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# A Direct Test for Changing Trend

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We consider tests for changing trend that do not require prior knowledge about the location of the changepoint. The limiting distribution is derived from the functional central limit theorem and the critical value from the hitting probability of a Brownian bridge. Applying a test sensitive to the alternative of trend stationarity with structural breaks, we find that for real gross national product, real per capita gross national product, and real wages before World War II the hypothesis of trend stationarity is rejected.

KEY WORDS: Brownian bridge; Functional central limit theorem; Hitting probability; Structural change; Trend stationarity.

Parameter constancy is a necessary condition for both accurate time series forecasting and the practice of policy evaluation. Because of the importance of parameter constancy, many tests have been suggested for it based on different formulations of the alternative hypothesis. Leading examples are the Chow (1960) test for a one-time structural change; the Brown, Durbin, and Evans (1975) CUSUM (cumulative sum) test for change at an unknown point in time; and the Lamotte and McWhorter (1978) test for time-varying parameters, where the changing parameters follow some stochastic process under the alternative, often the empirically appealing random-walk process (see also Nyblom 1989).

Recently, the theory of weak convergence and the functional central limit theorem (FCLT) have proved useful in obtaining limiting distributions of tests for parameter constancy. Successful efforts include (a) the fluctuation test of Sen (1980) and Ploberger, Krämer, and Kontrus (1989), which detects changing parameters in a linear regression model by comparing the parameter estimates from a partial sample with those from the whole sample; (b) the tests of James, James, and Siegmund (1987), who derived tests for one-time structural change in a simple location model and compared the power of their test to other competing tests; and (c) the tests of Andrews (1990), who derived the limiting distributions for the sequential likelihood ratio test (à la Quandt 1960), the maximal Wald-type test, and the Lagrange-multiplier-type test. These test statistics have been proved to have well-defined asymptotic distributions in the space of continuous function, and their critical values are determined from the hitting probability of the Browian bridge or tied down Bessel process. Quite significantly, Andrews also showed that these tests for parameter constancy against the alternative of one-time structural change in fact have nontrivial asymptotic local power against all alternatives of parameter nonconstancy. All of these results are limited to models with nontrending regressors.

For regression models with trending regressors, Kim and Siegmund (1989) considered tests for models with normal iid errors in which the regressor can be taken as a function of time. Hansen (1990) studied the Chow test for parameter constancy in regression models that include both trending and nontrending regressors. A novelty of Hansen's result is that instead of using the maximum Chow statistic as the basis for a test he proposed using the average of the sequence of Chow statistics. Following a limited Monte Carlo experiment, Hansen claimed that the mean–Chow test is preferable to the max–Chow test in terms of size accuracy and test power.

The main result of this article is that the same test as in the regression with nontrending regressors, with a different time scaling, can be employed for regression with deterministic and stochastic trends. We propose new tests for the stability of trend slope and intercept. Critical values of our new tests can be determined easily from the hitting probability of the Brownian-bridge process. A test for the constancy of cointegration is also derived as a straightforward extension.

The rest of this article is organized as follows. Section 1.1 focuses on a test for constant trend slope, in which a time rescaling technique due to Doob (1949) is employed to develop surprisingly simple asymptotics for a test of constancy in the trend coefficient. Section 1.2 contains a test for the constancy in trend intercept. Section 2 reports Monte Carlo experiments exploring the size and power of these tests. Section 3, as an illustration, applies the test for trend constancy to study the persistence of 14 U.S. macroeconomic time series. Section 4 ends the article with some brief remarks.

## 1. ASYMPTOTIC TESTING VIA BROWNIAN MOTION AND THE BROWNIAN BRIDGE

### 1.1 Testing for a Change in Trend Slope

Consider a trend-stationary process $H_0 : Y_t = \alpha_0 + \beta_0 t + \varepsilon_t$ $(t = 1, 2, \ldots, n)$. The goal is to detect a change in the trend function at some unknown point in time. The simplest alternative is that of a single shift at an unknown breakpoint $m$,

$$H_1 : Y_t = \alpha_1 + \beta_1 t + \varepsilon_t, \qquad t = 1, 2, \ldots, m$$

$$Y_t = \alpha_2 + \beta_2 t + \varepsilon_t, \qquad t = m + 1, \ldots, n.$$

We assume that $\{\varepsilon_t/\sigma_0\}$ satisfies the FCLT. By this we mean that $\sigma_0^{-1} n^{-1/2} S_{[n\lambda]} \equiv \sigma_0^{-1} n^{-1/2} \sum_{t=1}^{[n\lambda]} \varepsilon_t \Rightarrow W(\lambda)$, $\lambda \in [0, 1]$, where $W(.)$ is the standard Brownian motion and $\sigma_0^2 = \lim_{n\to\infty} n^{-1} E(S_n^2)$ is nonzero and finite. The notations $[x]$ and $\Rightarrow$ stand for the integer part of $x$ and weakly converges to, respectively. The FCLT requirement is weak and applies to a fairly wide class of sequences $\{\varepsilon_t/\sigma_0\}$ (see Phillips 1987; Wooldridge and White 1988).

A test with power against this and other alternatives can be based on the least squares estimators

$$\hat{\beta}_k = \Delta_k^{-1}\left[k \sum_{t=1}^{k} tY_t - \left(\sum_{t=1}^{k} t\right)\left(\sum_{t=1}^{k} Y_t\right)\right],$$

where $\Delta_k \equiv k \sum_{t=1}^{k} t^2 - (\sum_{t=1}^{k} t)^2 = (k^4 - k^2)/12$. Substituting $Y_t = \alpha_0 + \beta_0 t + \varepsilon_t$ into $\hat{\beta}_k$ gives

$$\hat{\beta}_k = \Delta_k^{-1}\left[k \sum_{t=1}^{k} t(\alpha_0 + \beta_0 t + \varepsilon_t) - \left(\sum_{t=1}^{k} t\right)\left[\sum_{t=1}^{k} (\alpha_0 + \beta_0 t + \varepsilon_t)\right]\right]$$

so that

$$(\hat{\beta}_k - \beta_0) = \Delta_k^{-1} k\left[\sum_{t=1}^{k} t\varepsilon_t - \frac{k+1}{2}\sum_{t=1}^{k} \varepsilon_t\right].$$

Suppose for the moment that $\beta_0$ and $\sigma_0^2$ are known. We will replace the unknown $\beta_0$ and $\sigma_0^2$ by consistent estimators later. Define a random function in the space $D[0, 1]$ with the Skorohod topology (Billingsley 1968, sec. 14) by

$$Z_n(\lambda) = (6\sigma_0)^{-1} \mathrm{n}^{3/2}([n\lambda]/n)^3(\hat{\beta}_{[n\lambda]} - \beta_0). \tag{1}$$

Now

$$n^{3/2}([n\lambda]/n)^3(\hat{\beta}_{[n\lambda]} - \beta_0) = n^{-3/2}[n\lambda]^3([n\lambda]/\Delta_{[n\lambda]}) \times \left[\sum_{t=1}^{[n\lambda]} t\varepsilon_t - (1/2)([n\lambda] + 1)\sum_{t=1}^{[n\lambda]} \varepsilon_t\right]$$

$$= \left[[n\lambda]^4/\Delta_{[n\lambda]}\right]\left[n^{-3/2}\sum_{t=1}^{[n\lambda]} t\varepsilon_t - (1/2)n^{-3/2}[n\lambda] \times \sum_{t=1}^{[n\lambda]} \varepsilon_t - (1/2)n^{-3/2}\sum_{t=1}^{[n\lambda]} \varepsilon_t\right].$$

Examining the terms of this expression, we note that $[n\lambda]^4/\Delta_{[n\lambda]} \to 12$ as $n \to \infty$ and that the FCLT governs the second and third terms inside the second brackets. The second term weakly converges to $-\lambda\, W(\lambda)/2$ by the FCLT, but the third term vanishes in probability. Combining these facts with the Lemma 1.1 that characterizes the limiting behavior of the first term, we obtain

$$Z_n(\lambda) \Rightarrow \left[\lambda W(\lambda) - 2\int_0^\lambda W(r)dr\right] \equiv G(\lambda).$$

*Lemma 1.1.* Suppose that $\{\varepsilon_t/\sigma_0\}$ obeys the FCLT. Then

$$n^{-3/2}\sum_{t=1}^{[n\lambda]} t\varepsilon_t \Rightarrow \sigma_0\left[\lambda W(\lambda) - \int_0^\lambda W(r)dr\right]$$

as $n \to \infty$.

*Lemma 1.2.* $G(\lambda)$ is a Gaussian process with zero mean and covariance function $E[G(\lambda)G(s)] = (1/3)(\lambda \wedge s)^3 \equiv (1/3)[\min(\lambda, s)]^3$, $\lambda, s \in (0, 1]$. Appendix A gives the proofs of Lemmas 1.1 and 1.2.

Although the limiting distribution of $Z_n(\lambda)$ is a functional of Brownian motion, it is analytically tractable because the covariance function of $G(\lambda)$ satisfies the Markov property by Lemma 1.2; that is, for $\lambda > s$, $E[G(\lambda)G(s)] = u(s)v(\lambda)$, where $u(s) = s^3/3$ and $v(\lambda) = 1$. We may therefore use the time-rescaling technique of Doob (1949) to transform $G(\lambda)$ to a standard Brownian motion.

For this, let $a(\lambda) \equiv u(\lambda)/v(\lambda) = \lambda^3/3$, which is monotonically increasing with inverse $b(\lambda) = (3\lambda)^{1/3}$. Then $G[b(\lambda)]$ is a standard Brownian motion, because $E\{G[b(\lambda)]\} = 0$ and $\mathrm{var}\{G[b(\lambda)]\} = [b(\lambda)]^3/3 = \lambda$, and for $\lambda > s$ $E\{G[b(\lambda)]\, G[b(s)]\} = [b(s)]^3/3 = s$. Since

$$\sup_{\lambda\in[0,1]}|G(\lambda)| \overset{d}{=} \sup_{\lambda\in[0,1]}|W(a(\lambda))| \overset{d}{=} \sup_{t\in[0,1/3]}|W(t)| \overset{d}{=} \sup_{\theta\in[0,1]}|W(\theta/3)| \overset{d}{=} \sup_{\theta\in[0,1]}|W(\theta)|/\sqrt{3},$$

where $\overset{d}{=}$ denotes equality in distribution. It follows that

$$P\{\sup_{\lambda\in[0,1]}|G(\lambda)| > c\} = P\{\sup_{\theta\in[0,1]}|W(\theta)| > \sqrt{3}\, c\}.$$

Defining

$$T_{1n} \equiv \max_{k\le n} (6\sigma_0)^{-1} n^{3/2}(k/n)^3|\hat{\beta}_k - \beta_0|, \tag{2}$$

we have that $\lim_{n\to\infty} P[T_{1n} > c] = P[\sup_{\theta\in[0,1]}|W(\theta)| > \sqrt{3}\, c]$ under $H_0$.

This permits construction of an asymptotic test for the constancy of the trend coefficient, because it establishes that using the hitting probability of Brownian motion to approximate a critical value of the $T_{1n}$ statistic in (2) delivers a test with the correct size asymptotically.

Let $N$ be a standard normal random variable;

the hitting probability of a Brownian motion is (Billingsley 1968, eq. 11.13)

$$P[\sup_{\lambda\in[0,1]}|W(\lambda)| > c] = 1 - \sum_{j=\infty}^{-\infty} (-1)^j P[(2j-1)c < N < (2j+1)c].$$

It is well known that $P[\sup_{\lambda\in[0,1]} W(\lambda) > c] = 2[1 - \Phi(c)]$, where $\Phi(.)$ is the cdf of a normal random variable. This provides a quick approximation to the critical values of the $T_{1n}$ test. For tests at the 10%, 5%, and 1% significance levels, the critical values are 1.13, 1.29, and 1.62, respectively.

When $\beta_0$ is unknown, we replace it with a consistent estimator under $H_0$, the least squares estimator for the entire sample, $\hat{\beta}_n$. Replacing $\beta_0$ with $\hat{\beta}_n$ in (1) gives

$$Z_n^0(\lambda) = (6\sigma_0)^{-1}n^{3/2}([n\lambda]/n)^3(\hat{\beta}_{[n\lambda]} - \hat{\beta}_n). \quad (3)$$

It follows that

$$\begin{aligned} Z_n^0(\lambda) &= (6\sigma_0)^{-1}n^{3/2}([n\lambda]/n)^3(\hat{\beta}_{[n\lambda]} - \beta_0) \\ &\quad - (6\sigma_0)^{-1}n^{3/2}([n\lambda]/n)^3(\hat{\beta}_n - \beta_0) \\ &= Z_n(\lambda) - ([n\lambda]/n)^3 Z_n(1) \Rightarrow G(\lambda) \\ &\quad - \lambda^3 G(1) \equiv G^0(\lambda), \end{aligned}$$

Routine computation shows that $G^0(\lambda)$ is a Gaussian process with zero mean and $\mathrm{cov}[G^0(\lambda), G^0(s)] = s^3(1 - \lambda^3)/3$, for $\lambda \geq s$. Because this covariance function also satisfies the Markov property, it can be verified that $\sqrt{3}\,(1+\lambda)G^0[(\lambda/(1+\lambda))^{1/3}]$ is a standard Brownian motion. The time-rescaling technique yields

$$\begin{aligned} &\sup_{\lambda\in[0,1]}|G^0(\lambda)| \\ &\overset{d}{=} \sup_{\lambda\in[0,1]}(1-\lambda^3)|W((\lambda/(1-\lambda))^3)|/\sqrt{3} \\ &\overset{d}{=} \sup_{t\in[0,1]}|(1-t)W(t/(1-t))|/\sqrt{3} \\ &\overset{d}{=} \sup_{t\in[0,1]}|W^0(t)|/\sqrt{3}, \end{aligned}$$

where $W^0$ is the standard Brownian bridge (Billingsley 1968, pp. 64–65). Defining

$$T_{1n}^0 \equiv \max_{1\leq k\leq n-1} (6\sigma_0)^{-1}n^{3/2}(k/n)^3|\hat{\beta}_k - \hat{\beta}_n|, \quad (4)$$

we obtain that $\lim_{n\to\infty} P[T_{1n}^0 > c] = P[\sup_{t\in[0,1]}|W^0(t)| > \sqrt{3}\,c]$ under $H_0$. Critical values for a test based on $T_{1n}^0$ can therefore be computed using the hitting probability of the Brownian bridge divided by $\sqrt{3}$.

From Billingsley (1968, eq. 11.39), the hitting probability is given by

$$P[\sup_{\lambda\in[0,1]}|W^0(\lambda)| > c] = 2\sum_{j=1}^{\infty}(-1)^{j+1}\exp(-2j^2c^2).$$

Kiefer (1959) gave an extensive table for this function. For $c = 1.223, 1.358$, and $1.628$, the hitting probabilities are .1, .05, and .01, respectively. Dividing these values by $\sqrt{3}$, the critical values of the $T_{1n}^0$ test at the 10%, 5%, and 1% levels are .708, .784, and .940, respectively.

In practice, $\sigma_0$ must also be replaced by a consistent estimator. There is a multitude of consistent estimators for $\sigma_0$ (e.g., Andrews 1991; Newey and West 1987). Use of any of these in place of $\sigma_0$ will leave the asymptotic null distribution of $T_{1n}$ and $T_{1n}^0$ unchanged, so we do not further consider the estimation of $\sigma_0$ here. Proper choice of an estimator for $\sigma_0$, however, is crucially important in practice, as our Monte Carlo results in Section 2 illustrate. We summarize the foregoing discussion by stating the following theorem.

*Theorem 1.3.* Let $Y_t = \alpha_0 + \beta_0 t + \varepsilon_t (t = 1, 2, \ldots, \alpha_0, \beta_0 \in \mathbb{R})$, and suppose that $\{\varepsilon_t/\sigma_0\}$ obeys the FCLT. Then (a) $T_{1n} \Rightarrow \sup_{\lambda\in[0,1]}|W(\lambda)|/\sqrt{3}$, and (b) $T_{1n}^0 \Rightarrow \sup_{\lambda\in[0,1]}|W^0(\lambda)|/\sqrt{3}$. The conclusions remain true if $\sigma_0$ is replaced by any estimator $\hat{\sigma}_n \xrightarrow{P} \sigma_0$ in computing $T_{1n}$, $T_{1n}^0$

Compared to results for nontrending regressors (Ploberger et al. 1989), we have entirely analogous results, the differences being that $T_{1n}$ and $T_{1n}^0$ are computed with scaling $n^{3/2}$ due to the presence of $t$ as a regressor (instead of $n^{1/2}$ for a nontrending regressor) and that a factor of $\sqrt{3}$ now appears to adjust the critical values used in stationary regression models.

The usefulness of these tests hinges on their power against alternatives of interest. Our focus here is the alternative $H_1$. Intuitively, any structural shift must be in effect for a nonnegligible proportion of the sample if the tests of this section are to have any power. This means that $m$ must be a function of $n$, leading us to consider a sequence of alternatives $\{H_{1n}\}$,

$$\begin{aligned} H_{1n}\colon Y_{nt} &= \alpha_1 + \beta_1 t + \varepsilon_t, \qquad t = 1, 2, \ldots, m_n \\ Y_{nt} &= \alpha_2 + \beta_2 t + \varepsilon_t, \qquad t = m_n + 1, \ldots, n, \\ &\delta \leq m_n/n \leq 1 - \delta, \quad 0 < \delta \leq \tfrac{1}{2}, \quad \beta_1 \neq \beta_2. \end{aligned}$$

We consider explicitly the consistency of the $T_{1n}^0$ test. Taking $k = m_n$ in (4), we have

$$\begin{aligned} T_{1n}^0 &\geq (6\sigma_0)^{-1}n^{3/2}(m_n/n)^3|\hat{\beta}_{m_n} - \hat{\beta}_n| \\ &\geq (6\sigma_0)^{-1}n^{3/2}\delta^3|\hat{\beta}_{m_n} - \hat{\beta}_n|. \end{aligned}$$

The consistency of the least squares estimator $\hat{\beta}_{m_n}$ establishes that $\hat{\beta}_{m_n}$ establishes that $\hat{\beta}_{m_n} = \beta_1 + o_p(1)$. To determine the convergence behavior of $\hat{\beta}_n$, we simplify the expression $\Delta_n\hat{\beta}_n = n\Sigma_{t=1}^n tY_t - (\Sigma_{t=1}^n \mathrm{t})(\Sigma_{t=1}^n \mathrm{Y}_t)$ with some lengthy algebra (provided in Appendix B) to obtain

$$\hat{\beta}_n = \beta_2 - d(4\theta^3 + 3\theta^2) + o_p(1), \quad (5)$$

where $d \equiv (\beta_2 - \beta_1)$, and $\theta = \lim_{n\to\infty}(m_n/n)$. The convergence behavior of $\hat{\beta}_{m_n}$ and $\hat{\beta}_n$ permits us to conclude that

$$\begin{aligned} T_{1n}^0 &\geq (6\sigma_0)^{-1}n^{3/2}\delta^3 \\ &\quad \times [(1-\delta)(4\delta^2 + \delta + 1)|\beta_2 - \beta_1| + o_p(1)] \to \infty \\ &\qquad \text{as } n \to \infty. \end{aligned}$$

The treatment for $T_{1n}$ is entirely analogous. This establishes the consistency of the tests as summarized in the following theorem.

*Theorem 1.4.* Suppose that $Y_{nt} = \alpha_1 + \beta_1 t + \varepsilon_t (t = 1, 2, \ldots m_n)$, $Y_{nt} = \alpha_2 + \beta_2 t + \varepsilon_t (t = m_n + 1, \ldots n, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R})$, $\delta \le m_n/n \le 1 - \delta, 0 < \delta \le \frac{1}{2}(n = 2, 3, \ldots)$, and that $\theta = \lim_{n\to\infty}(m_n/n)$ exists. If $\{\varepsilon_t/\sigma_0\}$ obeys the FCLT and $\beta_1 \ne \beta_2$, then for $c_n = o(n^{3/2})$ (a) $P[T_{1n} > c_n] \to 1$, as $n \to \infty$ and (b) $P[T_{1n}^0 > c_n] \to 1$, as $n \to \infty$. These conclusions remain true when $\sigma_0$ is replaced by $\hat{\sigma}_n = O_p(n^\kappa)$ and $\hat{\sigma}_n \ge \varepsilon > 0$ with $c_n = o(n^{3/2-\kappa})$, $\kappa \in [0, \frac{3}{2}]$.

Note that an increasing sequence of critical values can be used without destroying the consistency of the test, thus permitting the probabilities of both Type I and Type II errors to be driven to 0 as $n$ increases. Moreover, note that estimators $\hat{\sigma}_n$ consistent for $\sigma_0$ under $H_0$ will typically not be consistent for $\sigma_0$ under $\{H_{1n}\}$; however, this causes no difficulty asymptotically as long as $\hat{\sigma}_n$ does not increase too quickly. If $\{\varepsilon_t\}$ is a martingale difference sequence, then $\kappa = 0$ is appropriate. The case when $\kappa > 0$ arises in the use of heteroscedasticity and autocorrelation-consistent estimators, as pointed out by an anonymous referee.

In general, $T_{1n}$ and $T_{1n}^0$ have power against a range of other alternatives, as can be shown by extending Andrews's (1990) techniques to the present context. We omit treatment of these cases here. We proceed, however, to analyze the behavior of $T_{1n}^0$ when $\{\varepsilon_t/\sigma_0\}$ is integrated of order 1 ($I(1)$), violating the FCLT assumption. An interesting special case of this violation is that in which $Y_t$ is a drifted random walk.

Accordingly, let $Y_t = \alpha + \beta t + \varepsilon_t (\varepsilon_t = \Sigma_{\tau=1}^t \eta_\tau, t = 1, 2, \ldots)$, where now $\{\eta_t/\sigma\}$ obeys the FCLT. We refer to this alternative as $H_2$. Since $\sigma_n^2 = n^{-1}E(S_n^2)$ diverges (recall that $S_n = \Sigma_{t=1}^n \varepsilon_t$), $\sigma_0$ in (2) is no longer well defined. We proceed, however, to show that the $T_{1n}$ statistic is still bounded in probability provided the growth rate of $\sigma_n$ is scaled appropriately. For simplicity, we assume momentarily that $\{\eta_t/\sigma_1\}$ is an iid sequence. In this case,

$$\lim_{n\to\infty} n^{-3}E(S_n^2) = \tfrac{1}{6}\sigma_1^2 \lim_{n\to\infty} [n^{-3}n(n + 1)(2n + 1)] = (\tfrac{1}{3})\sigma_1^2.$$

Because

$$n^{-5/2}\sum_{t=1}^{[n\lambda]} t\varepsilon_t \Rightarrow \sigma_1 \int_0^\lambda rW(r)\,dr$$

and

$$n^{-3/2}\sum_{t=1}^{[n\lambda]} \varepsilon_t \Rightarrow \sigma_1 \int_0^\lambda W(r)\,dr,$$

it follows that

$$Z_n(\lambda) = \tfrac{1}{6}(\lim n^{-2}\sigma_n^2)^{-1/2}n^{1/2}([n\lambda]/n)^3(\hat{\beta}_{[n\lambda]} - \beta_0)$$

$$\Rightarrow \sqrt{3}\left[2\int_0^\lambda rW(r)\,dr - \lambda\int_0^\lambda w(r)\,dr\right].$$

Thus $T_{1n}$ has a well-defined asymptotic distribution even when the error process fails to obey the FCLT. Thus a test based on this scaled version of $T_{1n}$ against $H_2$ would not have power one asymptotically.

If the variance estimator used in constructing the $T_{1n}$ statistic is $\hat{\sigma}_n^2 = n^{-1}\Sigma_{t=1}^n(Y_t - \hat{\alpha}_n - \hat{\beta}_n t)^2$, however, an estimator appropriate when $\{\varepsilon\}$ is thought to be a martingale difference sequence, the resulting test does have power against $H_2$. To see this, let

$$\hat{Z}_n(\lambda) = (6\hat{\sigma}_n)^{-1}n^{3/2}([n\lambda]/n)^3(\hat{\beta}_{[n\lambda]} - \beta_0)$$

$$= (6\hat{\sigma}_n)^{-1}n^{-3/2}[n\lambda]^3(\hat{\beta}_{[n\lambda]} - \beta_0).$$

Because

$$n^{-5/2}[n\lambda]^3(\hat{\beta}_{[n\lambda]} - \beta_0) \Rightarrow 6\sigma_1\left[2\int_0^\lambda rW(r)\,dr - \lambda\int_0^\lambda W(r)\,dr\right]$$

and $n^{-1}\hat{\sigma}_n^2$ is $O_p(1)$ under $H_2$ (see Appendix C), we have

$$\hat{Z}_n(\lambda) = \tfrac{1}{6}n^{1/2}(n^{-1}\hat{\sigma}_n^2)^{-1/2}n^{-5/2}[n\lambda]^3(\hat{\beta}_{[n\lambda]} - \beta_0).$$

In other words, $n^{-1/2}\hat{Z}_n(\lambda) = O_p(1)$, so that $P[\hat{T}_{1n} > c_n] \to 1$ for $c_n = O(n^{1/2})$, where

$$\hat{T}_{1n} \equiv \max_{k\le n} (6\hat{\sigma}_n)^{-1}n^{3/2}(k/n)^3|\hat{\beta}_k - \beta_0|.$$

Similar reasoning applies to $T_{1n}^0$ and $\hat{T}_{1n}^0$ when $\{\varepsilon_t\}$ fails to obey the FCLT. We have the following result.

*Theorem 1.5.* Suppose that $Y_t = \alpha_0 + \beta_0 t + \varepsilon_t (\varepsilon_t = \Sigma_{\tau=1}^t \eta_\tau, \alpha_0 \in \mathbf{R}, t = 1, 2, \ldots)$, where $\{\eta_t/\sigma_1\}$ obeys the FCLT. Then, for $c_n = O(n^{1/2})$, (a) $P[\hat{T}_{1n} > c_n] \to 1$ as $n \to \infty$ and (b) $P[\hat{T}_{1n}^0 > c_n] \to 1$ as $n \to \infty$, where $\hat{\sigma}_n^2 = n^{-1}\Sigma_{t=1}^n(Y_t - \hat{\alpha}_n - \hat{\beta}_n t)^2$. The conclusions remain true when $\hat{\sigma}_n^2$ is replaced with any $O_p(n)$ statistic. If instead $\hat{\sigma}_n^2$ is replaced with $\tilde{\sigma}_n^2 = O_p(n^2)$, then $\hat{T}_{2n}$, $\hat{T}_{2n}^0$ are bounded in probability.

Thus $\hat{T}_{1n}$ and $\hat{T}_{1n}^0$ also yield consistent tests of the null hypothesis that a series is $I(0)$ against the alternative that there is no structural change but $\{\varepsilon_t\}$ is an $I(1)$ process. This reveals an empirical pitfall. If a given time series is a drifted random walk rather than a trend-stationary series, then we may easily have a significant $\hat{T}_{1n}$ or $\hat{T}_{1n}^0$ statistic. One should not draw the conclusion that there was a structural change on this basis, because in fact there may have been no structural break but instead the failure of the FCLT.

The last part of this theorem implies that the test will not detect $H_2$ with probability one when $\tilde{\sigma}_n^2 = O_p(n^2)$. Estimators with this property may result from use of certain of the *heteroscedasticity autocorrelation consistent* estimators mentioned previously. In this situation, failure to reject should not be interpreted solely as evidence consistent with a trend-stationary process. The data-generating process could also be $I(1)$.

Despite the simplicity of the tests of this section and the pitfalls just mentioned, their potential usefulness cannot be overstated. These or similar tests should be routinely applied whenever parameter constancy is needed to justify subsequent analysis. For example, if $Y_t$ is a variable that one is interested in forecasting using a potentially misspecified model (such as a time invariant model), then performing such a test can alert the investigator to the possibility of predictive failure.

### 1.2 Testing for a Change in Trend Intercept

The tests derived in Section 1.1 may have little power against trend intercept break alternatives of the form

$$H_3\colon Y_t = \alpha_1 + \beta_0 t + \varepsilon_t, \qquad t = 1, 2, \ldots m$$
$$Y_t = \alpha_2 + \beta_0 t + \varepsilon_t, \qquad t = m + 1, \ldots n,$$

where $\alpha_1 \neq \alpha_2$. To obtain a test sensitive to $H_3$, we consider the random function $Q_n^0(\lambda) \equiv (2\sigma_0)^{-1}([n\lambda]/n)n^{1/2}(\hat{\alpha}_{[n\lambda]} - \hat{\alpha}_n)(\lambda \in [\delta, 1], \delta > 0)$, where $\hat{\alpha}_k = \Delta_k^{-1}[(\Sigma_{t=1}^k t^2)(\Sigma_{t=1}^k y_t) - (\Sigma_{t=1}^k t)(\Sigma_{t=1}^k t y_t)]$. Given that $\{\varepsilon_t\}$ follows the FCLT and using Lemma 1.1, we find that

$$Q_n^0(\lambda) \Rightarrow \left[(3/\lambda)\int_0^\lambda W(r)\,dr - W(\lambda)\right]$$
$$\times \left[(3\lambda)\int_0^1 W(r)\,dr - \lambda W(1)\right] \equiv H^0(\lambda).$$

Examining the covariance structure of $H^0(\lambda)$, we see that $H^0(\lambda) \overset{d}{=} W^0(\lambda)$ for $\lambda \in [\delta, 1]$. This implies that $\lim P[T_{2n}^0 > c] = P[\sup_{\lambda\in[\delta,1]}|W^0(\lambda)| > c]$, where

$$T_{2n}^0 \equiv \max_{\delta n \le k \le n} (2\sigma_0)^{-1} n^{1/2}(k/n)|\hat{\alpha}_k - \hat{\alpha}_n|. \quad (6)$$

A test for a change in the trend intercept can be performed using the $T_{2n}^0$ statistic, where the critical value for the $T_{2n}^0$ test is approximated by the hitting probability of a Brownian bridge, without the $\sqrt{3}$ adjustment. Note that an $n^{1/2}$ scaling is used in $T_{2n}^0$ in contrast to the $n^{3/2}$ scaling obtained with trending regressor.

We conclude this section by noting that the previous analysis can be extended to test for structural change in the cointegration between two $I(1)$ series (Engle and Granger 1987). Let the null hypothesis be

$$H_0\colon Y_t = \alpha_0 + \beta_0 X_t + \varepsilon_t,$$
$$X_t = \gamma_0 + X_{t-1} + v_t;\ \gamma_0 \neq 0, \qquad t = 1, 2, \ldots, n.$$

The alternative of interest is

$$H_1\colon Y_t = \alpha_1 + \beta_1 X_t + \varepsilon_t, \qquad t = 1, 2 \ldots m$$
$$Y_t = \alpha_2 + \beta_2 X_t + \varepsilon_t, \qquad t = m + 1 \ldots n,$$

where $m$ is unknown. Now the regressor contains a stochastic trend rather than a deterministic trend. It was shown by Chu and White (1991) that the statistic

$$T_{3n}^0 \equiv \max_{2\le k\le n-1} (\gamma_0/6\sigma_0)n^{3/2}(k/n)^3|\hat{\beta}_k - \hat{\beta}_n|,$$

where

$$\hat{\beta}_k = \Delta_k^{-1}\left[k\sum_{t=1}^k X_t Y_t - \left(\sum_{t=1}^k X_t\right)\left(\sum_{t=1}^k Y_t\right)\right],$$
$$\Delta_k = k\sum_{t=1}^k X_t^2 - \left(\sum_{t=1}^k X_t\right)^2,$$

can be used to test the constancy of cointegration, based on the fact that $T_{3n}^0 \Rightarrow \sup_{\lambda\in[0,1]}|W^0(\lambda)|/\sqrt{3}$ under $H_0$. This result is to be expected because a drifted $I(1)$ series is dominated by its time-trend component asymptotically.

## 2. MONTE CARLO EXPERIMENTS

### 2.1 Size Simulations

The purpose of the foregoing theory is to construct tests based on $T_{in}$ and $T_{1n}^0 (i = 1, 2)$ having correct size asymptotically. The way we argue asymptotics is to let the time interval between observations converge to 0, and only by such a device can we have an asymptotic distribution in the space of continous functions. To see how relevant these asymptotic arguments are, we conduct some Monte Carlo experiments. We use the following notation: $\hat{T}_{in}^0$ is the statistic obtained by replacing the unknown $\sigma_0$ in $T_{in}^0$ with the estimator $\hat{\sigma}_n$, where $\hat{\sigma}_n^2$ is the usual estimated variance of the ordinary least squares residuals of the corresponding regression model and $\tilde{T}_{in}^0$ is obtained by replacing the unknown $\sigma_0$ in $T_{in}^0$ with $\tilde{\sigma}_n$, where $\tilde{\sigma}_n^2$ is a heteroscedasticity and autocorrelation consistent estimator of $\sigma_0^2$.

We start with the $\hat{T}_{1n}^0$ $(i = 1, 2)$ size simulations for an ideal normal iid error. We see from Table 1 that the empirical sizes of the $\hat{T}_{1n}^0$ test are satisfactory, though they are slightly smaller than the nominal significance level.

To examine a more interesting case of nonspherical error terms, we generate data from Gaussian (autoregressive) AR(1) processes with some selected autoregressive parameters. To estimate $\sigma_0^2$, we first use the Newey–West (1987) consistent estimator in implementing the $\tilde{T}_{1n}^0$ test. Table 2 displays the empirical size of the $\tilde{T}_{1n}^0$ test with a range of ad hoc choices of truncation lag.

We see from Table 2 that the quality of the asymptotic approximation in finite samples is generally acceptable,

*Table 1. Empirical Size of $\hat{T}_{1n}^0$ and $\hat{T}_{2n}^0$ Tests*

| | $\hat{T}_{1n}^0$ | | $\hat{T}_{2n}^0$ | |
|---|---|---|---|---|
| N | 10% | 5% | 10% | 5% |
| 100 | 9.80% | 5.10% | 9.70% | 4.65% |
| 300 | 9.60% | 4.85% | 9.35% | 4.45% |

NOTE: Data are generated by $Y_t = 5 + .2t + \varepsilon_t$; $\{\varepsilon_t\}$ is iid $N(0, 1)$. The number of replications = 5,000. $N$ denotes the sample size.

Table 2. Empirical Size of $\tilde{T}^0_{1n}$ Test With Serially Correlated Errors

| | TL = 2 | | TL = 5 | | TL = 8 | | TL = 10 | |
|---|---|---|---|---|---|---|---|---|
| N | 10% | 5% | 10% | 5% | 10% | 5% | 10% | 5% |
| 100 | 15.64% | 8.95% | 9.76% | 4.72% | 9.05% | 3.45% | 8.25% | 2.25% |
| 300 | 19.80% | 9.36% | 14.20% | 7.80% | 10.20% | 5.08% | 9.50% | 4.85% |

NOTE: Data are generated by $Y_t = 5 + .2t + \varepsilon_t$; $\varepsilon_t = .5\varepsilon_{t-1} + v_t$; $\{v_t\}$ is iid $N(0, 1)$. The number of replications = 5,000. TL denotes an ad hoc choice of truncation lag used in computing the Newey–West estimator.

given the proper choice of truncation lag. The size can be quite bad, however, for an improper lag truncation number. For example, in the case of sample size equal to 300, the choice of truncation lag 8 delivers satisfactory empirical size, but the choice of 5 results in 14.2% empirical size versus 10% nominal size. These findings underscore the importance of having a reliable method for choosing the truncation lag.

Recently, Andrews (1991) has suggested using an automatic bandwidth procedure to estimate the truncation lag. We use Andrews's automatic bandwidth estimator to simulate the $\tilde{T}^0_{1n}$ and the $\tilde{T}^0_{2n}$ statistic; results are summarized in Table 3.

Table 3 reveals how empirical size is affected by the underlying autoregressive coefficient $\rho$. In particular, the $\tilde{T}^0_{in}$ test becomes conservative as $\rho$ get closer to unity. In fact, this is theoretically expected. When $\rho$ = .9, Andrews's (1991) procedure yields an optimal fixed truncation lag of 34 for sample size 300. The choice 20, however, delivers more accurate empirical size (see Chu and White 1991). Thus the empirical size resulting from truncation lag 34 is too small relative to the nominal size. This observation also warns against indiscriminate use of the automatic bandwidth procedure to implement the $\tilde{T}^0_{1n}$ test when serious autocorrelation is present.

### 2.2 Power Simulations

To simulate the empirical power of the $\hat{T}^0_{1n}$ and $\tilde{T}^0_{1n}$ tests, we generate data from a piecewise trend-stationary process. Although there is a one-time structural change in the trend coefficient, the broken trend function is formulated to be continuous everywhere in our simulation. As for the power simulation of the $\hat{T}^0_{2n}$ and $\tilde{T}^0_{2n}$, we create data from a model that contains a one-time discrete jump in trend intercept. Results are summarized in Tables 4 and 5.

There are three parameters in Table 4—the break ratio, the magnitude of the change, and the variance-of-error term. We observe that the power of the $\hat{T}^0_{1n}$ test is quite impressive in the case of iid errors. Power of the $\hat{T}^0_{1n}$ test against alternatives in which the structural change occurs near the two ends of the sample is found to be low. This conforms well with intuition, given the weighting scheme $(k/n)^3$ in (4). Though we have fixed the magnitude of change in the simulation (from $\beta$ = .2 to .24; see the note to Table 4), results not reported here show that for the investigated range of values the larger the magnitude of change, the higher the power (see Chu and White 1991, fig. 1).

When positively autocorrelated errors are present, the power of the test decreases. Even a slightly autocorrelated error has nontrivial negative effect on the power (see Table 5). Heuristically, the structural change information contained in the sample is obscured by the autocorrelation, and the more seriously autocorrelated the error, the more slowly is the information revealed. This finding confirms the common knowledge that nonspherical errors need not destroy the size of the test but may affect the power of tests unfavorably.

## 3. EMPIRICAL APPLICATION

### 3.1 Broken Trends in Macroeconomic Time Series

Since Nelson and Plosser's (1982) influential studies on the nonstationary nature of major U.S. economic time series, the thought that most economic time series are well characterized as containing a unit root has been

Table 3. Empirical Size of $\tilde{T}^0_{1n}$ and $\tilde{T}^0_{2n}$ Tests With Automatic Bandwidth

| | $\tilde{T}^0_{1n}$ | | $\tilde{T}^0_{2n}$ | |
|---|---|---|---|---|
| $\rho$ | 10% | 5% | 10% | 5% |
| .1 | 10.25% | 4.95% | 10.10% | 5.03% |
| .3 | 9.75% | 5.10% | 11.98% | 6.05% |
| .5 | 9.84% | 4.42% | 12.45% | 5.18% |
| .7 | 9.03% | 3.56% | 11.25% | 4.50% |
| .9 | 7.04% | 1.74% | 10.75% | 2.20% |

NOTE: Data are generated by $Y_t = 5 + .2t + \varepsilon_t$; $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$; $\{v_t\}$ is iid $N(0, 1)$. The sample size = 100; the number of replications = 5,000.

Table 4. Empirical Power of $\hat{T}^0_{1n}$ and $\hat{T}^0_{2n}$ Tests

| Break ratio ($r$) | $\hat{T}^0_{1n}$ 10% | $\hat{T}^0_{1n}$ 5% | $\hat{T}^0_{2n}$ 10% | $\hat{T}^0_{2n}$ 5% |
|---|---|---|---|---|
| .2 | .70 | .55 | 1.00 | 1.00 |
| .4 | 1.00 | 1.00 | .99 | .95 |
| .6 | 1.00 | 1.00 | 1.00 | .99 |
| .8 | .99 | .96 | 1.00 | .99 |

NOTE: The data for the $\hat{T}^0_{1n}$ power simulation are generated from $Y_t = 5 + .2t + \varepsilon_t$, for $t < 100 \times r$ and $Y_t = 5 + .2t + .04(t - 100 \times r) + \varepsilon_t$, for $t \geq 100 \times r$. For the $\hat{T}_{2n}$ test, we generate data from $Y_t = 5 + .2t + \varepsilon_t$, for $t < 100 \times r$ and $Y_t = 3 + .2t + \varepsilon_t$, for $t \geq 100 \times r$. We have $\{\varepsilon_t\}$ iid $N(0, 1)$ in all cases. The sample size = 200; the number of replications = 2,000.

Table 5. Empirical Power of $\tilde{T}^0_{1n}$ with Autocorrelated Errors

| r/ρ | .0 | .1 | .3 | .5 | .7 |
|---|---|---|---|---|---|
| .2 | .70 | .50 | .34 | .19 | .09 |
| .4 | 1.00 | 1.00 | .99 | .84 | .21 |
| .6 | 1.00 | 1.00 | 1.00 | .96 | .35 |
| .8 | .99 | .96 | .83 | .56 | .18 |

NOTE: The sample size = 200; the number of replications = 2,000. $Y_t = 5 + .2t + \varepsilon_t$, for $t < 100 \times r$; $Y_t = 5 + .2t + .04(t - 100 \times r) + \varepsilon_t$, for $t \geq 100 \times r$; and $\varepsilon_t = \rho\varepsilon_{t-1} + v_t$; $\{v_t\}$ is iid $N(0, 1)$.

overwhelmingly accepted. Perron (1989) challenged this view by showing that the standard tests for the unit-root hypothesis against the trend-stationary alternative are inconsistent if the trend function contains a one-time break. Perron examined the evidence for the trend-shift hypothesis and obtained some surprising and controversial results—namely, that for 11 out of 14 economic time series studied by Nelson and Plosser the unit-root hypothesis is rejected rather decisively. Hence Perron strongly argued that many time series have been mistakenly thought to contain a unit root when in fact they are better characterized by trend-stationary series with structural change.

In Perron's work, however, the breaks in the trend function were assumed known to occur at either the Great Depression or the Oil Shock. Such an ad hoc specification of breakpoints is problematic and controversial. Christiano (1988) and Banerjee, Lumsdaine, and Stock (1992) argued that it is more appropriate not to assume that the breakpoint is known but rather to let the estimation of the breakpoint be an integral part of the econometric procedure. Applying their tests for the unit root against trend stationarity with unknown breakpoint, Banerjee et al. rejected Perron's hypothesis and concluded that there is no statistical evidence against the unit-root hypothesis for postwar U.S. gross national product (GNP).

We shall use the test statistics $\tilde{T}^0_{1n}$ and $\tilde{T}^0_{2n}$ to test Perron's hypothesis directly. The logic is simply that if the hypothesis of trend stationary with structural break is correct, we should be able to reject the null hypothesis using our $\tilde{T}^0_{1n}$ or $\tilde{T}^0_{2n}$ tests. Note that Banerjee et al. (1992) tested for the unit-root hypothesis against an explicit alternative of trend break, whereas we are testing for departure from the trend-stationarity hypothesis.

### 3.2 Data and Results

We apply our tests to the data set used by Nelson and Plosser (1982) and Perron (1989). The data set consists of 14 major macroeconomic series (see Tables 6 and 7). A detailed description of the data can be found in the article by Perron (1989). Following convention, we take natural logarithms of each time series except the interest rate.

For each time series $Y_t$, we fit a linear time-trend model $Y_t = a + bt + e_t$ and compute the $\tilde{T}^0_{1n}$ and $\tilde{T}^0_{2n}$ statistics as in (4) and (6) with

$$\tilde{\sigma}^2_n = n^{-1} \sum_{t=1}^{n} e_t^2 + 2n^{-1} \sum_{\tau=1}^{l} \omega_{\tau l} \sum_{t=\tau+1}^{n} e_t e_{t-\tau},$$

where $\omega_{\tau l} = 1 - \tau/(l + 1)$. Since we use Andrews's (1991) automatic bandwidth procedure with AR(1) specification for $e_t$ to determine the truncation lag $l$, we refer to the resulting truncation lag as ABE (automatic bandwidth estimator). Tables 6 and 7 summarize the corresponding results for each series.

Of the 11 series reported in Table 6, the null hypothesis of trend-intercept constancy is rejected by the $\tilde{T}^0_{2n}$ test at the 10% level or better in 7 cases—real GNP, nominal GNP, real per capita GNP, GNP deflator, consumer prices, wages, and money stock. The $\tilde{T}^0_{1n}$ statistics indicate that GNP deflator and consumer prices exhibit trend-slope changes as well. Since our $\tilde{T}^0_{2n}$ test is sensitive to structural change in trend intercept, we tentatively conclude that for these seven series Perron's "Great Crash" hypothesis is not contradicted. We do not imply that the Great Crash hypothesis is correct because other alternatives are pos-

Table 6. Tests for Trend Constancy of Eleven Time Series

| Time series | N | ABE | $\tilde{T}^0_{1n}$ | $\tilde{T}^0_{2n}$ | a | b |
|---|---|---|---|---|---|---|
| Real GNP | 62 | 17 | .54 | 1.38[b] | 4.58 | .031 |
| Nominal GNP | 62 | 27 | .54 | 1.34[a] | 10.29 | .053 |
| Real per capita GNP | 62 | 16 | .55 | 1.38[b] | 6.98 | .018 |
| Industrial production | 111 | 17 | .52 | 1.15 | .008 | .042 |
| Employment | 81 | 20 | .45 | 1.22 | 10.08 | .015 |
| GNP deflator | 82 | 29 | .74[a] | 1.25[a] | 2.98 | .022 |
| Consumer prices | 111 | 92 | .91[b] | 2.03[c] | 3.15 | .011 |
| Wages | 71 | 29 | .60 | 1.24[a] | 6.04 | .040 |
| Money stock | 82 | 35 | .69 | 1.67[c] | 1.28 | .058 |
| Velocity | 102 | 33 | .67 | 1.06 | 1.42 | −.012 |
| Interest rate | 71 | 31 | .68 | .98 | 3.68 | .005 |

NOTE: The columns a and b are the least squares intercept and slope estimates using $N$ observations. ABE is the truncation lag determined from Andrews's automatic bandwidth estimation procedure.

[a] Statistical significance at the 10% level.

[b] Statistical significance at the 5% level.

[c] Statistical significance at the 1% level.

*Table 7. Tests for Trend Constancy of Three Time Series*

| Time series | N | ABE | $\tilde{T}^0_{1n}$ | $\tilde{T}^0_{2n}$ | a | b |
|---|---|---|---|---|---|---|
| Common stock prices | 100 | 27 | 0.79[a] | 1.11 | 1.04 | .028 |
| Real wages | 71 | 17 | 0.62 | 1.36[b] | 2.83 | .020 |
| Postwar quarterly real GNP | 159 | 60 | 0.66 | 1.14 | 7.02 | .008 |

NOTE: See the note to Table 6.
[a] Statistical significance at the 1% level.
[b] Statistical significance at the 5% level.

sible. Our use of $\tilde{\sigma}^2_n$, however, provides some insurance against our being deceived by drifted random walks.

We report both the $\tilde{T}^0_{1n}$ and $\tilde{T}^0_{2n}$ statistics for each of the remaining three series in Table 7. Only two of the six statistics are significant, the $\tilde{T}^0_{1n}$ statistic for the common-stock price index at the 1% level and the $\tilde{T}^0_{2n}$ statistic for the real wage at the 5% level.

As previously emphasized, the choice of truncation lag is crucial for ensuring proper size of our tests. Estimating the truncation lag using an AR(1) specification for residuals $e_t$ is convenient but need not be optimal (see Andrews [1991] for discussion). It is thus useful to see how sensitive our previous conclusions are to the choice of the truncation lag. In results not tabulated here, we computed the $\tilde{T}^0_{2n}$ statistic for all 14 series using every possible truncation lag. Our findings can be summarized as follows.

First, for real GNP, real per capita GNP, consumer prices, and real wages, the previous conclusions do not hinge on the choice of the truncation lag. Specifically, we can reject the null hypothesis for real GNP at the 5% level or better for all possible choices of the truncation lag. We also reject the null for the rest of three series at the 10% level or better for all truncation lags. Our evidence is thus consistent with the Great Crash hypothesis (i.e., a trend with a break in level). In contrast with Perron's (1989) findings in which the unit-root hypothesis for consumer prices is not rejected, we find that the consumer price series is potentially characterized by a trend with break in trend intercept.

Next, we have some series semirobust to the choice of truncation lag—money stock, velocity, interest rates, common stock, and postwar GNP. Given $l$ = ABE = 35 in Table 6, we reject the null hypothesis for money stock at the 1% level; the rejection of the null hypothesis remains justified at the 10% level or better for all other choices of the truncation lag except for $5 \leq l \leq 20$. As for velocity, interest rates, common stock, and post-war GNP, we cannot reject the null at the 10% level for $l$ = ABE = 33, $l$ = ABE = 31, $l$ = ABE = 60, and $l$ = ABE = 27, respectively (see Tables 6 and 7). The previous conclusions hold at the 10% level or better for all truncation lags $12 \leq l \leq 61$, $4 \leq l \leq 39$, $11 \leq l \leq 44$, and $16 \leq l \leq 77$, respectively.

Finally, we have five series very sensitive to the choice of the truncation lag—nominal GNP, industrial production, employment, GNP deflator, and wages. For GNP deflator and wages, we reject the null hypothesis at the 10% level for $l$ = ABE = 29. If we change the truncation lag from 29 to 28, however, we fail to reject the trend-constancy hypothesis. The employment series is equally sensitive. We fail to reject the null hypothesis at $l$ = ABE = 20, but do reject the null at $l = 21$. Although nominal GNP and industrial production are a little less sensitive, the previous rejection for nominal GNP does not occur for $7 \leq l \leq 22$, but the previous nonrejection for industrial production occurs only for $3 \leq l \leq 40$.

We also computed the $\tilde{T}^0_{1n}$ statistic for a variety of values of the truncation lag for common stock, real wages, and postwar GNP. Previously, we failed to reject the null hypothesis of trend constancy at $l$ = ABE = 17 and $l$ = ABE = 60 for real wages and postwar GNP, respectively. This finding is not reversed for $8 \leq l \leq 37$ and $16 \leq l \leq 77$, respectively. For common stock, the rejection at $l$ = ABE = 27 is not maintained for $20 \leq l \leq 25$.

For velocity, interest rates, and postwar GNP, failure to reject the null hypothesis of trend-slope constancy is not very sensitive to the choice of the truncation lag. Although the null hypothesis cannot be rejected for these three series, care is needed in interpreting such results. Our failure to reject the null hypothesis may result from a lack of power due to a small magnitude of change or severe autocorrelation or both. Moreover, if the error process for the given time series contains a unit root (i.e., a drifted random walk), Andrews's automatic bandwidth estimator $\tilde{\sigma}^2_n$ appears to be $O_p(n^2)$. As a consequence of Theorem 1.5, $\tilde{T}^0_{1n}$ will be bounded in probability. Instead of a complicated theoretical analysis for the $\tilde{T}^0_{1n}$ statistic when the FCLT fails, we have generated data from a drifted random walk for 10,000 replications and obtained the empirical distribution of the $\tilde{T}^0_{1n}$ statistic (not reported here). Our simulation results show that the $\tilde{T}^0_{1n}$ test is likely to be insignificant if $Y_t$ is in fact a drifted random walk. Thus failure to reject the null hypothesis using the $\tilde{T}^0_{1n}$ statistic does not necessarily imply that the given time series is constant-trend stationary. An insignificant $\tilde{T}^0_{1n}$ statistic is consistent with either (a) trend stationarity with $I(0)$ error term or (b) a unit root. Discrimination between trend stationarity and a unit root is a separate issue, beyond the scope of our analysis here.

To summarize, we find that for real capita GNP, consumer prices, and real wages before World War II Perron's Great Crash hypothesis is not contradicted. The trend-shift hypothesis is an attractive characterization for pre-war real GNP but not for post-war real GNP. Money stock seems to exhibit a trend break in intercept. For velocity, interest rates, and common stock, the evidence of trend break is weak. Our test is not definitive for the five series—nominal GNP, industrial production, employment, GNP deflator, and wages—because the test outcomes of these series are very sensitive to the truncation lag.

## 4. CONCLUDING REMARKS

In this article, we have considered tests for structural change in trend slope and intercept. Perhaps surprisingly, the critical values from the asymptotic null distribution for testing structural change in nonstationary time series (e.g., testing for change in trend slope or cointegration) differs from those in the stationary time series models merely by a constant of proportionality, $\sqrt{3}$.

The tests proposed in this article are generally conservative. This is perhaps due to our weak requirements on the error process. To remedy this, prewhitening the residuals may be helpful. This prewhitening introduces its own complications, so we leave the study of such methods to other work.

Finally, we caution that the empirical maximum or the empirical first hitting time of the $\tilde{T}^0_{1n}$ statistic is not very accurate in locating the breakpoint, unless the structural break occurs around the middle of the sample. Proper parameterization of the error process can improve accuracy in locating the true break (see Hinkley 1970; Kim and Siegmund 1989). Nevertheless, the tests considered in this article appear to have nontrivial power against general alternatives of parameter nonconstancy. Because rejecting the null hypothesis does not necessarily imply one-time structural change, care is warranted in drawing conclusions about the true data-generating process.


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## APPENDIX A: PROOFS OF LEMMAS 1.1 and 1.2

### A.1 Proof of Lemma 1.1

Let $R_n(\lambda) = \sigma^{-1}n^{-1/2}S_{[n\lambda]} \equiv \sigma^{-1}n^{-1/2}\sum_{j=1}^{[n\lambda]}\varepsilon_j$. Observe that

$$\sum_{i=1}^{k}\int_{i-1/n}^{i/n}\sigma^{-1}n^{-1/2}S_{[n\lambda]}d\lambda = \int_0^{k/n}R_n(r)\,dr.$$

The left side of the preceding equation is equal to

$$\begin{aligned}\sum_{i=1}^{k}\sigma^{-1}n^{-3/2}S_{i-1} &= n^{-3/2}\sigma^{-1}\sum_{i=2}^{k}\sum_{t=1}^{i-1}\varepsilon_t \\ &= n^{-3/2}\sigma^{-1}\sum_{t=1}^{k-1}(k-t)\varepsilon_t \\ &= n^{-3/2}\sigma^{-1}\left[\sum_{t=1}^{k-1}k\varepsilon_t - \sum_{t=1}^{k-1}t\varepsilon_t\right] \\ &= n^{-3/2}\sigma^{-1}\left[\sum_{t=1}^{k}k\varepsilon_t - \sum_{t=1}^{k}t\varepsilon_t\right].\end{aligned}$$

Hence

$$\begin{aligned}n^{-3/2}\sigma^{-1}\sum_{t=1}^{k}t\varepsilon_t &= n^{-3/2}\sigma^{-1}\sum_{t=1}^{k}k\varepsilon_t - \int_0^{k/n}R_n(r)\,dr \\ &= (k/n)^{-3/2}\sigma^{-1}(1/\sqrt{k})\sum_{t=1}^{k}\varepsilon_t - \int_0^{k/n}R_n(r)\,dr \Rightarrow \lambda W(\lambda) - \int_0^{\lambda}W(r)\,dr.\end{aligned}$$

### A.2 Proof of Lemma 1.2

The covariance function of $G(\lambda)$ is given by

$$\begin{aligned}E[G(\lambda)G(s)] &= E\left[\lambda W(\lambda) - 2\int_0^{\lambda}W(r)\,dr\right] \times \left[sW(s) - 2\int_0^{s}W(r)\,dr\right] \\ &= E\Big[\lambda sW(\lambda)W(s) - 2sW(s)\int_0^{\lambda}W(r)\,dr - 2\lambda W(\lambda) \\ &\quad \times \int_0^{s}W(r)\,dr + 4\int_0^{\lambda}W(r)\,dr\int_0^{s}W(r)\,dr\Big]. \qquad (A.1)\end{aligned}$$

It suffices to perform the analysis using the following conditions. Assume that $\{\varepsilon_t\}$ is iid with mean 0 and variance 1 and that $q$ and $k$ are such that $q/n \to s$, $k/n \to \lambda$, and $k > q$. Then

$$\begin{aligned}&E\left[n^{-1/2}\sum_{j=1}^{q}\varepsilon_j\right]\left[n^{-1}\sum_{j=1}^{k}n^{-1/2}S_{j-1}\right] \\ &= n^{-1}E[n^{-1/2}S_q(n^{-1/2}S_1 + n^{-1/2}S_2 + \cdots + n^{-1/2}S_{k-1})] \\ &= (1/n)[(1/n) + 2(1/n) + \cdots + \underbrace{q(1/n) + \cdots + q(1/n)}_{(k-q)\text{ terms}}] \\ &= (1/n^2)(1 + 2 + \cdots + q) + (1/n^2)q(k-q-1) \\ &= (1/n^2)q(q+1)/2 + (1/n^2)(kq - q^2 - q) \\ &\to s^2/2 + \lambda s - s^2 = \lambda s - s^2/2. \qquad (A.2)\end{aligned}$$

Similarly, we compute that

$$E\left[n^{-1/2}\sum_{j=1}^{k} e_j\right]\left[(1/n)\sum_{j=1}^{q} n^{-1/2}S_{j-1}\right] \to s^2/2 \quad \text{(A.3)}$$

and

$$E\left[(1/n)\sum_{j=1}^{q} n^{-1/2}S_{j-1}\right]\left[(1/n)\sum_{j=1}^{k} n^{-1/2}S_{j-1}\right] \to s^3/3 + (s^2\lambda - s^3)/2. \quad \text{(A.4)}$$

Using (A.2)–(A.4) to evaluate (A.1), we obtain

$$\text{(A1)} = \lambda s^2 - 2s(\lambda s - s^2/2) - 2\lambda s^2/2 + 4[s^3/3 + (s^2\lambda - s^3)/2] = s^3/3.$$

## APPENDIX B: ALGEBRAIC DERIVATION OF EQUATION (5)

For notational convenience, we sometimes write $m_n$ as $m(n)$. Because

$$\sum_{t=1}^{n} tY_t = \sum_{t=1}^{m(n)} tY_t + \sum_{t=m(n)+1}^{n} tY_t$$
$$= \alpha_1 \sum_{t=1}^{m(n)} t + \alpha_2 \sum_{t=m(n)+1}^{n} t + \beta_1 \sum_{t=1}^{m(n)} t^2 + \beta_2 \sum_{t=m(n)+1}^{n} t^2 + \sum_{t=1}^{n} t\varepsilon_t,$$

and

$$\sum_{t=1}^{n} Y_t = \sum_{t=1}^{m(n)} Y_t + \sum_{t=m(n)+1}^{n} Y_t$$
$$= m_n\alpha_1 + (n - m_n)\alpha_2 + \beta_1 \sum_{t=1}^{m(n)} t + \beta_2 \sum_{t=m(n)+1}^{n} t + \sum_{t=1}^{n} \varepsilon_t,$$

we may substitute these results into the expression for $\Delta_n\hat{\beta}_n$ and obtain

$$\Delta_n\hat{\beta}_n = n\alpha_1 \sum_{t=1}^{m(n)} t + n\alpha_2 \sum_{t=m(n)+1}^{n} t + \left(\sum_{t=1}^{n} t\right)(m_n\alpha_1 + (n - m_n)\alpha_2) + n\beta_1 \sum_{t=1}^{m(n)} t^2$$
$$+ n\beta_2 \sum_{t=m(n)+1}^{n} t^2 + n\sum_{t=1}^{n} t\varepsilon_t - \left(\sum_{t=1}^{n} t\right)\left(\beta_1 \sum_{t=1}^{m(n)} t\right) - \left(\sum_{t=1}^{n} t\right)\left(\beta_2 \sum_{t=m(n)+1}^{n} t\right) - \left(\sum_{t=1}^{n} t\right)\left(\sum_{t=1}^{n} \varepsilon_t\right).$$

It follows that

$$\Delta_n\hat{\beta}_n = n\alpha_1 \sum_{t=1}^{m(n)} t + n\alpha_2 \sum_{t=m(n)+1}^{n} t + \left(\sum_{t=1}^{n} t\right)(m_n\alpha_1 + (n - m_n)\alpha_2)$$
$$+ n\beta_2 \sum_{t=1}^{n} t^2 - nd \sum_{t=1}^{m(n)} t^2 + d\left(\sum_{t=1}^{n}\right)\left(\sum_{t=1}^{m(n)} t\right)$$
$$- \beta_2\left(\sum_{t=1}^{n} t\right)^2 + n\sum_{t=1}^{n} t\varepsilon_t - \left(\sum_{t=1}^{n} t\right)\left(\sum_{t=1}^{n} \varepsilon_t\right),$$

where $d = (\beta_2 - \beta_1)$. Because $n\Sigma_{t=1}^{m(n)} t/\Delta_n$, $n\Sigma_{t=m(n)+1}^{n} t/\Delta_n$, and $m_n(\Sigma_{t=1}^{n} t)/\Delta_n$ all converge to 0 as $n \to \infty$ and because $n\Sigma_{t=1}^{n} t^2/\Delta_n \to 4$, $(\Sigma_{t=1}^{n} t)^2/\Delta_n \to 3$, $n\Sigma_{t=1}^{n} t\varepsilon_t/\Delta_n \xrightarrow{p} 0$, $(\Sigma_{t=1}^{n} t)(\Sigma_{t=1}^{n} \varepsilon_t)/\Delta_n \xrightarrow{p} 0$, $\lim_{n\to\infty} n\Sigma_{t=1}^{m(n)} t^2/\Delta_n = 4\theta^3$, and $\lim_{n\to\infty} (\Sigma_{t=1}^{n} t)(\Sigma_{t=1}^{m(n)} t)/\Delta_n = 3\theta^2$, where $\theta = \lim_{n\to\infty} m_n/n$, it follows that $\hat{\beta}_n = \beta_2 - 4d\theta^3 + 3d\theta^2 + o_p(1)$.

## APPENDIX C: PROOF THAT $n^{-1}\hat{\sigma}_n^2$ IS $O_p(1)$

All of the following summations are indexed by $t$ running from 1 to $n$. By definition,

$$\hat{\sigma}_n^2 = \sum (Y_t - \hat{\alpha}_n - \hat{\beta}_n t)^2/n$$

$$n\hat{\sigma}_n^2 = \sum \varepsilon_t^2 - \Delta_n^{-1}\left[\left(\sum \varepsilon_t\right)^2\left(\sum t^2\right) - 2\left(\sum \varepsilon_t\right)\left(\sum t\varepsilon_t\right)\left(\sum t\right) + n\left(\sum t\varepsilon_t\right)^2\right].$$

Straightforward algebra gives

$$\Delta_n^{-1}\left(\sum t^2\right) = (12/n^4 - n^2)(n(n + 1)(2n + 1)/6]$$
$$= 2[(2n + 1)/n(n - 1)]$$
$$\Delta_n^{-1} = 12/n(n + 1)(n - 1)$$
$$\Delta_n^{-1}\left(\sum \mathrm{t}\right) = (12/n^4 - n^2)[n(n + 1)/2] = 6/n(n - 1).$$

Hence

$$\sigma_n^2 = \sum \varepsilon_t^2 - [(4n + 2)/n^2 - n)]\left(\sum \varepsilon_t\right)^2 - [12/n(n + 1)(n - 1)]\left(\sum t\varepsilon\right)^2 + [12/n(n - 1)]\left(\sum \varepsilon_t\right)\left(\sum t\varepsilon_t\right).$$

Multiplying $n^{-2}$ through the preceding equation, we have

$$n^{-1}\hat{\sigma}_n^2 = \sum \varepsilon_t^2 - [(4n+2)/n^2 - n)]n\left(n^{-3/2}\sum \varepsilon_t\right)^2$$
$$- [12/n(n+1)(n-1)]n^3\left(n^{-5/2}\sum t\varepsilon_t\right)^2$$
$$+ [12/n\left(n-1)\right]n(n^{-1/2}\sum \varepsilon_t\Big)\left(n^{-5/2}\sum t\varepsilon_t\right)$$
$$\Rightarrow \sigma_0^2\left[\int W^2(r)\,dr - 4\left[\int W(r)\,dr\right]^2 - 12\int rW(r)\,dr\right].$$

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