

Seasonality Tests

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This article modifies and extends the test against nonstationary stochastic seasonality proposed by Canova and Hansen. A simplified form of the test statistic in which the nonparametric correction for serial correlation is based on estimates of the spectrum at the seasonal frequencies is considered and shown to have the same asymptotic distribution as the original formulation. Under the null hypothesis, the distribution of the seasonality test statistics is not affected by the inclusion of trends, even when modified to allow for structural breaks, or by the inclusion of regressors with nonseasonal unit roots. A parametric version of the test is proposed, and its performance is compared with that of the nonparametric test using Monte Carlo experiments. A test that allows for breaks in the seasonal pattern is then derived. It is shown that its asymptotic distribution is independent of the break point, and its use is illustrated with a series on U.K. marriages. A general test against any form of permanent seasonality, deterministic or stochastic, is suggested and compared with a Wald test for the significance of fixed seasonal dummies. It is noted that tests constructed in a similar way can be used to detect trading-day effects. An appealing feature of the proposed test statistics is that under the null hypothesis, they all have asymptotic distributions belonging to the Cramér–von Mises family.

KEY WORDS: Cramér–von Mises distribution; Locally best invariant test; Seasonal breaks; Structural time series model; Trading-day effects; Unobserved components.

1. INTRODUCTION

Monthly and quarterly economic time series are often subject to seasonal movements. These seasonal patterns tend to evolve over time, and most seasonal adjustment procedures assume that this is the case. However, if the seasonal pattern does not change, it can be modeled by a set of dummy variables. Indeed, a deterministic seasonal pattern can be removed without even estimating a time series model. All that needs to be assumed is the number of times the series needs to be differenced to make it stationary (see Pierce 1978; Harvey 1989, p. 203). Adjusting series in this way may simplify the exploration of relationships between time series.

Canova and Hansen (1995), hereafter denoted by CH, proposed a test of the null hypothesis that the seasonal pattern is deterministic against the alternative that it evolves as a nonstationary stochastic process. The test includes a nonparametric correction for serial correlation and seasonal heteroscedasticity. The aim of this article is to extend the CH test in various directions. We show how to modify the test so as to allow for the effect of modeling breaks in the seasonal pattern. We then consider a different, but related testing problem—namely, testing for the presence of any kind of seasonal effects, whether deterministic, stochastic, or both. Similar techniques can be used for detecting trading-day effects. Before describing these extensions, we examine the CH test in more detail, look at some alternative formulations, and compare the performance of the nonparametric and parametric tests.

Section 2 shows how a nonparametric correction for serial correlation can be set up in terms of the spectrum at seasonal frequencies. This formulation is more restrictive than the CH test insofar as it does not allow for seasonal heteroscedasticity. Subject to this proviso, its asymptotic distribution under the null hypothesis is the same as in the original formulation, but its

interpretation is more transparent. In deriving asymptotic distributions, we relax the conditions of CH by showing that the distribution is unaffected when a deterministic trend is included in the model; regressors with unit roots can also be included provided that they do not have seasonal unit roots.

Parametric versions of the tests against nonstationary seasonality can be based on structural time series models. Such models are set up in terms of unobserved components, such as trends and cycles, which have a direct interpretation (see Harvey 1989; Kitagawa and Gersch 1996). The use of autoregressive models (as in Caner 1998) is less appealing in this context, because they may yield a poor approximation to the moving average terms typically found in the reduced forms of structural models (see also Taylor 2002a). The evidence provided by Leybourne and McCabe (1994) and Harvey and Streibel (1997) suggests that when testing against the presence of a random walk component, a parametric approach will usually give tests with a higher power and more reliable size. We investigate whether this is the case for seasonality tests through a series of Monte Carlo experiments. The results are reported in the final section. Along with casting light on the relationship between parametric and nonparametric tests, these experiments provide information on the robustness of the nonparametric test to the order of differencing.

Breaks in the trend leave the asymptotic distribution unaffected if they are correctly modeled by the inclusion of dummy variables; this is proved in Section 2. Structural breaks in the seasonal pattern are also considered. Empirical results of Canova and Ghysels (1994) suggest that seasonal mean shifts are not uncommon in U.S. quarterly series. Neglecting these

shifts will bias the nonstationarity tests at both the 0 and the seasonal frequencies. Modeling breaks in the seasonal pattern will, however, affect the distribution of the seasonality test statistics. Section 3 shows how to construct a test statistic against stochastic seasonality, the asymptotic distribution of which is independent of the breakpoint location.

The parametric and nonparametric tests, with the breakpoint modification, are illustrated by an application to a quarterly series on U.K. marriages. The point about this example is that there is an identifiable break in the seasonal pattern because of a known change in the tax laws. The modified test is trying to assess whether it is necessary to allow for stochastic seasonality once the deterministic break has been accounted for by intervention dummy variables.

Section 4 suggests a general test for seasonality. This takes the same form as the test against nonstationary seasonality, except that seasonal dummies are not fitted. The asymptotic distribution is given, and the performance of the test is compared with that of a Wald test for the significance of fixed seasonal dummies. Similar techniques are used to construct a test for the presence of trading-day effects.

A unifying feature of the test statistics is that under the null hypothesis, they all have asymptotic distributions belonging to the Cramér–von Mises family. These distributions differ according to the deterministic components fitted and a degrees of freedom parameter. The same distributions arise in tests against nonstationary trends as noted by Harvey (2001).

2. TESTING AGAINST THE PRESENCE OF A NONSTATIONARY SEASONAL COMPONENT

In this section we develop the trigonometric form of the test against nonstationary stochastic seasonality, show that it is locally best invariant, give nonparametric corrections for serial correlation, and show how to set up a parametric test. We then use Monte Carlo simulation experiments to compare the performance of the parametric and nonparametric tests in small samples.

2.1 The Testing Framework

Let y_t be a scalar time series, let s denote the number of seasons in a year and let $Z_t = (z'_{1t}, \dots, z'_{[s/2]t})'$ be an $(s - 1)$ vector of trigonometric seasonal variables, that is, $z_{jt} = (\cos 2\pi jt/s, \sin 2\pi jt/s)'$, $j = 1, \dots, s^*$, where $s^* = s/2 - 1$ if s is even or $[s/2]$ if s is odd, whereas $z_{s/2,t} = (-1)^t$ if s is even. The j th pair of trigonometric terms, z_{jt} , corresponds to the j th harmonic seasonal frequency, $\lambda_j \equiv 2\pi j/s$, $j = 1, \dots, s^*$. When s is even, the last element of Z_t , $z_{[s/2]t}$, corresponds to the Nyquist frequency, $\lambda_{[s/2]} \equiv \pi$.

The test against stochastic seasonality is developed in the context of the following unobserved components model:

$$y_t = \mu_t + s_t + \varepsilon_t, \quad t = 1, \dots, T, \quad (1)$$

$$\mu_t = X_t' \beta, \quad (2)$$

$$s_t = Z_t' \gamma_t, \quad (3)$$

and

$$A' \gamma_t = A' \gamma_{t-1} + \kappa_t, \quad (4)$$

where X_t is a $p \times 1$ vector of linearly independent deterministic regressors with associated coefficient vector β , s_t is a time-varying seasonal component with Z_t defined as earlier, A is a known $(s - 1) \times k$ selection matrix with rank $k \leq s - 1$, and ε_t and κ_t are mutually uncorrelated mean 0 disturbances with variances σ_ε^2 and $\sigma_\kappa^2 W$. The initial value γ_0 is assumed to be fixed.

The aim is to test the null hypothesis that the seasonal component is deterministic, $H_0 : \sigma_\kappa^2 = 0$, against the alternative that it has unit root behavior, $H_1 : \sigma_\kappa^2 > 0$. Following CH, the matrix A is used to formulate tests at subsets of the seasonal frequencies $\lambda_1, \dots, \lambda_{[s/2]}$. If the test is to be carried out at the single frequency λ_j , then we let A_j denote the $(2j - 1)$ th and $(2j)$ th columns of I_{s-1} for $j < s/2$ and the $(2j - 1)$ th column of I_{s-1} if $j = s/2$; I_k denotes an identity matrix of dimension k . When all of the seasonal frequencies are considered, $A = I_{s-1}$.

2.2 Locally Best Invariant Test

Under Gaussianity, the locally best invariant (LBI) test for the null hypothesis of deterministic seasonality for the model (1)–(4) can be easily obtained from the results of King and Hillier (1985) [which also show that the test is a one-sided Lagrange Multiplier (LM) test] and Taylor (2003a).

Specifically, let e_t , $t = 1, \dots, T$, be the ordinary least squares (OLS) residuals from regressing y_t on $(X_t', Z_t)'$ and let $\hat{\sigma}^2 = T^{-1} \sum_{t=1}^T e_t^2$ be their sample variance. Let $a_j = 1$ if $j = s/2$ and $a_j = 2$ otherwise.

First, consider each seasonal frequency, λ_j , $j = 1, \dots, [s/2]$ in turn, that is $A = A_j$, in the model (1)–(4). Then, under Gaussianity, the LBI test for testing $H_0 : \sigma_\kappa^2 = 0$ against $H_1 : \sigma_\kappa^2 > 0$ rejects for large values of the statistic ω_j , defined as

$$\omega_j = a_j T^{-2} \hat{\sigma}^{-2} \sum_{t=1}^T \left[\left(\sum_{i=1}^t e_i \cos \lambda_j i \right)^2 + \left(\sum_{i=1}^t e_i \sin \lambda_j i \right)^2 \right], \quad j = 1, \dots, [s/2]. \quad (5)$$

Note that when s is even, the test statistic at the Nyquist frequency, $\omega_{s/2}$, can be written without the terms $e_i \sin \lambda_{s/2} i$, because they are identically 0.

A complete test against nonstationary seasonality at all frequencies, (i.e., $A = I_{s-1}$) rejects for large values of the statistic obtained by adding up the test statistics for each individual frequency, namely

$$\omega = \sum_{j=1}^{[s/2]} \omega_j. \quad (6)$$

This test is LBI for a model where under the alternative hypothesis, the coefficients corresponding to each seasonal frequency λ_j , $j = 1, \dots, s^*$, evolve as mutually independent random walks with variances σ_κ^2 , and if s is even, then the coefficient for frequency π is a random walk with variance $\sigma_\kappa^2/2$. Thus W is an identity matrix unless s is even, in which case the last element in the main diagonal is $1/2$.

Under H_0 , $\omega_j \xrightarrow{d} CvM(a_j)$ and $\omega \xrightarrow{d} CvM(s - 1)$, where $CvM(k)$ denotes a Cramér–von Mises random variable with k degrees of freedom and $\sum_{j=1}^{[s/2]} a_j = s - 1$. Gaussianity of the ε_t 's

is not needed; all that is required is that they be martingale differences satisfying the conditions of Andrews (1991, p. 823) or Stock (1994, p. 2745). The proof is a special case of Proposition 1 in the next section.

2.3 Nonparametric Correction for Serial Correlation Based on the Spectrum at Seasonal Frequencies

Serial correlation in the stationary component of (1)–(4) can be treated nonparametrically by replacing the sample variance, $\hat{\sigma}^2$, in ω_j with an estimator of the spectrum of ε_t at frequency λ_j . We denote this *spectral nonparametric* test statistic by

$$\omega_j(m) = \frac{a_j \sum_{t=1}^T [(\sum_{i=1}^t e_i \cos \lambda_j i)^2 + (\sum_{i=1}^t e_i \sin \lambda_j i)^2]}{T^2 \hat{g}(\lambda_j; m)}, \quad j = 1, \dots, [s/2], \quad (7)$$

where

$$\hat{g}(\lambda_j; m) = \sum_{\tau=-m}^m w(\tau, m) \hat{\gamma}_e(\tau) \cos \lambda_k \tau \quad (8)$$

is the estimator of the spectral generating function, $w(\tau, m)$ is a weighting function or kernel, such as $w(\tau, m) = 1 - |\tau|/(m+1)$, and $\hat{\gamma}_e(\tau) = T^{-1} \sum_{t=\tau+1}^T e_t e_{t-\tau}$ is the sample autocovariance of the OLS residuals at lag τ . Alternative options for the kernel $w(\cdot, \cdot)$ have been examined by Andrews (1991, p. 821). For testing all of the seasonal frequencies, the spectral nonparametric statistic is

$$\omega(m) = \sum_{j=1}^{[s/2]} \omega_j(m).$$

Under the assumptions set out later, the asymptotic distributions of the foregoing test statistics under the null hypothesis are the same as given in the preceding section.

Assumption A1. X_t is a $p \times 1$ vector of deterministic regressors, and D_T is a (diagonal) scaling matrix such that

$$(a) \quad \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T D_T^{-1} X_t X_t' D_T^{-1} = Q_x$$

and

$$(b) \quad \lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T D_T^{-1} X_t Z_t' = 0,$$

where Q_x is a positive definite matrix.

Assumption A2. The ε_t 's have the structure of a linear process, $\varepsilon_t = \psi(L)\varepsilon_t^*$, where $\{\varepsilon_t^*\}$ is a martingale difference sequence satisfying the conditions of Stock (1994, p. 2745) and $\psi(L) \equiv 1 + \sum_{i=1}^{\infty} \psi_i L^i$ is a polynomial in L , the conventional lag operator, $L^k y_t \equiv y_{t-k}$, $k = 0, 1, \dots$, satisfying (a) $\psi(\exp\{\pm i2\pi\lambda_j/s\}) \neq 0$ for all $j = 1, \dots, [s/2]$ and (b) $\sum_{k=1}^{\infty} k|\psi_k| < \infty$.

Assumption A3. The lag truncation parameter, m , is such that, as $T \rightarrow \infty$, $m \rightarrow \infty$ and $m/T^{1/2} \rightarrow 0$.

Proposition 1. Let y_t be generated by the model (1)–(4) under the assumptions A1–A3. Then, under $H_0 : \sigma_\kappa^2 = 0$, when $A = A_j$, $j = 1, \dots, [s/2]$,

$$\omega_j(m) \xrightarrow{d} \int_0^1 B_{a_j}(r)' B_{a_j}(r) \equiv CvM(a_j),$$

when $A = I_{s-1}$, $\omega(m) \xrightarrow{d} CvM(s-1)$, where $B_k(r) = W_k(r) - rW_k(1)$, and $W_k(r)$, $r \in [0, 1]$, denotes a k -dimensional standard Wiener process. Under $H_A : \sigma_\kappa^2 > 0$ and when $A = A_j$, $j = 1, \dots, [s/2]$, $\omega_j(m)$ and $\omega(m)$ are $O_p(T/m)$.

Proposition 1 extends the results given by CH to allow for a general specification for the deterministic trend μ_t ; CH gave μ_t as a constant level. The limiting distribution is unchanged provided that the regressors X_t satisfy assumption A1. In particular, contrary to what CH (p. 238) stated, the model can include linear trends. Structural breaks in the trend at known points can also be included. Thus if $X_t = (1, t, d_t(\alpha))$, where $d_t(\alpha)$ is a dummy variable equal to 1 for $t > \alpha T$ with $0 < \alpha < 1$ and equal to 0 for $t \leq \alpha T$, then assumption A1 is satisfied by choosing $D_T = \text{diag}(1, T, 1)$.

Concerning the properties of the disturbances, ε_t , condition (a) of assumption A2 rules out a 0 in the spectrum at any of the seasonal frequencies λ_j , $j = 1, \dots, [s/2]$, whereas condition (b) ensures that poles do not exist in the spectrum. These conditions are satisfied by any finite-order stationary and invertible ARMA processes. Assumption A3 is required to achieve consistency of the test under the (fixed) alternative $H_A : \sigma_\kappa^2 > 0$ (see Stock 1994, pp. 2797–2799).

Model (1)–(4) can be extended by including stochastic regressors with nonseasonal unit roots, and following the line of the proof of Proposition 1, it can be shown that the asymptotic critical values for the tests in the augmented model are unchanged; the proof is omitted here but is available from the authors on request. Such regressors were ruled out by CH, who stated that “the explanatory variables may be any non-trending variables that satisfy weak dependence conditions.” The generalization is of some practical importance. Note that the presence of cross-correlation between ε_t and the disturbance vector driving the stochastic regressors is not important for our testing seasonality; unless we are interested in making inferences on the coefficient vector of the regressors, there is no need to use, say, a fully modified least squares procedure instead of OLS.

In summary, the inclusion of deterministic trends and stochastic integrated regressors does not affect the asymptotic distribution of the seasonal test statistics. However, the inclusion of seasonal slopes does affect the distribution, just as it does for tests of seasonal unit roots, as discussed by Smith and Taylor (1998) and Taylor (2003a). Given the obvious parallels with stationarity tests, it is not difficult to see that the change in distribution is the same as when a time trend is fitted before a stationarity test statistic is constructed. The critical values, once again from the Cramér–von Mises family, are as in table 2 of Nyblom and Harvey (2000). For quarterly data, the 5% critical value for an overall test, based on 3 df, is .337. (There is actually a typographical error in the published table, with the value printed as .332.)

Dummy variables introduced to capture breaks in the seasonal pattern will also affect the distribution of test statistics. This is the subject of Section 3.

2.4 The Canova–Hansen Test Statistic

The test statistic proposed by CH in the framework of (1)–(4) takes the form

$$\omega_A(m) = T^{-2} \text{trace} \left((A' \hat{\Omega}(m) A)^{-1} A' \sum_{t=1}^T S_t S_t' A \right), \quad (9)$$

where $S_t = \sum_{i=1}^t Z_i e_i$ and $\hat{\Omega}(m)$ is a nonparametric estimator of the “long run variance” of $Z_t e_t$; that is,

$$\hat{\Omega}(m) = \sum_{\tau=-m}^m w(\tau, m) \hat{\Gamma}(\tau), \quad (10)$$

where $w(\tau, m)$ is a kernel as in (8), and $\hat{\Gamma}(\tau) = T^{-1} \sum_{t=\tau+1}^T Z_t e_t e_{t-\tau}' Z_{t-\tau}'$ is the sample autocovariance matrix at lag τ formed from $Z_t e_t$. The main difference between (9) and $\omega(m)$ is that (9) allows for seasonal heteroscedasticity as in Andrews (1991, p. 839); see CH (p. 240). Under the null hypothesis $H_0 : \sigma_\kappa^2 = 0$, the asymptotic representation of (9) corresponds to that of Proposition 2, that is, $\omega_A(m) \xrightarrow{d} CvM(\text{rank}(A))$.

2.5 Parametric Tests

A structural time series model typically contains stochastic trend and seasonal components, together with an irregular. This model can be extended in various ways; for example, by including a cycle. However, for many economic time series, the flexibility of the stochastic trend is such that the model can still adequately capture seasonal movements if a cycle is excluded. We now consider how to set up a parametric test of whether the seasonal component in a structural time series model is stochastic.

If the process generating the nonseasonal part of the model is known, then the LBI test against stochastic seasonality is constructed from a set of “smoothing errors.” As shown in Appendix B, the smoothing errors are in general serially correlated, but the form of this serial correlation may be deduced from the specification of the model, thereby allowing construction of a statistic with a Cramér–von Mises distribution asymptotically, under the null hypothesis. The computation of smoothing errors has been discussed by de Jong and Penzer (1998), but if the model contains a serially uncorrelated irregular component, then it can be shown that the smoothing errors are proportional to the optimal estimates of this component.

An alternative possibility is to use the T standardized one-step-ahead prediction errors, the innovations, calculated by treating nonstationary and deterministic components as having fixed initial conditions. No correction is then needed; the statistic is of the form (5) and has the same asymptotic distribution. Calculating innovations under the assumption that the initial conditions are fixed requires that the initial conditions be estimated, but a backward smoothing recursion can be avoided simply by reversing the order of the observations and calculating a set of innovations starting from the filtered estimator of the state at the end of the sample. Actually, the forward and backward innovations are not the same, and in neither case do the sums, weighted by $\cos \lambda_j t$ and $\sin \lambda_j t$, equal 0, so statistics

formed from forward and backward sums are different. Fortunately, the asymptotic properties are unaffected. Smoothing errors do not suffer from these ambiguities.

For both the smoothing error and innovation forms of the test, nuisance parameters normally will have to be estimated. For stationarity tests, Leybourne and McCabe (1994) argued that this is best done under the alternative using maximum likelihood. Proceeding in this way has the compensating advantage that because there often will be some doubt about a suitable model specification, estimation of the unrestricted model affords the opportunity to check its suitability by the usual diagnostics and goodness-of-fit tests. Once the nuisance parameters have been estimated, the test statistic is calculated from the innovations obtained with σ_κ^2 set to 0. The asymptotic distribution under the null is unaffected.

2.6 Monte Carlo Experiments

This section compares the probability of rejecting the null hypothesis of constant seasonality for the nonparametric tests of Sections 2.3–2.4 and for the parametric test of Section 2.5 based on a correctly specified model. The results offer guidance in assessing the effectiveness of the two approaches as well as establishing the reliability of the tests in terms of actual, as opposed to nominal, size.

The data-generation process is the basic structural model (BSM), consisting of seasonal and stationary components combined with a local linear trend, μ_t . Thus

$$y_t = \mu_t + Z_t' \gamma_t + \varepsilon_t, \quad \varepsilon_t \sim NID(0, \sigma_\varepsilon^2), \quad t = 1, \dots, T, \quad (11)$$

where

$$\mu_t = \mu_{t-1} + \beta_{t-1} + \eta_t, \quad \eta_t \sim NID(0, \sigma_\eta^2) \quad (12)$$

and

$$\beta_t = \beta_{t-1} + \zeta_t, \quad \zeta_t \sim NID(0, \sigma_\zeta^2), \quad (13)$$

with γ_t as in (4) with $A = W = I$. The relationship between this seasonal model and the one of Harvey (1989, chap. 2) has been discussed by Proietti (2000).

The probability of rejection depends on the seasonal signal-to-noise ratio, $q_\kappa = \sigma_\kappa^2 / \sigma_\varepsilon^2$, although in the tables we prefer to report the square root. Empirical size and power of the tests are computed for $\sigma_\kappa / \sigma_\varepsilon$ taking the values 0, .01, .025, .05, .1, and .5. The results are for quarterly series of length $T = 200$. The empirical rejection frequencies, reported in percentages, are based on 50,000 replications and refer to tests run at the 5% significance level. Results for testing at a single frequency, $\pi/2$ and π in turn, and for the joint test at both frequencies are provided. The program was written in Ox using the SSF-pack set of subroutines of Koopman, Shephard, and Doornik (1999).

The first set of experiments, the results of which are given in Tables 1, 2, and 3, is for the data-generating process (11)–(13) with $\sigma_\zeta^2 = 0$, so the trend is a random walk with constant drift, β . The performance of the tests does not depend on the value assigned to β , and so it can be equal to 0. The critical factor in the nuisance parameters is the level signal-to-noise ratio, $q_\eta = \sigma_\eta^2 / \sigma_\varepsilon^2$, and $q_\eta^{1/2}$ is set to .1 in Table 1, .5 in Table 2, and 1.0 in Table 3.

Table 1. Rejection Frequencies for a Random Walk Drift Plus Noise Model With $\sigma_\eta/\sigma_\varepsilon = .1$

$\sigma_\kappa/\sigma_\varepsilon$		Parametric tests						Nonparametric tests							
		Model known		Model estimated		BSM estimated		Spectral: Levels		Spectral: FD		CH: Levels		CH: FD	
		Innovations	Smoothing errors	Innovations	Smoothing errors	Innovations	Smoothing errors	m = 4	m = 8	m = 4	m = 8	m = 4	m = 8	m = 4	m = 8
$\lambda = \pi/2$	0	5.10	4.82	5.63	5.85	5.56	5.42	4.29	4.30	4.75	4.47	3.84	3.24	4.16	3.49
	.010	8.52	8.10	9.36	9.97	9.23	9.90	7.31	7.23	7.77	7.31	6.32	5.49	6.74	5.61
	.025	26.78	27.50	31.52	33.47	31.23	33.19	24.75	23.93	25.33	24.36	22.52	20.24	23.29	20.42
	.050	60.37	64.35	74.13	77.96	74.07	77.78	58.80	56.68	58.78	56.56	56.36	52.43	56.44	52.01
	.100	86.27	92.61	95.54	98.31	95.40	98.15	86.92	83.86	85.95	83.27	85.49	81.42	84.54	80.45
	.500	96.75	100.00	98.70	100.00	98.70	100.00	98.70	96.71	97.72	95.89	98.41	95.50	97.21	94.37
$\lambda = \pi$	0	5.13	4.89	8.58	8.83	6.92	6.96	4.53	4.46	6.46	5.35	4.53	4.46	6.46	5.35
	.010	10.01	9.53	14.92	15.26	14.67	15.04	8.61	8.35	11.24	9.71	8.61	8.35	11.24	9.71
	.025	30.44	30.77	40.51	41.64	40.33	41.44	27.82	26.54	31.49	28.10	27.82	26.54	31.49	28.10
	.050	58.05	60.71	74.66	77.40	74.28	77.04	54.48	50.80	57.46	52.47	54.48	50.80	57.46	52.47
	.100	80.64	86.91	91.25	96.00	91.09	95.79	76.73	69.92	78.89	71.23	76.73	69.92	78.89	71.23
	.500	94.05	100.00	96.43	100.00	96.50	100.00	92.12	83.20	93.08	83.84	92.12	83.20	93.08	83.84
Joint	0	5.01	4.60	9.58	9.79	7.00	6.91	3.98	3.95	5.28	4.55	3.11	2.45	4.29	2.83
	.010	10.32	10.24	15.76	16.23	15.59	16.05	8.61	8.14	10.63	9.28	7.13	5.27	8.93	8.02
	.025	40.93	41.66	47.25	48.18	46.99	47.90	36.71	34.61	40.00	36.52	32.84	26.80	35.94	28.88
	.050	81.31	82.98	87.19	87.78	86.86	87.38	77.68	74.84	79.27	75.70	75.07	68.70	76.94	69.47
	.100	97.76	98.77	99.46	99.52	99.29	99.34	96.32	94.47	96.63	94.63	95.56	92.11	95.85	92.25
	.500	100.00	100.00	100.00	100.00	100.00	100.00	99.95	99.53	99.93	99.48	99.96	99.13	99.93	99.07

The spectral nonparametric test and the CH test are applied both to levels and to first differences with the lag truncation parameter, m , set to 4 and 8; CH used 3 and 5 for $T = 50$ and 150. The parametric test statistics are computed first assuming that q_η is known (with results in the columns headed "Model known") and then with σ_η^2 and σ_ε^2 estimated by maximum likelihood under the alternative hypothesis (in columns headed "Model estimated"). In addition, the BSM model is also estimated; that is, the constraint that the slope variance is 0 is not imposed (with the relevant columns headed "BSM estimated"). The parametric test results are shown for innovations, computed starting from a smoothed estimate of the initial conditions, and for smoothing errors; compare Section 2.5.

The main findings of Tables 1, 2, and 3 are as follows:

- a. Although the empirical sizes (rows with $\sigma_\kappa/\sigma_\varepsilon$ equal to 0) of the parametric tests with q_η known are very close to the nominal 5%, the tests with q_η estimated are somewhat oversized at frequency π . The parametric joint tests for overall seasonality are similarly oversized, with the actual sizes being around .09 in most cases. It is interesting to note that the autoregression-based tests reported by Caner (1998, table 1) display even more oversizing.
- b. There appears a slight power advantage for a parametric test constructed from smoothing errors, rather than from innovations, when testing at a single frequency (particularly when the smaller empirical size of the parametric test is taken into account); for example, for $\lambda = \pi/2$ in Table 1, the estimated

Table 2. Rejection Frequencies for a Random Walk Drift Plus Noise Model With $\sigma_\eta/\sigma_\varepsilon = .5$

$\sigma_\kappa/\sigma_\varepsilon$		Parametric tests						Nonparametric tests							
		Model known		Model estimated		BSM estimated		Spectral: Levels		Spectral: FD		CH: Levels		CH: FD	
		Innovations	Smoothing errors	Innovations	Smoothing errors	Innovations	Smoothing errors	m = 4	m = 8	m = 4	m = 8	m = 4	m = 8	m = 4	m = 8
$\lambda = \pi/2$	0	5.34	4.90	5.54	5.98	5.42	5.71	1.06	1.92	4.77	4.63	.86	1.24	4.23	3.42
	.010	7.73	7.35	8.74	9.28	8.58	9.13	2.01	3.21	7.16	6.86	1.63	2.29	6.35	5.28
	.025	24.57	24.96	28.68	30.49	28.50	30.20	10.41	13.56	23.13	22.03	8.98	10.37	21.35	18.63
	.050	57.16	60.63	70.62	74.25	70.46	73.99	39.18	43.43	55.88	54.00	36.51	38.30	53.47	49.22
	.100	84.46	90.98	94.42	97.66	94.37	97.64	76.24	76.88	84.41	81.60	74.37	73.24	82.81	78.89
	.500	96.58	100.00	98.65	100.00	98.63	100.00	98.30	96.36	97.67	95.77	97.97	95.06	97.20	94.31
$\lambda = \pi$	0	5.18	4.93	9.26	9.42	8.28	8.47	1.47	2.40	6.13	5.18	1.47	2.40	6.13	5.18
	.010	9.67	9.39	14.91	15.18	14.65	15.01	3.39	4.60	10.80	9.30	3.39	4.60	10.80	9.30
	.025	29.39	29.62	38.97	40.02	38.75	39.76	16.26	19.20	30.42	27.26	16.26	19.20	30.42	27.26
	.050	56.78	59.37	72.99	75.27	72.28	74.61	42.75	44.25	56.52	51.52	42.75	44.25	56.52	51.52
	.100	80.15	85.93	91.14	95.19	90.91	95.08	68.89	65.94	77.91	70.39	68.89	65.94	77.91	70.39
	.500	94.41	99.99	96.83	100.00	96.72	100.00	91.44	82.98	93.01	83.77	91.44	82.98	93.01	83.77
Joint	0	5.19	4.60	9.27	9.51	8.57	8.73	.80	1.42	5.31	4.59	.55	.68	4.29	2.79
	.010	9.71	9.49	15.49	15.88	15.22	15.66	1.98	3.44	10.03	8.73	1.40	1.77	8.26	5.85
	.025	38.25	38.58	44.78	45.73	44.38	45.31	15.72	20.19	37.40	33.73	12.50	13.57	33.64	26.27
	.050	78.81	80.40	84.65	85.30	84.18	84.81	57.70	61.84	76.92	73.18	53.30	53.24	74.14	66.60
	.100	97.29	98.27	99.26	99.29	99.20	99.22	91.16	90.61	95.99	93.90	89.07	87.13	95.01	91.56
	.500	100.00	100.00	100.00	100.00	100.00	100.00	99.93	99.47	99.92	99.49	99.87	98.94	99.93	99.00

Table 3. Rejection Frequencies for a Random Walk Drift Plus Noise Model With $\sigma_\eta/\sigma_\varepsilon = 1$

		Parametric tests						Nonparametric tests															
		Model known		Model estimated		BSM estimated		Spectral: Levels				Spectral: FD				CH: Levels				CH: FD			
		Smoothing errors		Smoothing errors		Smoothing errors		m = 4		m = 8		m = 4		m = 8		m = 4		m = 8		m = 4		m = 8	
λ	$\sigma_\kappa/\sigma_\varepsilon$	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	Innovations	
$\lambda = \pi/2$	0	5.33	4.83	5.33	5.69	4.99	5.35	.16	.54	4.90	4.56	.10	.27	4.17	3.44								
	.010	7.08	6.60	7.46	7.94	7.07	7.61	.30	1.02	6.61	6.50	.24	.61	5.64	4.64								
	.025	19.11	19.46	22.73	24.38	22.32	24.22	1.98	4.35	18.50	17.66	1.56	2.76	16.80	14.65								
	.050	49.25	52.78	61.01	65.18	61.67	65.66	15.46	22.57	48.37	46.47	13.51	18.16	46.35	41.83								
	.100	79.83	86.35	91.44	95.48	91.64	95.43	52.77	60.19	79.78	77.15	50.13	54.99	78.28	73.97								
	.500	95.77	100.00	98.26	100.00	98.24	100.00	96.92	95.14	97.52	95.48	96.32	93.81	96.94	94.14								
$\lambda = \pi$	0	5.04	4.82	9.79	9.98	9.45	9.69	.35	.87	5.83	4.91	.35	.87	5.83	4.91								
	.010	9.20	8.90	14.65	14.97	14.32	14.55	.91	1.77	9.66	8.60	.91	1.77	9.66	8.60								
	.025	26.84	26.54	35.90	36.82	35.48	36.23	5.92	9.60	27.23	24.57	5.92	9.60	27.23	24.57								
	.050	53.92	55.70	68.22	70.61	68.29	70.52	24.40	30.80	53.04	48.54	24.40	30.80	53.04	48.54								
	.100	78.30	83.23	89.99	93.54	90.54	93.51	53.58	56.48	75.69	68.50	53.58	56.48	75.69	68.50								
	.500	95.01	99.98	96.95	100.00	97.04	99.99	89.56	81.93	92.89	83.65	89.56	81.93	92.89	83.65								
Joint	0	4.93	4.49	9.68	9.95	9.27	9.48	.10	.36	4.99	4.54	.03	.12	4.08	2.76								
	.010	8.61	8.22	14.42	14.91	14.27	14.61	.19	.69	8.96	7.72	.09	.23	7.13	4.76								
	.025	32.04	31.82	38.92	39.91	38.92	39.65	2.93	6.41	31.43	28.24	2.00	3.45	27.84	21.55								
	.050	72.27	73.83	78.72	79.39	78.93	79.71	25.60	36.60	70.26	66.48	21.36	27.01	67.22	59.91								
	.100	95.79	97.10	98.48	98.65	98.16	98.33	72.44	78.29	94.22	91.67	68.82	71.67	92.94	88.96								
	.500	100.00	100.00	100.00	100.00	100.00	100.00	99.74	99.25	99.92	99.47	99.60	98.58	99.89	98.95								

rejection probability is .64 versus .60 when the model is known and .78 versus .74 when the model is estimated. This advantage seems to vanish for the joint tests.

c. Estimating the extra parameter, $q_\zeta = \sigma_\zeta^2/\sigma_\varepsilon^2$ in the more general BSM model has no adverse effect on the performance of the parametric test. On the contrary, there is a slight improvement in size in that it is closer to the nominal; for example, in Table 1 the empirical size of the joint test is .07 when a BSM is estimated, compared with .10 when the trend is a random walk with drift.

d. Regarding the nonparametric tests in first differences, only a small drop in the rejection probabilities occurs when moving from $m = 4$ to $m = 8$. The seasonality test seems less sensitive to lag length than the test of Kwiatkowski, Phillips, Schmidt and Shin (1992); compare the comments of CH (p. 246). However, m should not be set too small. Our simulations for the joint test with $m = 0$ (unreported in the tables) showed rejection frequencies of 14.5, 13.4, and 11.0 for the models in Tables 1, 2, and 3.

e. The nonparametric test in levels has a much lower probability of rejection than the corresponding test in first differences when $q_\eta^{1/2}$ is .5 or 1 (see Tables 2 and 3). This is due to the presence of a so-called *unattended unit root* at frequency 0 when the tests are run in levels. In fact, Busetti and Taylor (2003) and Taylor (2003b) have demonstrated that, although they are consistent, the nonparametric tests would suffer from a great loss of power if that unit root were not removed by taking first differences of the data. Indeed, these authors showed that under the null hypothesis of deterministic seasonality, the test statistics (5), (7), and (9) all converge in probability to 0. This is also confirmed in Tables 1, 2, and 3; in Table 3, where $\sigma_\eta/\sigma_\varepsilon$ is equal to 1, the joint test in levels has a size of nearly 0.

f. The parametric tests show higher rejection frequencies than the nonparametric tests, but any assessment of power must be considered in terms of the larger size. Overall, the loss in power resulting from using nonparametric tests is not great. For example, in Table 3, $\lambda = \pi/2$, where the empirical size of

the innovation-based parametric test is broadly comparable to that of the spectral nonparametric test in first differences with $m = 4$, the powers for a seasonal signal-to-noise ratio of .05² are around .60 and .50.

g. The nonparametric tests perform relatively better than the results for stationarity tests (at the 0 frequency) would suggest. This is because in the experiments reported in the literature (e.g., Leybourne and McCabe 1994), the process generating the stationary part of the model—typically a first-order autoregression—interferes with the unit root process. This problem does not arise with the data-generating processes considered here and it would be unlikely to arise even if a first-order autoregressive process were to replace the white noise irregular.

The second group of experiments, reported in Tables 4 and 5, is for the so-called *smooth trend* model; the data-generating process is given by (11)–(13) with $\sigma_\eta^2 = 0$. The relevant signal-to-noise ratio is now q_ζ . The parametric model is estimated both with and without σ_η^2 set to 0; the results are given in the columns labeled “Model estimated” and “BSM estimated.” The nonparametric tests are run after taking first differences and after taking differences a second time. Although in theory, the first difference operator should be applied twice to avoid the power loss induced by the presence of unattended unit roots, Table 4 indicates that for $q_\zeta^{1/2} = .1$, no significant loss occurs if only first differences are taken. In fact, in this case first differences may be preferable, and because $q_\zeta^{1/2}$ is often smaller than .1 for economic time series, basing tests on first differences may be a good strategy.

The conclusions with respect to size and power that emerge from Tables 4 and 5 are similar to those reached for the random walk plus drift model of Tables 1, 2, and 3. The parametric tests are somewhat oversized but have a higher probability of rejection under the alternative. Again, there is no disadvantage to fitting a more general local linear trend model when carrying out the parametric tests.

Table 4. Rejection Frequencies for a Smooth Trend Plus Noise Model With $\sigma_{\xi}/\sigma_{\varepsilon} = .1$

		Parametric tests						Nonparametric tests							
		Model known		Model estimated		BSM estimated		Spectral: FD		Spectral: Double FD		CH: FD		CH: Double FD	
$\sigma_{\kappa}/\sigma_{\varepsilon}$		Innovations	Smoothing errors	Innovations	Smoothing errors	Innovations	Smoothing errors	m = 4	m = 8	m = 4	m = 8	m = 4	m = 8	m = 4	m = 8
$\lambda = \pi/2$	0	5.36	4.77	5.58	5.86	6.19	6.43	4.22	4.22	2.35	2.96	3.65	3.29	1.92	2.22
	.010	8.76	8.11	9.32	9.83	10.06	10.57	7.15	7.05	4.20	5.13	6.17	5.18	3.39	3.71
	.025	27.18	27.64	31.82	33.62	33.02	34.94	24.11	23.86	18.19	19.98	22.35	19.59	15.79	15.72
	.050	60.56	64.32	74.27	77.68	75.18	78.57	57.76	55.99	50.73	51.80	55.52	51.45	47.95	46.82
	.100	86.37	92.50	95.54	98.30	95.59	98.36	85.53	83.07	80.76	79.93	84.15	80.35	79.06	76.93
$\lambda = \pi$	0	5.13	4.82	8.66	8.81	9.28	9.44	6.16	5.27	7.41	5.87	6.16	5.27	7.41	5.87
	.010	9.99	9.43	14.83	15.12	15.85	16.16	10.85	9.60	12.53	10.52	10.85	9.60	12.53	10.52
	.025	30.59	30.82	40.69	41.68	41.78	42.78	30.97	27.80	33.17	29.07	30.97	27.80	33.17	29.07
	.050	58.03	60.74	74.38	77.03	74.98	77.95	57.15	52.30	58.90	53.12	57.15	52.30	58.90	53.12
	.100	80.72	86.97	91.31	95.73	91.52	95.91	78.63	71.02	79.90	71.60	78.63	71.02	79.90	71.60
Joint	0	5.21	4.56	9.49	9.69	10.23	10.53	4.68	4.19	3.67	3.51	3.82	2.64	2.89	2.14
	.010	10.65	10.06	15.66	16.02	16.79	17.22	9.82	8.85	8.26	7.81	8.18	5.78	6.91	4.85
	.025	41.15	41.44	47.45	48.25	48.95	49.87	38.70	35.77	35.10	32.90	34.31	27.98	31.12	25.48
	.050	81.04	82.75	87.35	87.88	87.87	88.42	78.58	75.29	75.91	73.16	75.93	68.87	72.82	66.50
	.100	97.82	98.77	99.46	99.53	99.46	99.54	96.48	94.47	95.72	93.86	95.62	92.17	94.70	91.15
	.500	100.00	100.00	100.00	100.00	100.00	100.00	99.93	99.48	99.91	99.34	99.92	99.06	99.81	98.73

Because the models used in the foregoing simulations do not exhibit seasonal heteroscedasticity, it is not surprising that the spectral nonparametric test performs slightly better than the CH test. However, once seasonal heteroscedasticity is present, the situation changes. For example, with a model consisting of a seasonal plus white noise with variance in the four quarters of 1, 3, 5, and 7, the size of a $\omega(4)$ test for $T = 1,000$ was estimated to be .080, and that of a $\omega_A(4)$ test was .038. Thus the CH size is closer to the nominal .05. However, with $q = .025$, the estimated probability of rejection was .458 for $\omega(4)$ and .662 for $\omega_A(4)$. Although this is a rather extreme case, it does illustrate the point that when seasonal heteroscedasticity is present, the CH test not only is more robust with respect to size, but also may show a higher probability of rejection away from the null.

3. DETERMINISTIC BREAKS IN THE SEASONAL PATTERN

In this section we consider testing against nonstationary stochastic seasonality when there is a break in the seasonal pattern at time $[\alpha T]$, $\alpha \in [0, 1]$; that is, we replace (3) with

$$s_t = Z_t' \gamma_t + d_t(\alpha) Z_t' \theta, \tag{14}$$

where $d_t(\alpha) = 1$ ($t > \alpha T$) is a break dummy variable. The model now implies that the coefficients of the seasonal terms have changed from γ_t when $t \leq [\alpha T]$ to $\gamma_t + \theta$ when $t > [\alpha T]$. We focus on the nonparametric tests, although of course the same issues arise with the parametric versions.

Table 5. Rejection Frequencies for a Smooth Trend Plus Noise Model With $\sigma_{\xi}/\sigma_{\varepsilon} = .5$

		Parametric tests						Nonparametric tests							
		Model known		Model estimated		BSM estimated		Spectral: FD		Spectral: Double FD		CH: FD		CH: Double FD	
$\sigma_{\kappa}/\sigma_{\varepsilon}$		Innovations	Smoothing errors	Innovations	Smoothing errors	Innovations	Smoothing errors	m = 4	m = 8	m = 4	m = 8	m = 4	m = 8	m = 4	m = 8
$\lambda = \pi/2$	0	5.33	4.69	5.53	5.84	5.59	5.90	.86	1.48	2.58	3.15	.72	.96	2.18	2.19
	.010	8.47	7.63	8.95	9.38	9.05	9.57	1.67	2.65	4.16	5.21	1.40	1.98	3.53	3.79
	.025	25.64	25.78	30.50	32.16	30.74	32.52	9.62	12.99	17.74	19.08	8.24	9.65	15.46	15.15
	.050	58.43	61.89	71.84	75.58	71.94	75.88	37.57	42.20	49.42	50.46	34.94	37.30	46.87	45.59
	.100	85.49	91.48	94.79	97.81	94.70	97.84	73.46	75.22	80.14	79.29	71.77	71.76	78.33	76.27
$\lambda = \pi$	0	5.16	4.85	8.78	8.96	9.63	9.87	2.77	3.26	7.29	5.77	2.77	3.26	7.29	5.77
	.010	10.06	9.51	15.07	15.42	16.24	16.51	5.98	6.58	12.36	10.29	5.98	6.58	12.36	10.29
	.025	30.02	30.11	40.20	41.30	41.63	42.65	22.20	22.92	32.78	28.82	22.20	22.92	32.78	28.82
	.050	57.88	60.12	74.13	76.49	74.78	77.29	48.58	47.34	58.51	52.94	48.58	47.34	58.51	52.94
	.100	80.89	86.57	91.24	95.43	91.61	95.66	72.90	68.13	79.73	71.47	72.90	68.13	79.73	71.47
Joint	0	5.20	4.22	9.47	9.80	10.10	10.37	1.07	1.42	3.67	3.50	.69	.73	3.04	2.09
	.010	10.32	9.40	15.88	16.28	16.67	17.03	2.54	3.63	8.24	7.74	2.03	2.09	6.73	4.78
	.025	39.61	39.80	46.24	47.09	47.20	48.03	18.81	21.88	34.10	32.14	15.90	15.37	30.36	24.80
	.050	79.96	81.51	86.02	86.59	86.47	87.06	59.98	62.82	74.86	72.25	55.93	54.82	72.02	65.50
	.100	97.52	98.50	99.36	99.38	99.37	99.40	91.15	90.81	95.53	93.34	89.11	87.18	94.44	90.94
	.500	100.00	100.00	100.00	100.00	100.00	100.00	99.88	99.43	99.91	99.34	99.82	98.92	99.82	98.71

It is initially assumed that the breakpoint parameter, α , is known. Extensions to situations in which the breakpoint is unknown are discussed in Section 3.3.

When there is a break in the seasonal pattern, the nonparametric test statistics $\omega_j(m)$, $\omega(m)$, and $\omega_A(m)$ of the preceding section must be constructed using the OLS residuals from regressing y_t on $(X'_t, Z'_t, d_t(\alpha)Z'_t)'$. Their asymptotic representations under the null hypothesis are no longer Cramér–von Mises with a_j , $s - 1$, $rank(A)$ df, but rather they depend in a rather complicated way on the breakpoint parameter α . However, a simple modification yields test statistics, the null limiting distributions of which are still Cramér–von Mises but with degrees of freedom equal to $2a_j$, $2(s - 1)$ and $2 rank(A)$. This extends to the seasonal case the modification to the LBI test at frequency 0 suggested by Busetti and Harvey (2001). The parametric test can be modified along the same lines.

3.1 Modified Test With Seasonal Break

The modified seasonal break spectral nonparametric statistic for testing against nonstationary seasonality at frequency λ_j is defined as

$$\omega_j^*(\alpha; m) = \frac{a_j \sum_{t=1}^T c_{jt} k_t}{\hat{g}(\lambda_j; m)}, \quad j = 1, \dots, [s/2], \quad (15)$$

where

$$c_{jt} = \left(\sum_{i=1}^t e_i \cos \lambda_j i \right)^2 + \left(\sum_{i=1}^t e_i \sin \lambda_j i \right)^2,$$

$$k_t = [\alpha T]^{-2} (1 - d_t(\alpha)) + [(1 - \alpha)T]^{-2} d_t(\alpha),$$

a_j and $\hat{g}(\lambda_j; m)$ are defined as in the preceding section, and the e_t 's are the OLS residuals from regressing y_t on $(X'_t, Z'_t, d_t(\alpha)Z'_t)'$. The corresponding statistic for the test at all frequencies is then $\omega^*(\alpha; m) = \sum_{j=1}^{[s/2]} \omega_j^*(\alpha; m)$. The modifications to

the CH statistic, (9), and the parametric statistic are carried out in a similar way.

Proposition 2. Let y_t be generated by the model (1), (2), (14), (4) under the assumptions A1–A3. Then, under $H_0 : \sigma_\kappa^2 = 0$, when $A = A_j, j = 1, \dots, [s/2]$, $\omega_j^*(\alpha; m) \xrightarrow{d} CvM(2a_j)$, when $A = I_{s-1}$, $\omega^*(\alpha; m) \xrightarrow{d} CvM(2s - 2)$. Under $H_A : \sigma_\kappa^2 > 0$ and when $A = A_j, j = 1, \dots, [s/2]$, $\omega_j^*(\alpha; m)$ and $\omega^*(\alpha; m)$ are $O_p(T/m)$.

The idea behind the construction of (15) is to combine the evidence in the two subsamples, $\{1, \dots, [\alpha T]\}$ and $\{[\alpha T] + 1, \dots, T\}$. Note that $\omega_j^*(.5; m) = .25\omega_j(m)$; thus when the breakpoint is in the middle of the sample, the tests defined by the two statistics are the same. This is important, because the latter has properties of optimality obtained by extending the LBI/LM test to deal with serial correlation in the stationary component. Furthermore, for the case of testing at frequency 0, Busetti and Harvey (2001) have shown via simulation experiments that for $\alpha \neq .5$, the loss of power of the modified test with respect to the LBI test is not great.

3.2 U.K. Marriages

The quarterly series of marriages registered in the United Kingdom from 1958Q1 to 1982Q4 was extracted from various issues of the *U.K. Monthly Digest of Statistics*. It is shown in Figure 1(a). The spectral nonparametric test statistic, $\omega(m)$, calculated from first differences, is 4.18, 2.74, and 2.11 for lags of 4, 8, and 12. This leads to a rejection of the null hypothesis as the 5% critical value for the $CvM(3)$ distribution is 1.00. The original CH statistic, $\omega_A(m)$, gave smaller values: 1.78, 1.20, and .96.

Estimating (11) with a random walk trend using the STAMP 6 program of Koopman, Harvey, Doornik, and Shephard (2000)

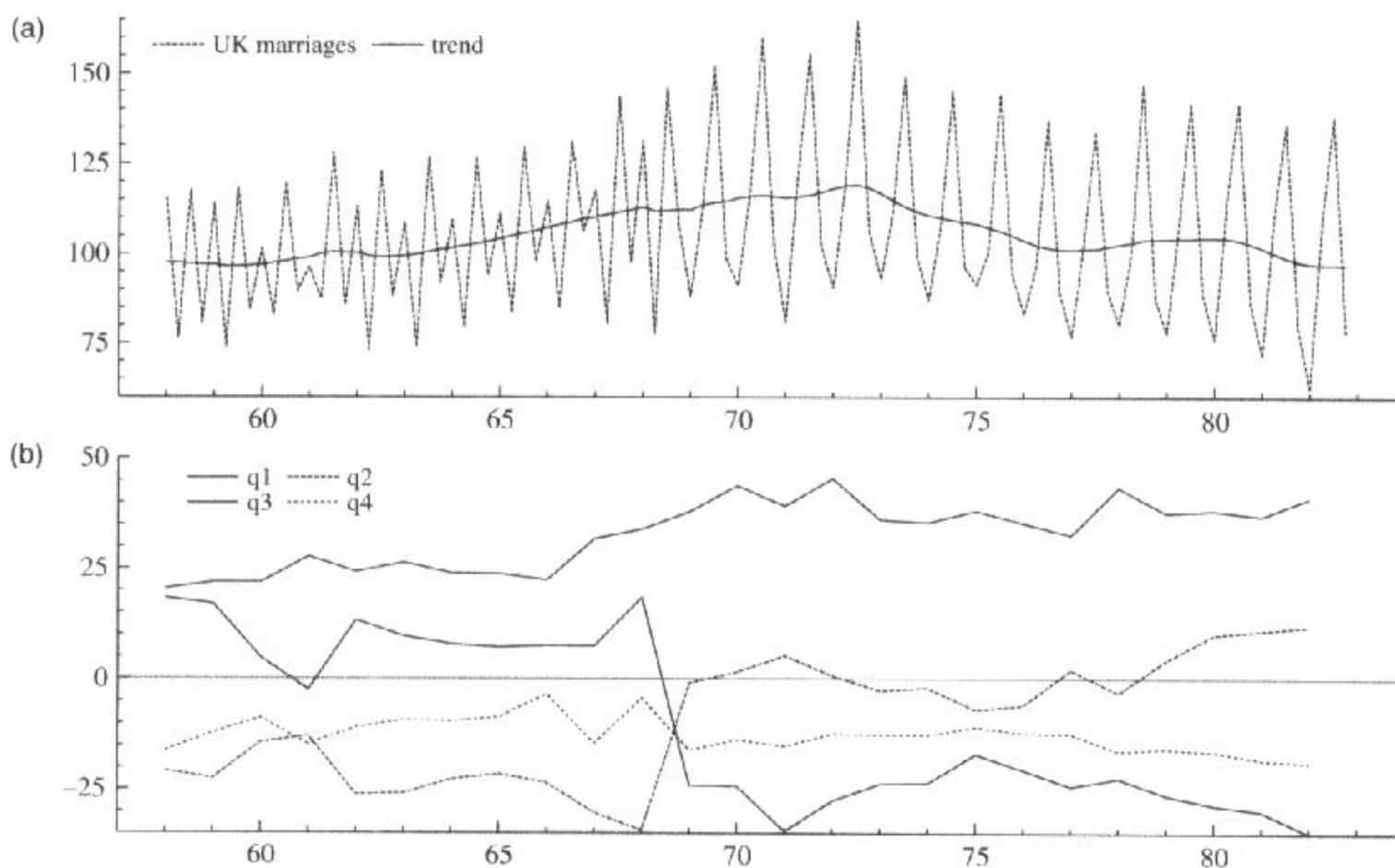


Figure 1. (a) Number of Marriages (in thousands) in the United Kingdom, 1958Q1–1982Q4 and (b) Estimates of the Individual Seasonals.

gives

$$\tilde{\sigma}_\varepsilon = 0, \quad \tilde{\sigma}_\eta = 1.61, \quad \text{and} \quad \tilde{\sigma}_\kappa = 2.69,$$

with an equation standard error (the standard deviation of the innovations), $\tilde{\sigma}$, of 7.91. The parametric test statistic, constructed from the Kalman filter innovations, is 6.96 which is a much firmer rejection of the null hypothesis than was given by the nonparametric test. The reason for the rejection can be seen in Figure 1(a): there appears to be a break in the seasonal pattern at the beginning of 1969. The plot of the individual seasons in Figure 1(b) reveals a switch from winter marriages to marriages in the spring quarter. This happened because of a change in the tax law. Up to the end of 1968, couples were allowed to claim the married persons tax allowance retrospectively for the entire year in which they married. Because the tax year begins in April, this arrangement provided an incentive to marry in the first quarter of the calendar year rather than in the spring.

Adding a set of three seasonal break dummy variables, starting in the first quarter of 1969, to take into account a complete change in the seasonal pattern leads to the following estimates of the parameters:

$$\tilde{\sigma}_\varepsilon = 2.42, \quad \tilde{\sigma}_\eta = 1.59, \quad \text{and} \quad \tilde{\sigma}_\kappa = 1.36,$$

with

$$Q(9, 7) = 12.54 \quad \text{and} \quad \tilde{\sigma} = 5.66,$$

where $Q(P, f)$ is the Box–Ljung statistic based on P residual autocorrelations but with f degrees of freedom (see Koopman et al. 2000). The t statistics for the seasonal break dummies are -8.33 , 7.58 , and 2.09 . There is a large reduction in the estimate of the seasonal parameter, σ_κ , which no longer needs to allow the stochastic seasonal model to accommodate the change; the equation standard error, $\tilde{\sigma}$, also is considerably reduced.

The modified seasonal break nonparametric test statistics carried out on the residuals obtained from regressing first differences on seasonal means and the seasonal break dummies are 2.06, 1.69, and 1.58 for $m = 4, 8$, and 12 for ω^* and 1.80, 1.57, and 1.50 for the CH form, ω_A^* . Thus for $m = 4$ and 8, the null of a constant seasonal pattern is rejected by the ω^* test at the 5% level of significance because the critical value for $CvM(6)$ is 1.69. However, the smaller values for ω_A^* lead only to a rejection for $m = 4$. The corresponding parametric test statistic, calculated from the Kalman filter innovations, is 2.42, giving a stronger indication that there is still stochastic seasonality present. This is supported by the fact that estimating the model with a fixed seasonal gives a significant Box–Ljung statistic of $Q(9, 8) = 22.38$, whereas the fourth-order residual autocorrelation, $r(4)$, is .33.

3.3 Unknown Breakpoint

If the breakpoint parameter α is unknown, then the two-step strategy of Buseti and Harvey (2003) for testing stationarity in the presence of a structural break can be adapted in a straightforward manner. The idea is to estimate the breakpoint under the null by minimizing, over α , the error variance of an OLS regression of y_t on $(X_t', Z_t', d_t(\alpha)Z_t)'$, that is, $\hat{\alpha} = \arg \min_\alpha T^{-1} \sum_{t=1}^T e_t^2$, where e_t are the OLS residuals. Bai (1997) showed that under the null hypothesis of deterministic

seasonality this estimator is superconsistent in the sense that it converges to the true value at rate T instead of the usual rate $T^{1/2}$. Therefore, the null asymptotic distributions of $\omega_j^*(\hat{\alpha}; m)$ and $\omega^*(\hat{\alpha}; m)$ are the same as if the true value α were used, whereas under the alternative hypothesis, the statistics diverge regardless of the break date used due to the presence of the seasonal unit roots. Hence running the seasonal break tests with an estimated breakpoint leads to an asymptotically valid procedure. Clearly, some loss of power with respect to a test based on a known α is to be expected. Buseti and Harvey (2003) used Monte Carlo simulation experiments to evaluate this power loss for the zero frequency stationarity tests.

4. TESTING AGAINST A PERMANENT SEASONAL COMPONENT

The test against nonstationary seasonal components takes the null hypothesis to be a model in which seasonality is deterministic. Sometimes we may wish to test whether there is any seasonality at all, irrespective of whether it is deterministic or stochastic. One strategy, implemented in the STAMP package, is to fit a structural time series model and then perform a test of significance on the seasonal coefficients as estimated at the end of the period. However, this has the disadvantage of not being able to indicate seasonal effects in a situation where seasonality has become less pronounced over time. This is precisely the kind of behavior noted by CH (pp. 24–50) in their analysis of U.S. macroeconomic series.

4.1 Tests and Their Power

Our aim is to develop tests that are powerful against the presence of deterministic seasonality and/or stochastic seasonality. The stochastic seasonality is taken to be nonstationary, so that the effects are *permanent* in that forecasts of the seasonal pattern remain constant rather than dying away. We thus consider the data-generating process (1)–(4) with $A = I_{s-1}$ and split the seasonal component, s_t , into deterministic and stochastic parts,

$$s_t = s_t^D + s_t^S, \quad (16)$$

$$s_t^D = Z_t' \gamma_0, \quad (17)$$

and

$$s_t^S = Z_t' \sum_{i=1}^l \kappa_i, \quad (18)$$

where γ_0 is a $(s-1) \times 1$ vector of fixed coefficients and κ_i is a vector of mean 0, serially independent disturbances with covariance matrix $\sigma_\kappa^2 W$, independent of ε_t . The null hypothesis of no seasonality is

$$H_0 : \gamma_0 = 0, \quad \sigma_\kappa^2 = 0,$$

whereas the alternative hypotheses are of deterministic seasonality,

$$H_1^D : \gamma_0 \neq 0, \quad \sigma_\kappa^2 = 0,$$

and stochastic seasonality,

$$H_1^S : \gamma_0 = 0, \quad \sigma_\kappa^2 > 0.$$

We show in Proposition 3 that the standard Wald test on fixed seasonal coefficients is consistent against both alternative hypotheses. We also show that a test constructed in a similar way to the tests of Section 2, but without fitting seasonal regressors, is consistent against both hypotheses. More specifically let e_t be the OLS residuals from regressing y_t on $(X_t', Z_t)'$ and let \underline{e}_t be the residual from regressing y_t on X_t only. Because assumption A1 requires that the two sets of regressors X_t and Z_t be orthogonal in large samples, the X_t 's can be ignored in the analysis of the Wald statistic, which may be written as

$$F = \hat{\gamma}'_0 \left(\hat{\sigma}^{-2} \sum_{t=1}^T Z_t Z_t' \right) \hat{\gamma}_0, \tag{19}$$

where $\hat{\gamma}_0$ is the OLS estimator of γ_0 and $\hat{\sigma}^2 = T^{-1} \sum_{t=1}^T e_t^2$ is the estimator of σ_ε^2 . Note that the usual form of the F statistic is our F multiplied by $(T - s + 1)/2T$.

The new statistic is

$$\underline{\omega} = \sum_{j=1}^{\lfloor s/2 \rfloor} \underline{\omega}_j, \tag{20}$$

where each $\underline{\omega}_j, j = 1, \dots, \lfloor s/2 \rfloor$, is defined as in Proposition 1 but using the residuals \underline{e}_t and with the summation running in the reverse order, that is,

$$\underline{\omega}_j = a_j T^{-2} \underline{\sigma}^{-2} \sum_{t=1}^T \left[\left(\sum_{i=t}^T e_i \cos \lambda_j i \right)^2 + \left(\sum_{i=t}^T e_i \sin \lambda_j i \right)^2 \right], \tag{21}$$

$j = 1, \dots, \lfloor s/2 \rfloor,$

where $\underline{\sigma}^2 = T^{-1} \sum_{t=1}^T \underline{e}_t^2$. Using the arguments of King and Hillier (1985) and Taylor (2003a), it is easy to show that when W is specified as after formula (6), $\underline{\omega}$ is the LBI test statistic against H_1^S for the model in (1)–(4) with $\gamma_0 = 0$. Note that in the LBI statistics the summations run in reverse order, from t to T as opposed from 1 to t ; in the tests of Section 1 the reverse summations can be replaced by the more usual direct summations, because fitting the seasonal regressors implies that $\sum_{i=1}^T Z_i e_i = 0$.

The following proposition provides the asymptotic distribution of F and $\underline{\omega}$ under the local alternative hypotheses $H_{1,T}^D$ and $H_{1,T}^S$, where $H_{1,T}^D: \gamma_0 = c_D \iota / \sqrt{T}, \sigma_\varepsilon^2 = 0$ and $H_{1,T}^S: \gamma_0 = 0, \sigma_\varepsilon^2 = c_S^2 / T^2$, where ι is an $s - 1$ vector of ones and c_D and c_S are fixed constants. This provides the basis of a power comparison between the two tests.

Proposition 3. Let y_t be generated by the model (1)–(2), (16)–(18) with $W = I_{s-1}$ and with the nonseasonal regressors X_t satisfying assumption A1 of Section 1. Let $W_{0,s-1}(r), W_{1,s-1}(r)$ be independent standard Wiener processes of dimension $s - 1$, and let $\Lambda = \text{diag}(1/2, \dots, 1/2, 1)$ when s is even and $\Lambda = 1/2 I_{s-1}$ when s is odd. Then the following results hold:

a. Under $H_{1,T}^D$,

$$F \xrightarrow{d} V_{D,s-1}(0; c_D)' V_{D,s-1}(0; c_D),$$

$$\underline{\omega} \xrightarrow{d} \int_0^1 V_{D,s-1}(r; c_D)' V_{D,s-1}(r; c_D) dr,$$

where $V_{D,s-1}(r; c_D) = W_{0,s-1}(1 - r) + c_D \sigma_\varepsilon^{-1} \Lambda^{1/2} (1 - r) \iota, r \in [0, 1]$.

b. Under $H_{1,T}^S$,

$$F \xrightarrow{d} V_{S,s-1}(0; c_S)' V_{S,s-1}(0; c_S),$$

$$\underline{\omega} \xrightarrow{d} \int_0^1 V_{S,s-1}(r; c_S)' V_{S,s-1}(r; c_S) dr,$$

where $V_{S,s-1}(r; c_S) = W_{0,s-1}(1 - r) + c_S \sigma_\varepsilon^{-1} \Lambda^{1/2} \int_r^1 W_{1,s-1}(s) ds, r \in [0, 1]$.

c. Under either H_1^D or H_1^S , F and $\underline{\omega}$ are $O_p(T)$.

Remark 1. The asymptotic distribution of F under $H_{1,T}^D$ is a noncentral chi-squared distribution with $s - 1$ df and noncentrality parameter equal to $c_D^2 \iota' \Lambda \iota / \sigma_\varepsilon^2$.

Remark 2. Under both local alternatives (i.e., when $\gamma_0 = c_D \iota / \sqrt{T}$ and $\sigma_\varepsilon^2 = c_S^2 / T^2$), the asymptotic distribution of F and $\underline{\omega}$ are constructed using the process $V(r; c_D, c_S) = W_{0,s-1}(1 - r) + c_D \Lambda^{1/2} (1 - r) \iota + c_S \sigma_\varepsilon^{-1} \Lambda^{1/2} \int_r^1 W_{1,s-1}(s) ds, r \in [0, 1]$, instead of either $V_{D,s-1}(r; c_D)$ or $V_{S,s-1}(r; c_S)$.

Remark 3. A modification of $\underline{\omega}$ would be to replace $\underline{\sigma}^2$ with $\hat{\sigma}^2$, that is, to fit seasonal dummies when calculating the denominator of the statistics in (21). This makes no difference to the asymptotic distribution under the null and the local alternative hypotheses.

Although the asymptotic distribution of $\underline{\omega}$ under the null hypothesis differs from that of ω , it still belongs to the Cramér-von Mises family. The 5% critical values for 1, 2, and 3 df—the latter appropriate for a full test on quarterly data—are 1.65, 2.63, and 3.46 (see table 1 of Nyblom 1989, p. 227). The 5% critical value for 11 df (kindly supplied by J. Nyblom) is 9.03. For the reasons given in subsection 2.3, the asymptotic distribution is unaffected by the inclusion of a constant or a constant and a time trend.

The asymptotic distributions of F and $\underline{\omega}$ under the local alternatives $H_{1,T}^D$ and $H_{1,T}^S$ can be used to compare the power performance of the two tests. This is done in Table 6. Specifically, for a quarterly model, $s = 4$, we have generated 50,000 replications of the limiting random variables defined in Proposition 3 by replacing the continuous-time Wiener processes $W_{0,s-1}$, and $W_{1,s-1}$ by their discrete counterparts (dividing the unit interval into 1,000 parts). We have also considered the limiting behavior of the ω test, invariant to the presence of deterministic seasonality; its asymptotic distribution against $H_{1,T}^S$ was given by Taylor (2003a). Note that under both the fixed and local alternative H_1^D and $H_{1,T}^D$, the asymptotic power of this test is equal to its nominal size.

Thus Table 6 reports, for a quarterly model, the local asymptotic power of the three tests at the nominal 5% significance level across a range of values for the parameters c_D and c_S (with σ_ε^2 set equal to 1). As expected, the Wald test is more powerful under the local alternative of deterministic seasonality, whereas $\underline{\omega}$ achieves the highest power under pure stochastic seasonality, being the LBI test for this case. For example, for $c_D = 2$ and $c_S = 0$, the asymptotic power of the Wald test is .652, as opposed to .559 for the test based on $\underline{\omega}$. In contrast, for $c_S = 4$ and $c_D = 0$, the power of the $\underline{\omega}$ test is .668, whereas that of the Wald test is .627. Finally, note that under pure stochastic seasonality, the power of the ω test of Section 2 is considerably lower than that of $\underline{\omega}$.

Table 6. Local Asymptotic Power Against Deterministic and/or Pure Stochastic Seasonality of the F , $\underline{\omega}$, and ω Tests

		$c_D = 0$	$c_D = .5$	$c_D = 1.0$	$c_D = 1.5$	$c_D = 2.0$	$c_D = 2.5$	$c_D = 3.0$	$c_D = 3.5$
$c_S = 0$	F	4.90	7.98	18.86	39.72	65.21	85.52	95.94	99.26
	$\underline{\omega}$	4.90	7.46	16.01	32.90	55.85	77.37	91.33	97.74
	ω	4.89	4.89	4.89	4.89	4.89	4.89	4.89	4.89
$c_S = 1$	F	9.34	12.94	24.52	44.17	66.36	84.43	94.68	98.75
	$\underline{\omega}$	10.53	13.60	23.03	39.08	58.60	77.02	89.81	96.60
	ω	6.08	6.08	6.08	6.08	6.08	6.08	6.08	6.08
$c_S = 2$	F	24.52	28.19	38.61	53.51	69.58	83.02	92.33	97.10
	$\underline{\omega}$	29.07	31.83	39.84	51.65	65.16	77.85	87.75	94.21
	ω	10.32	10.32	10.32	10.32	10.32	10.32	10.32	10.32
$c_S = 3$	F	45.34	47.74	54.35	63.94	74.19	83.43	90.61	95.34
	$\underline{\omega}$	51.46	53.38	58.28	64.92	72.91	80.79	87.45	92.66
	ω	18.13	18.13	18.13	18.13	18.13	18.13	18.13	18.13
$c_S = 4$	F	62.72	63.97	67.81	73.12	79.32	85.26	90.33	94.16
	$\underline{\omega}$	68.83	69.70	72.13	75.80	80.10	84.69	88.87	92.51
	ω	29.22	29.22	29.22	29.22	29.22	29.22	29.22	29.22
$c_S = 5$	F	74.79	75.58	77.52	80.63	84.07	87.66	91.06	93.96
	$\underline{\omega}$	80.10	80.58	81.76	83.56	85.84	88.55	90.97	93.16
	ω	41.62	41.62	41.62	41.62	41.62	41.62	41.62	41.62
$c_S = 7.5$	F	89.59	89.82	90.27	90.93	91.87	93.01	94.15	95.31
	$\underline{\omega}$	92.79	92.84	93.00	93.46	94.02	94.68	95.29	95.97
	ω	68.11	68.11	68.11	68.11	68.11	68.11	68.11	68.11
$c_S = 10$	F	95.09	95.21	95.25	95.49	95.70	96.05	96.45	96.90
	$\underline{\omega}$	97.03	97.05	97.08	97.17	97.27	97.44	97.64	97.87
	ω	83.73	83.73	83.73	83.73	83.73	83.73	83.73	83.73

The local power of the modified test, using $\hat{\sigma}^2$ rather than $\underline{\sigma}^2$, is, as suggested in Remark 3, the same as that of $\underline{\omega}$. However, in practice it may well have a higher power than $\underline{\omega}$ against deterministic seasonality. This is because when $\sigma_k^2 = 0$, the probability limit of $\underline{\sigma}^2$ exceeds that of $\hat{\sigma}^2$; indeed, because $\underline{\sigma}^2 \geq \hat{\sigma}^2$, the modified statistic will always be greater than or equal to $\underline{\omega}$. There is a parallel with the test on $\hat{\gamma}_0$ in that using $\underline{\sigma}^2$ instead of $\hat{\sigma}^2$ would give the LM statistic.

A second modification is also in order. As formulated in (21), the test is LBI against stochastic seasonality with $\gamma_0 = 0$. In practice, we are more concerned with seasonal patterns that diminish over time. Thus our recommendation is to use the *forward* summation just as in the ω test of Section 2, because this would be the LBI test if the data were generated backward starting with $\gamma_{T+1} = 0$. Taking these points into consideration, our preferred statistic, $\underline{\omega}$, is constructed using

$$\underline{\omega}_j = a_j T^{-2} \hat{\sigma}^{-2} \sum_{i=1}^T \left[\left(\sum_{i=1}^i e_i \cos \lambda_j i \right)^2 + \left(\sum_{i=1}^i e_i \sin \lambda_j i \right)^2 \right],$$

$$j = 1, \dots, [s/2]. \quad (22)$$

When ε_t is serially correlated, the $\underline{\omega}$ test can be modified as in Section 2.3. If the spectrum is computed using the residuals after fitting the seasonal regressors, then the statistic is denoted by $\underline{\omega}^*(m)$. The test can be extended to deal with both serial correlation and heteroscedasticity by making the amendment of (9).

The Wald test can be carried out by fitting a model that is unrestricted except insofar as the seasonal component is taken to be nonstochastic; that is, σ_k^2 is set to 0. Alternatively, a nonparametric test can be set up using a nonparametric covariance

matrix estimator, as was done by Andrews (1991). This is essentially the same correction as in (9). To be specific,

$$F(m) = T \hat{\gamma}_0' [Q^{-1} \hat{\Omega}(m) Q^{-1}]^{-1} \hat{\gamma}_0, \quad (23)$$

where $Q = T^{-1} \sum_{i=1}^T Z_i Z_i'$. If there is no need to guard against heteroscedasticity, the modifications are made simply using estimates of the spectrum as for $\underline{\omega}^*(m)$.

4.2 A Diminishing Seasonal Pattern in Spanish Interest Rates

As an example, we consider the logarithm of 3-month money market interest rate in Spain for the period 1977Q1–2001Q4; the source is the Bank of International Settlements macroeconomic series database. The series is depicted in Figure 2(a). It is difficult to detect a seasonal pattern from a casual glance at the graph, and one would not normally expect such a pattern to be present in an interest rate series; however, the functioning of the interbank loans market may imply some seasonality (see, e.g., Hamilton 1996).

Fitting the BSM to the series gives a seasonal component, as shown in Figure 2(b); the slope variance is estimated to be 0, and the estimate of the (fixed) slope is small and insignificant. We have used logarithms of the data only because the diagnostics are better; if the raw series is used, then the resulting seasonal pattern is similar.

The chi-squared statistic for the seasonals at the end of the series is only .09, which is clearly not significant, because the 5% critical value for a χ_3^2 is 7.81. However, the graph shows a fairly strong seasonal pattern until the mid-1980s. The question is whether the pattern as a whole is in any sense significant.

Setting the seasonal variance to 0 and reestimating the BSM gives a Wald statistic of 4.76, with a p value of .19. This is still

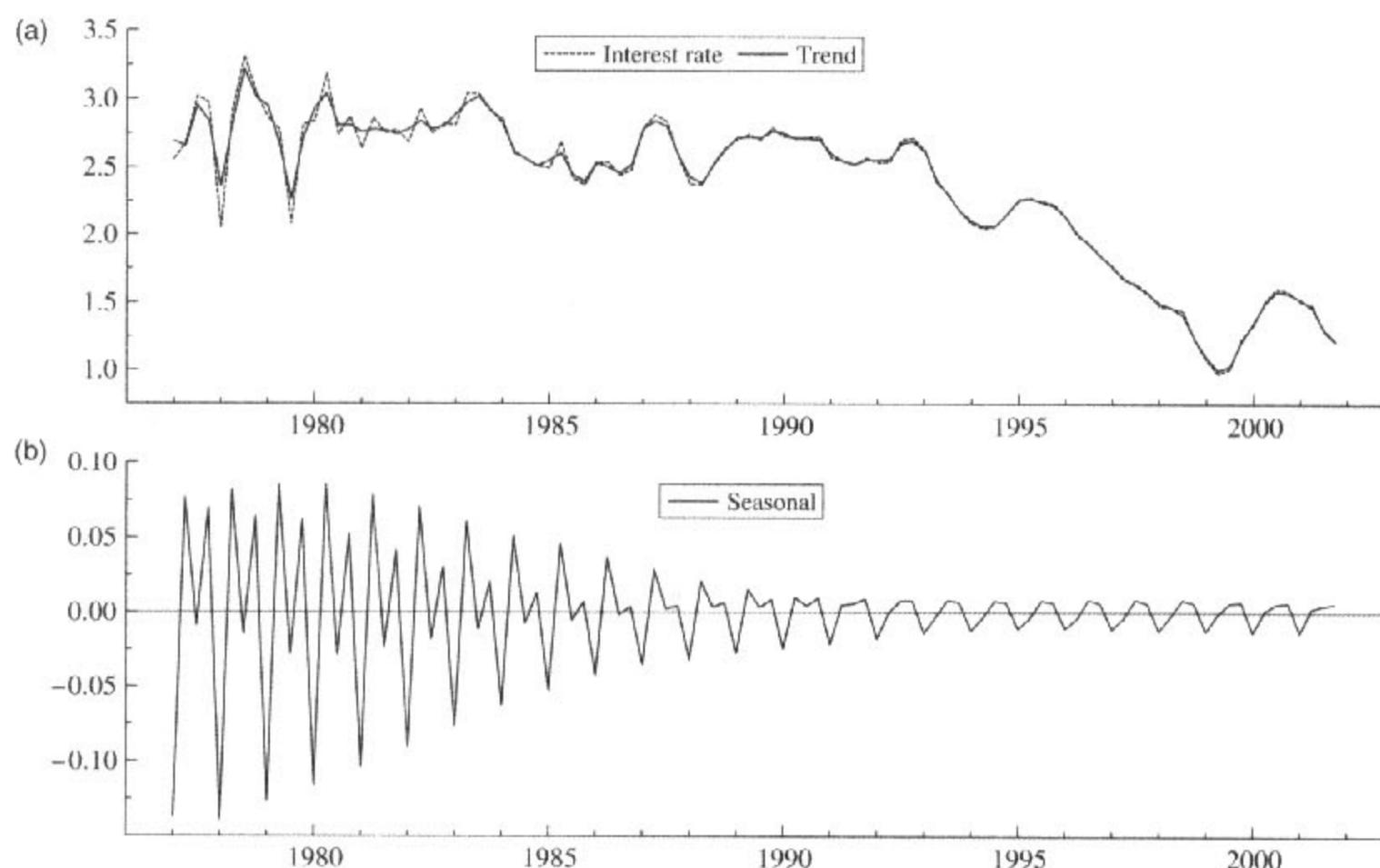


Figure 2. (a) Logarithm of the 3-Month Spanish Interest Rate, 1977Q1–2001Q4 and (b) Estimate of the Seasonal Component.

not significant. If the series is differenced and a nonparametric Wald test, (23), is computed using the Newey–West covariance matrix estimator with three lags, then a similar p value (.17) is obtained. On the other hand, the spectral nonparametric statistic computed using forward summations takes the values 3.83 and 3.01 for $m = 3$ and 6, rising to 4.64 and 3.89 for $\hat{\omega}^*(m)$, the preferred form in which the spectrum is estimated after fitting seasonal regressors. Because the 5% critical value is 3.46, this test provides a firm rejection of the hypothesis that there is no seasonality in the series.

Finally, for $m = 3$ and 6, the $\omega(m)$ statistic of Section 2 takes the values 1.17 and 1.02 (against a 5% critical value of 1.00). This confirms the presence of stochastic seasonality.

4.3 Seasonal Adjustment

The foregoing tests can be applied to a seasonally adjusted series to check whether the adjustment has been effective. This assumes that the adjustment has been done by means of moving averages, rather than by regressing on seasonal dummies. If dummies have been used, then the ω test statistics have the asymptotic distributions of Section 2.

4.4 Detection of Trading-Day Effects

Cleveland and Devlin (1980) showed that peaks at certain frequencies in the estimated spectra of monthly time series indicate the presence of trading-day effects. Specifically, there is a peak at a frequency of $.348 \times 2\pi$ radians, with the possibility of subsidiary peaks at $.432 \times 2\pi$ and $.304 \times 2\pi$ radians. An option in the output of the X-12-ARIMA program provides a comparison of the estimates of these frequencies with the adjacent frequencies (see Soukup and Findley 2000). However, there is no formal test. One possibility is to construct parametric or nonparametric statistics analogous to $\hat{\omega}_j$ and $\hat{\omega}_j^*$ so as to

carry out tests for permanent cyclical effects at one or all of the three trading-day frequencies. Assuming that no (deterministic) trading-day model has been fitted, the asymptotic distributions under the null will be CvM_0 , with a 5% critical value of 2.63 for a test at a single frequency and 5.68 for a test at all three frequencies.

As an example, we took the irregular component, obtained from X12-ARIMA, of series s0b56ym, *U.S. Retail Sales of Children's, Family, and Miscellaneous Apparel*, as supplied by the Bureau of the Census. Because the process followed by this irregular component cannot be derived, we decided to use the nonparametric test. The test statistic with 10 lags was 7.03 for the single main frequency and 8.21 for all three frequencies. Both give a clear rejection of the null hypothesis that there is no trading-day effect.

5. CONCLUSION

The seasonality test statistic proposed by CH may be simplified so that a nonparametric correction for serial correlation is based on estimating the spectrum of the series at the relevant seasonal frequency or frequencies. This test statistic then has a very straightforward interpretation. As might be expected, Monte Carlo experiments show a slight gain in power over the original CH test for homoscedastic series, but a size distortion and lower power when there is seasonal heteroscedasticity.

If a model is fitted, then a parametric seasonality test may be based on the innovations or smoothing errors, but Monte Carlo experiments show that they have similar properties. If the main reason for fitting a model is to investigate seasonality, then a basic structural time series model consisting of stochastic trend, seasonal, and irregular components usually will be adequate. However, it is worth noting that the innovations test can be implemented for any structural time series model, including

those that do not have a time invariant structure. Nonparametric tests require a decision about lag truncation, but our Monte Carlo experiments show that in samples of size 200 the rejection probabilities do not fall by very much when the lag length is increased from 4 to 8. Nonparametric tests are also dependent on decisions regarding differencing, but an important practical finding to emerge from the Monte Carlo experiments is that for most economic time series, taking first differences is likely to be a good strategy.

The Monte Carlo experiments indicate higher probabilities of rejection from parametric tests, but any assessment of power must be offset against the higher size. For quarterly data, there is a tendency for parametric tests to be oversized at frequency π , and this carries over to the joint test. The actual sizes of the joint tests at the 5% level of significance in a sample of size 200 are nearly 10%. On the other hand, with a lag length of 4, the actual size of the corresponding nonparametric tests never exceeded 5.5%. Parametric tests are attractive within the context of a model-building exercise, but if the sole focus is on testing for stochastic seasonality, then there is no overwhelming case for preferring them to nonparametric tests.

If there are breaks in the seasonal pattern, then the seasonality test may be modified so as to have an asymptotic distribution that is independent of the position of the break points under the null hypothesis that the seasonal pattern is deterministic. The U.K. marriages example yields much greater values for the parametric test statistics both with and without the seasonal break dummy variables. In the modeled break case, the parametric test indicates a rejection of the null hypothesis, whereas the conclusions from the nonparametric tests are ambivalent.

Although a fixed seasonal component is normally estimated under the null hypothesis, there may be situations in which the researcher wishes to carry out a general test against a permanent seasonal component, regardless of whether it is deterministic and stochastic. We propose the use of test statistics that have the same form as the tests against nonstationary seasonality except insofar as no fixed seasonal effects are removed when the residuals used to construct the partial sums are formed. The asymptotic critical values are easily obtained. We compared the test with a Wald test carried out to determine the joint significance of a set of seasonal dummies assumed to be constant over time. This test can also be carried out nonparametrically. When only deterministic seasonality is present, an analysis of local power shows the Wald test to be more powerful than our modification of the test against nonstationary seasonality, but not by very much. In the not uncommon situation when the seasonal pattern is diminishing over time, the modified test against nonstationary seasonality is slightly more attractive. Indeed, in the example of Spanish interest rates, this test shows a clear rejection of the null hypothesis of no permanent seasonality, whereas the Wald test does not reject. Tests against permanent seasonal effects can also be used to detect trading-day effects by exploiting the fact that these give rise to cycles at known frequencies.

An appealing feature of the proposed test statistics is that under the null hypothesis, they all have asymptotic distributions belonging to the Cramér-von Mises family. Thus they provide an integrated approach to testing a wide range of hypotheses that arise in the context of seasonal time series.

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APPENDIX A: ASYMPTOTIC REPRESENTATIONS FOR THE SPECTRAL NONPARAMETRIC TESTS

Proof of Proposition 1

From assumptions A1–A2, we have that under H_0 , $T^{\frac{1}{2}}D_T(\hat{\beta} - \beta)$ and $T^{\frac{1}{2}}(\hat{\gamma}_0 - \gamma_0)$ are $O_p(1)$ and asymptotically orthogonal. In particular,

$$T^{\frac{1}{2}}(\hat{\gamma}_0 - \gamma_0) \xrightarrow{d} N(0, G^{-1}), \quad (\text{A.1})$$

where G is an $(s-1) \times (s-1)$ diagonal matrix whose elements are proportional to the spectral-generating function evaluated at the seasonal frequencies $\lambda_j, j = 1, \dots, [s/2]$; when s is even,

$$G = \text{diag} \left(\frac{1}{2}g(\lambda_1), \frac{1}{2}g(\lambda_1), \dots, \frac{1}{2}g(\lambda_{s/2-1}), \frac{1}{2}g(\lambda_{s/2-1}), g(\lambda_{s/2}) \right),$$

whereas when s is odd, the last two diagonal elements of G are both equal to $\frac{1}{2}g(\lambda_{[s/2]})$.

Note that asymptotic orthogonality is a direct consequence of assumption A2(b), whereas the result (A.1) follows mainly from the central limit theorem of Brillinger (1975, thm. 4.4.1)—namely, for $j = 1, \dots, s^*$,

$$\begin{pmatrix} T^{-\frac{1}{2}} \sum_{t=1}^T \varepsilon_t \cos \lambda_j t \\ T^{-\frac{1}{2}} \sum_{t=1}^T \varepsilon_t \sin \lambda_j t \end{pmatrix} \xrightarrow{d} N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}g(\lambda_j) & 0 \\ 0 & \frac{1}{2}g(\lambda_j) \end{pmatrix} \right), \quad (\text{A.2})$$

whereas when s is even, $T^{-\frac{1}{2}} \sum_{t=1}^T \varepsilon_t \cos \lambda_{s/2} t \xrightarrow{d} N(0, g(\lambda_{s/2}))$. In addition, the limiting random vectors of (A.2) are also independent across j . Furthermore, a functional central limit theorem also holds; that is, the partial sums of $\varepsilon_t \cos \lambda_j t$ and $\varepsilon_t \sin \lambda_j t$ weakly converge to independent Wiener processes (see Chan and Wei 1988).

Now write the OLS residuals as

$$e_t = \varepsilon_t - X_t'(\hat{\beta} - \beta) - Z_t'(\hat{\gamma}_0 - \gamma_0), \quad t = 1, \dots, T,$$

and for $j = 1, \dots, [s/2]$, consider the (normalized) partial sum process $S_{j,T}(r) = T^{-\frac{1}{2}} \sum_{t=1}^{[Tr]} z_{jt} e_t, r \in [0, 1]$.

We then have that, under H_0 ,

$$\begin{aligned}
 S_{j,T}(r) &= T^{-\frac{1}{2}} \sum_{t=1}^{[Tr]} z_{jt} \varepsilon_t - T^{-1} \sum_{t=1}^{[Tr]} z_{jt} X_t' D_T^{-1} D_T T^{\frac{1}{2}} (\hat{\beta} - \beta) \\
 &\quad - T^{-1} \sum_{t=1}^{[Tr]} z_{jt} Z_t' T^{\frac{1}{2}} (\hat{\gamma}_0 - \gamma_0) \\
 &= T^{-\frac{1}{2}} \sum_{t=1}^{[Tr]} z_{jt} \varepsilon_t - T^{-1} \sum_{t=1}^{[Tr]} z_{jt} Z_t' \left(T^{-1} \sum_{t=1}^T Z_t Z_t' \right)^{-1} \\
 &\quad \times T^{-\frac{1}{2}} \sum_{t=1}^T Z_t \varepsilon_t + o_p(1) \\
 &= T^{-\frac{1}{2}} \sum_{t=1}^{[Tr]} z_{jt} \varepsilon_t - r T^{-\frac{1}{2}} \sum_{t=1}^T z_{jt} \varepsilon_t + o_p(1),
 \end{aligned}$$

where the last expression is due to the orthogonality relation $\lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T z_{jt} z_{ht} = 0$ for $j \neq h$.

Then, using the functional central limit theorem of Chan and Wei (1988) and the continuous mapping theorem, we have that under H_0 ,

$$(a_j g(\lambda_j))^{-\frac{1}{2}} S_{j,T}(r) \Rightarrow B_{a_j}(r), \quad r \in [0, 1], \quad (\text{A.3})$$

where $B_k(r) = W_k(r) - rW_k(1)$ is a k -dimensional standard Brownian bridge, $W_k(r)$ is a k -dimensional standard Wiener process, and \Rightarrow denotes weak convergence; furthermore $S_{j,T}(r)$ is asymptotically independent of $S_{h,T}(r)$ for $j \neq h$. As under H_0 and by assumption A3, $\hat{g}(\lambda_j; m) \xrightarrow{p} g(\lambda_j)$, it then follows by the continuous mapping theorem that the null limiting distributions of $\omega_j(m)$ when $A = A_j$, $j = 1, \dots, [s/2]$ and $\omega(m)$ when $A = I_{s-1}$ are $CvM(a_j)$ and $CvM(s-1)$.

Under $H_1 : \sigma_\kappa^2 > 0$, when $A = A_j$ it is easily seen that the partial sum $S_{j,T}(r)$ is $O_p(T^{\frac{1}{2}})$, whereas assumption A3 implies that $\hat{g}(\lambda_j; m)$ is $O_p(m)$ (see Stock 1994, pp. 2797–2799). Thus both $\omega_j(m)$ and $\omega(m)$ are $O_p(T/m)$.

Proof of Proposition 2

Consider the two subsamples $\{1, \dots, [\alpha T]\}$, $\{[\alpha T] + 1, \dots, T\}$, and let $S_{j,T}^A(r) = [\alpha T]^{-\frac{1}{2}} \sum_{t=1}^{[\alpha Tr]} z_{jt} \varepsilon_t$, $r \in [0, 1]$, and $S_{j,T}^B(r) = [(1 - \alpha)T]^{-\frac{1}{2}} \sum_{t=[\alpha T]+1}^{[Tr]} z_{jt} \varepsilon_t$, $r \in [0, 1]$, be the partial sum processes in the first and second subsamples. Using a functional central limit theorem as in (A.3), we have that under H_0 ,

$$(a_j g(\lambda_j))^{-\frac{1}{2}} (S_{j,T}^A(r)', S_{j,T}^B(r)')' \Rightarrow (B_{a_j}^A(r)', B_{a_j}^B(r)')', \quad r \in [0, 1],$$

where $B_{a_j}^A(r)$ and $B_{a_j}^B(r)$ are independent k -dimensional standard Brownian bridges. Noticing that

$$\begin{aligned}
 \sum_{t=1}^T c_{jt} k_t &= [\alpha T]^{-1} \sum_{t=1}^{[\alpha T]} \text{trace} \left(S_{j,T}^A \left(\frac{t}{T} \right) S_{j,T}^A \left(\frac{t}{T} \right)' \right) \\
 &\quad + [(1 - \alpha)T]^{-1} \sum_{t=[\alpha T]+1}^T \text{trace} \left(S_{j,T}^B \left(\frac{t}{T} \right) S_{j,T}^B \left(\frac{t}{T} \right)' \right),
 \end{aligned}$$

it then follows by the functional central limit theorem of Chan and Wei (1988), the continuous mapping theorem, and the additivity property of independent Cramér–von Mises random variables that $\omega_j^*(\alpha; m) \xrightarrow{d} CvM(2a_j)$ and $\omega^*(\alpha; m) \xrightarrow{d} CvM(2s - 2)$ under H_0 . Then, from the same arguments as in Proposition 2, under $H_1 : \sigma_\kappa^2 > 0$, when $A = A_j$, both $\omega_j^*(\alpha; m)$ and $\omega^*(\alpha; m)$ are $O_p(T/m)$.

APPENDIX B: LBI TEST

When ε_t in (1) is generalized so as to be a linear stationary process, possibly consisting of more than one component, its $T \times T$ matrix covariance matrix will be denoted $V = \sigma_*^2 V_*$, where σ_*^2 is a variance parameter. If V_* is known, then it follows from King and Hillier (1985) that the LBI test is of the form (5) with the OLS residuals replaced by the elements of the $T \times 1$ vector, $V_*^{-1} \tilde{e}$, where \tilde{e} is the $T \times 1$ vector of generalized least squares residuals. The LBI test against stochastic seasonality at all frequencies is constructed similarly. If ε_t contains a white noise component with variance σ_*^2 then it is straightforward to show that $V_*^{-1} \tilde{e}$ is equal to the smoothed estimator of the vector of the white noise series. More generally, when multiplied by σ_*^{-2} it becomes the $T \times 1$ vector of smoothing errors, denoted by $u = V_*^{-1} \tilde{e}$. The smoothing errors are produced as a byproduct of the smoother applied to the state-space form of the model (see de Jong and Penzer 1998; Harvey and Streibel 1997).

With V_* known, an exact test can be carried out using numerical inversion to construct critical values or probability values. However, V_* normally will depend on unknown parameters, so there are good reasons for wishing to use a statistic with a known asymptotic distribution. If the test statistic is formed from smoothing errors, then it is necessary to take into account their serial correlation. Following a argument similar to that used to give (7), the denominator needs an estimator of the spectral generating function of $V_*^{-1} \varepsilon$. This is equal to $1/g_\varepsilon(\lambda)$, where $g_\varepsilon(\lambda)$ is the spectral-generating function of ε_t . The parametric test statistic is, therefore,

$$\omega_j = a_j T^{-2} \tilde{g}_\varepsilon(\lambda_j) \sum_{i=1}^T \left[\left(\sum_{i=1}^i u_i \cos \lambda_j i \right)^2 + \left(\sum_{i=1}^i u_i \sin \lambda_j i \right)^2 \right], \quad j = 1, \dots, [s/2], \quad (\text{B.1})$$

where u_i is the i th smoothing error. The test statistic has the same asymptotic distribution as (5), namely, $CvM(2)$. This remains true when parameters in V_* are estimated (cf. Leybourne and McCabe 1994; Saikkonen and Luukkonen 1993).

The foregoing correction can be made even if the model contains a stochastic trend. The smoothing error series is stationary, and although it is not (strictly) invertible, the noninvertibility affects only the zero frequency, and the “quasi” Spectral Generating Function (sgf) of the nonseasonal part of the model can be inverted at λ_j . Thus for the special case of (11) in which the trend is a random walk, $\tilde{g}_\varepsilon(\lambda_j)$ in (B.1) is replaced by

$$\left[\frac{\tilde{\sigma}_\eta^2 + 2(1 - \cos \lambda) \tilde{\sigma}_\varepsilon^2}{2(1 - \cos \lambda)} \right], \quad 0 < \lambda \leq \pi. \quad (\text{B.2})$$

If instead of the smoothing errors, the smoothed estimator of an irregular component, ε_t , is used, the foregoing correction factor must be divided by $\tilde{\sigma}_\varepsilon^4$.

APPENDIX C: ASYMPTOTIC REPRESENTATIONS FOR THE TESTS AGAINST PERMANENT SEASONALITY

To prove Proposition 3, we need the following lemma.

Lemma C.1. (a) Under $H_{1,T}^S : \gamma_0 = 0, \sigma_\kappa^2 = c_S^2/T^2 > 0,$

$$T^{-\frac{1}{2}} \sum_{t=[Tr]}^T Z_t Z_t' \sum_{i=1}^t \kappa_i \Rightarrow c_S \Lambda \int_r^1 W_{1,s-1}(r) dr, \quad r \in [0, 1].$$

(b) Under $H_1^S : \gamma_0 = 0, \sigma_\kappa^2 > 0,$

$$T^{-1} \sum_{t=1}^T D_T^{-1} X_t Z_t' \sum_{i=1}^t \kappa_i \xrightarrow{p} 0.$$

Proof of the Lemma

To avoid unnecessary complications in the notations, we assume that $T_s \equiv T/s$ is an integer. Let $t^* = (\bar{t} - 1)/s + 1,$ where $\bar{t} = 1, \dots, T.$ For part (i), first note that

$$\sum_{t=\bar{t}}^T \sum_{i=1}^t \kappa_i = \begin{cases} \sum_{j=1}^s \sum_{u=t^*}^{T_s} (B_u + R_{j,u}) & \text{if } t^* \text{ is an integer} \\ \sum_{j=1}^s \sum_{u=[t^*]}^{T_s} (B_u + R_{j,u}) - \sum_{i=[t^*]}^{t^*-1} \kappa_i & \text{otherwise,} \end{cases}$$

where, for $u = 1, \dots, T_s,$

$$B_u = \begin{cases} 0 & u = 1 \\ \sum_{i=1}^{s(u-1)} \kappa_i & u > 1, \end{cases}$$

$$R_{j,u} = \sum_{i=s(u-1)+1}^{s(u-1)+j} \kappa_i, \quad j = 1, \dots, s.$$

Note that we can write, for $t = 1, \dots, T,$

$$Z_t = \begin{cases} Z_j & \text{if } 1 \leq j \leq s-1 \\ Z_s & \text{if } j = 0 \end{cases}$$

where $j = t \bmod s.$ Then, using the decomposition above, we have that under the local alternative $H_{1,T}^S,$ for $r \in [0, 1],$

$$\begin{aligned} T^{-\frac{1}{2}} \sum_{t=[Tr]}^T Z_t Z_t' \sum_{i=1}^t \kappa_i &= \sum_{j=1}^s \frac{1}{sT_s} \sum_{u=[T_s r]}^{T_s} Z_j Z_j' T^{\frac{1}{2}} (B_u + R_{j,u}) + o_p(1) \\ &= \sum_{j=1}^s Z_j Z_j' \frac{1}{sT_s} \sum_{u=[T_s r]}^{T_s} T^{\frac{1}{2}} (B_u + R_{j,u}) + o_p(1). \end{aligned}$$

By a standard functional central limit theorem, we obtain that under $H_{1,T}^S,$

$$T^{\frac{1}{2}} B_{[T_s r]} \Rightarrow c_S W_{1,s-1}(r), \quad r \in [0, 1],$$

with $W_{1,s-1}(r)$ being an $s - 1$ dimensional standard Wiener process, whereas, for all $j = 1, \dots, s,$

$$T^{-\frac{1}{2}} \sum_{u=[T_s r]}^{T_s} R_{j,u} \xrightarrow{p} 0.$$

Thus, by an application of the continuous mapping theorem, for $r \in [0, 1],$

$$T^{-\frac{1}{2}} \sum_{t=[Tr]}^T Z_t Z_t' \sum_{i=1}^t \kappa_i \xrightarrow{d} c_S \Lambda \int_r^1 W_{1,s-1}(r) dr,$$

where $\Lambda = s^{-1} \sum_{j=1}^s Z_j Z_j'$ is the diagonal matrix defined in the statement of the proposition.

For part (b), first note that

$$\sum_{j=1}^s Z_j' = 0. \tag{C.1}$$

Then, proceeding similarly as before, we have that

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T D_T^{-1} X_t Z_t' \sum_{i=1}^t \kappa_i &= \sum_{j=1}^s \frac{1}{sT_s} \sum_{u=1}^{T_s} D_T^{-1} X_{su} Z_j' (B_u + R_{j,u}) + o_p(1) \tag{C.2} \\ &= \frac{1}{sT_s} \sum_{u=1}^{T_s} D_T^{-1} X_{su} \tilde{\kappa}_u + o_p(1), \tag{C.3} \end{aligned}$$

where the second equality uses (C.1) and

$$\tilde{\kappa}_u \equiv \sum_{j=1}^s Z_j' R_{j,u}.$$

Now $\tilde{\kappa}_u, u = 1, \dots, T_s$ is an independent sequence, because each element is made as a weighted sum of s nonoverlapping disturbances $\kappa_t, t = 1, \dots, T.$ Thus part (b) of the lemma is proved by applying to (C.3) a law of large numbers for independent heteroscedastic sequences.

Proof of Proposition 3

Let y_t be generated by the model (1)–(2), (16)–(18) with $W = I_{s-1};$ and let $\hat{\beta}$ and $\hat{\gamma}_0$ be the OLS estimators of β and γ_0 from regressing y_t on $(X_t', Z_t)';$ and let $\underline{\beta}$ be the OLS estimator from regressing y_t on $X_t.$ Then

$$\begin{pmatrix} \hat{\beta} - \beta \\ \hat{\gamma}_0 - \gamma_0 \end{pmatrix} = A^{-1} b,$$

where

$$\begin{aligned} A &= \begin{pmatrix} T^{-1} \sum_{t=1}^T D_T^{-1} X_t X_t' D_T^{-1} & T^{-1} \sum_{t=1}^T D_T^{-1} X_t Z_t' \\ T^{-1} \sum_{t=1}^T Z_t X_t' D_T^{-1} & T^{-1} \sum_{t=1}^T Z_t Z_t' \end{pmatrix}, \\ b &= \begin{pmatrix} T^{-1} \sum_{t=1}^T D_T^{-1} X_t \left(\varepsilon_t + Z_t' \sum_{i=1}^t \kappa_i \right) \\ T^{-1} \sum_{t=1}^T Z_t \left(\varepsilon_t + Z_t' \sum_{i=1}^t \kappa_i \right) \end{pmatrix}, \end{aligned}$$

and

$$(\underline{\beta} - \beta) = \left(T^{-1} \sum_{t=1}^T D_T^{-1} X_t X_t' D_T^{-1} \right)^{-1} \times T^{-1} \sum_{t=1}^T D_T^{-1} X_t \left(\varepsilon_t + Z_t' \sum_{i=1}^t \kappa_i + Z_t' \gamma_0 \right).$$

The OLS residuals for the two regressions can be written as

$$e_t = \varepsilon_t + Z_t' \sum_{i=1}^t \kappa_i - X_t'(\hat{\beta} - \beta) - Z_t'(\hat{\gamma}_0 - \gamma_0) \quad (C.4)$$

and

$$\underline{e}_t = \varepsilon_t + Z_t' \sum_{i=1}^t \kappa_i + Z_t' \gamma_0 - X_t'(\underline{\beta} - \beta). \quad (C.5)$$

By the orthogonality of the regressors Z_t and X_t of assumption A1, it can be straightforwardly shown that $\underline{\beta} = \hat{\beta} + o_p(1)$. Furthermore, by the functional central limit theorem of Chan and Wei (1988), we have the following weak convergence result:

$$\sigma_\varepsilon^{-1} \Lambda^{-\frac{1}{2}} T^{-\frac{1}{2}} \sum_{t=[Tr]}^T Z_t \varepsilon_t \Rightarrow W_{0,s-1}(1-r), \quad r \in [0, 1], \quad (C.6)$$

where $W_{0,s-1}(r)$ and Λ are defined in the statement of the proposition. Note that

$$\lim_{T \rightarrow \infty} T^{-1} \sum_{t=1}^T Z_t Z_t' = \Lambda.$$

Consider the local alternative $H_{1,T}^D : \gamma_0 = c_D t / \sqrt{T}, \sigma_\kappa^2 = 0$. We have that $\hat{\beta} = \beta + o_p(1)$,

$$\sigma_\varepsilon^{-1} \Lambda^{\frac{1}{2}} T^{\frac{1}{2}} \hat{\gamma}_0 \xrightarrow{d} W_{0,s-1}(1) + c_D \sigma_\varepsilon^{-1} \Lambda^{\frac{1}{2}} t = V_{D,s-1}(0; c_D),$$

and $e_t = \varepsilon_t + o_p(1)$. Then $F \xrightarrow{d} V_{D,s-1}(0; c_D)' V_{D,s-1}(0; c_D)$; note that because $W_{0,s-1}(1)$ is a standard $s-1$ dimensional normal, the asymptotic distribution of F is a noncentral chi-squared distribution with $s-1$ degrees of freedom and noncentrality parameter equal to $c_D^2 \sigma_\varepsilon^{-2} t' \Lambda t$.

Furthermore, from (C.6) and (C.5),

$$\begin{aligned} \sigma_\varepsilon^{-1} \Lambda^{-\frac{1}{2}} T^{-\frac{1}{2}} \sum_{t=[Tr]}^T Z_t \underline{e}_t &\Rightarrow W_{0,s-1}(1-r) + c_D \sigma_\varepsilon^{-1} \Lambda^{\frac{1}{2}} (1-r)t \\ &= V_{D,s-1}(r; c_D), \quad r \in [0, 1], \end{aligned}$$

and because $\underline{e}_t = \varepsilon_t + o_p(1)$, $\underline{\omega} \xrightarrow{d} \int_0^1 V_{D,s-1}(r; c_D)' V_{D,s-1}(r; c_D) dr$.

Under the fixed alternative H_1^D , using similar arguments it is not difficult to show that $\hat{\gamma}_0$ is $O_p(1)$, $\hat{\sigma}^2$ and $\underline{\sigma}^2$ are $O_p(1)$, and $\sum_{i=t}^T Z_i \underline{e}_i$ is $O_p(T)$. Thus both F and $\underline{\omega}$ are $O_p(T)$.

Consider the local alternative $H_{1,T}^S$. By (C.6) and using part (a) of the lemma, we have that

$$\begin{aligned} \sigma_\varepsilon^{-1} \Lambda^{\frac{1}{2}} T^{\frac{1}{2}} \hat{\gamma}_0 &\xrightarrow{d} W_{0,s-1}(1) + c_S \sigma_\varepsilon^{-1} \Lambda^{\frac{1}{2}} \int_0^1 W_{1,s-1}(r) dr \\ &= V_{S,s-1}(0; c_S). \end{aligned}$$

As under $H_{1,T}^S, e_t = \varepsilon_t + o_p(1)$, it easily follows that $F \xrightarrow{d} V_{S,s-1}(0; c_S)' V_{S,s-1}(0; c_S)$.

Using similar arguments, we also have that

$$\begin{aligned} \sigma_\varepsilon^{-1} \Lambda^{-\frac{1}{2}} T^{-\frac{1}{2}} \sum_{t=[Tr]}^T Z_t \underline{e}_t &\Rightarrow W_{0,s-1}(1-r) \\ &\quad + c_S \sigma_\varepsilon^{-1} \Lambda^{\frac{1}{2}} \int_r^1 W_{1,s-1}(s) ds \\ &= V_{S,s-1}(r; c_S), \quad r \in [0, 1] \end{aligned}$$

and $\underline{\sigma}^2 = T^{-1} \sum_{t=1}^T \underline{e}_t^2 \xrightarrow{p} \sigma_\varepsilon^2$. Thus

$$\begin{aligned} \underline{\omega} &= \sum_{j=1}^{[s/2]} \underline{\omega}_j \\ &= T^{-2} \underline{\sigma}^{-2} \left(\sum_{t=1}^T \text{trace} \left(\sum_{i=t}^T Z_i e_i \right) \left(\sum_{i=t}^T Z_i e_i \right)' \right) \\ &\xrightarrow{d} \int_0^1 V_{S,s-1}(r; c_S)' V_{S,s-1}(r; c_S) dr. \end{aligned}$$

Finally, under the fixed alternative H_1^S , from part (b) of the lemma, we first obtain that both $\hat{\beta} - \beta$ and $\underline{\beta} - \beta$ are $o_p(1)$. Then, as $\sum_{i=t}^T Z_i \underline{e}_i$ is $O_p(T^{\frac{1}{2}})$ and $\hat{\gamma}_0$ is $O_p(T^{\frac{1}{2}})$, and $\hat{\sigma}^2$ and $\underline{\sigma}^2$ are $O_p(1)$, it easily follows that both the Wald statistic F and the statistic $\underline{\omega}$ are $O_p(T)$.

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