIONS OF LINEAR RANK STATISTICS

Let $u_1, u_2, \ldots, u_n$ be independent, identically distributed random variables, uniformly distributed on $[0, 1]$, and $r_1, r_2, \ldots, r_n$ be the corresponding ranks. The scores are $a_n(1), \ldots, a_n(n)$ and we assume that

$$\bar{a}_n = \frac{1}{n} \sum_{1 \le i \le n} a_n(i) = 0.$$

We obtain upper bounds for the difference between

$$V_k = \sum_{1 \le i \le k} a_n(r_i)$$

and

$$Z_k = \sum_{1 \le i \le k} a_n([nu_i] + 1) - \frac{k}{n} \sum_{1 \le i \le n} a_n([nu_i] + 1).$$

LEMMA 6.1 *For any $x > 0$*

$$P\left\{\max_{1 \le k \le n} |V_k - Z_k| \ge x\right\} \le \frac{C_1}{x^2} \max_{1 \le i \le n} |a_n(i)| \left(\sum_{1 \le i \le n} a_n^2(i)\right)^{1/2} \tag{78}$$

with some constant C_1.

Proof Let $u_{(\cdot)} = (u_{1,n}, u_{2,n}, \ldots, u_{n,n})$ be the order statistics of $u_1, u_2, \ldots, u_n$. The σ-algebra generated by $r_1, \ldots, r_k$ is denoted by $\mathcal{B}_{n,k}$. Following the arguments in Hušková (1997), we use the fact that conditionally on $u_{(\cdot)}$, $\{(V_k - Z_k)/(n - k), \mathcal{B}_{n,k}, k = 1, 2, \ldots, n - 1\}$ is a martingale. By the Hájek-Rényi inequality for martingales (cf. Chow, 1960) we have

$$\begin{aligned}
&P\left\{\max_{1 \le k \le n-1} |V_k - Z_k| \ge x | u_{(\cdot)}\right\} \\
&\quad = P\left\{\max_{1 \le k \le n-1} (n - k) \left|\frac{1}{n - k}(V_k - Z_k)\right| \ge x | u_{(\cdot)}\right\} \\
&\quad \le \frac{1}{x^2}\left\{\sum_{1 \le k \le n-1} ((n - k)^2 - (n - k - 1)^2) \frac{1}{(n - k)^2} \operatorname{var}(V_k - Z_k | u_{(\cdot)}) \right. \\
&\qquad \left. + \operatorname{var}(V_{n-1} - Z_{n-1} | u_{(\cdot)})\right\}.
\end{aligned} \tag{79}$$

Lemma 2.1 of Hájek (1961) yields

$$\begin{aligned}
E\operatorname{var}(V_k - Z_k | u^{(\cdot)}) &\le (E(\operatorname{var}(V_k - Z_k | u_{(\cdot)}))^2)^{1/2} \\
&\le C_2 \frac{k(n - k)}{n} \max_{1 \le i \le n} |a_n(i)| \left(\sum_{1 \le i \le n} a_n^2(i)\right)^{1/2}
\end{aligned} \tag{80}$$

with some constant C_2. Hence Eq. (79) yields Eq. (80).

Next we consider the standardized version of Lemma 6.1.

LEMMA 6.2 *For any $x > 0$ we have*

$$P\left\{\max_{1\le k\le n-1}\frac{n^{1/2}}{(k(n-k))^{1/2}}|V_k - Z_k| \ge x\right\} \le \frac{C_3 \log n}{nx^2}\max_{1\le i\le n}|a_n(i)|\left(\sum_{1\le i\le n}a_n^2(i)\right)^{1/2} \tag{81}$$

with some constant C_3.

Proof Following the proof of Lemma 6.1, the Hájek–Rényi–Chow (cf. Chow, 1960) inequality implies

$$\begin{aligned}
&P\left\{\max_{1\le k\le n-1}\frac{n^{1/2}}{(k(n-k))^{1/2}}|V_k - Z_k| \ge x|u_{(\cdot)}\right\}\\
&\quad = P\left\{\max_{1\le k\le n-1}\left(\frac{n(n-k)}{k}\right)^{1/2}\left|\frac{1}{n-k}(V_k - Z_k)\right| \ge x|u_{(\cdot)}\right\}\\
&\quad \le \frac{1}{x^2}\left\{\sum_{1\le k\le n-1}\left(\frac{n(n-k)}{k} - \frac{n(n-k-1)}{k+1}\right)\frac{1}{(n-k)^2}\mathrm{var}(V_k - Z_k|u_{(\cdot)})\right.\\
&\qquad \left. + \frac{n}{n-1}\mathrm{var}(V_{n-1} - Z_{n-1}|u_{(\cdot)})\right\}\\
&\quad \le \frac{C_3 \log n}{nx^2}\max_{1\le i\le n}|a_n(i)|\left(\sum_{1\le i\le n}a_n^2(i)\right)^{1/2},
\end{aligned} \tag{82}$$

where in the last step we used Eq. (80).

Acknowledgement

István Berkes is supported by the Hungarian National Foundation for Scientific Research, Grants T29621, 37866 and NSF grant INT-0223262. Lajos Horvàth is supported by NSF grant INT-0223262 and NATO grant PST. CLG. 977607. Marie Hušková is supported by NATO grant PST. CLG. 977607 and GAČR 201/03/0945 and MSM 113200008. Josef Steinebach is supported by NATO grant PST.CLG.977607.

References

Antoch, J. and Hušková, M. (1989). Some *M*-tests for detection of a change in linear models. In: *Proceedings of the Fourth Prague Symposium on Asymptotic Statistics*, Charles University, Prague, pp. 123–136.

Antoch, J. and Hušková, M. (2001). Permutation tests in change point analysis. *Stat. Prob. Letters*, **53**, 37–46.

Billingsley, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

Borovkov, A. A. (1973). On the rate of convergence for the invariance principle. *Theory Probab. Appl.*, **18**, 207–225.

Chow, Y. S. (1960). A martingale inequality and the law of large numbers. *Proc. Amer. Math. Soc.*, **11**, 107–111.

Csörgő, M. and Horváth, L. (1997). *Limit Theorems in Change-Point Analysis*. Wiley, Chichester.

Csörgő, M., and Révész, P. (1981). *Strong Approximations in Probability and Statistics*. Academic Press, New York.

Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*, Vol. I. Wiley, New York.

Fernique, X. (1975). Regularité des trajectoires des fonctions aléatoires Gaussiennes. *Lecture Notes in Math.*, Vol. 480. Springer-Verlag, Berlin, pp. 1–96.

Gombay, E. and Horváth, L. (2002). Rates of convergence for *U*-statistic processes and their bootstrapped versions. *J. Stat. Plan. Inf.*, **102**, 247–272.

Hájek, J. (1961). Some extensions of the Wald-Wolfowitz-Noether theorem. *Ann. Math. Statist.*, **32**, 506–523.
Hušková, M. (1994). Miscellaneous procedures connected with the change point problem. In: *COMPSTAT*, Physica-Verlag, Heidelberg, pp. 443–457.
Hušková, M. (1997). Limit theorems for rank statistics. *Stat. Prob. Letters*, **32**, 45–55.
Komlós, J., Major, P. and Tusnàdy, G. (1975a). An approximation of partial sums of independent R.V.'s and the sample DF.-I. *Z. Wahrschein. Verw. Gebiete*, **32**, 111–131.
Komlós, J., Major, P. and Tusnády, G. (1975b). Weak convergence and embedding. In: *Colloq. Math. Soc. János Bolyai*. Vol. 11, Amsterdam, North Holland.
Komlós, J., Major, P. and Tusnády, G. (1976). An approximation of partial sums of independent R.V.'s and the sample DF.-II. *Z. Wahrschein. Verw. Gebiete*, **34**, 33–58.
Leadbetter, M. R., Lindgren, G. and Rootzén, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer-Verlag, New York.
Petrov, V. V. (1995). *Limit Theorems of Probability Theory*. Clarendon Press, Oxford.
Sakhanenko, A. I. (1980). On unimprovable estimates of the rate of convergence in the invariance principle. In: *Colloquia Math. Soc. J. Bolyai 32 Nonparametric Statistical Inference*, Budapest, Hungary, pp. 779–783.
Sakhanenko, A. I. (1984). On estimates of the rate of convergence in the invariance principle. In: *Advances in Probability Theory: Limit Theorems and Related Problems*. Springer-Verlag, New York, pp. 124–135.
Sakhanenko, A. I. (1985). Convergence rate in the invariance principle for non-identically distributed variable with exponential moment. In: *Advances in Probability Theory: Limit Theorems for Sums of Random Variables*. Springer-Verlag, New York, pp. 2–73.
Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*. Wiley, New York.
Steinebach, J. (1994). Change-point and jump estimates in an AMOC renewal model. In: *Contributions to Statistics*. Physica-Verlag, Heidelberg, pp. 447–457.
Steinebach, J. and Eastwood, V. R. (1996). Extreme value asymptotics for increments of renewal processes. *J. Multivariate Analysis*, **56**, 284–302.