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NONPARAMETRIC TESTS FOR THE CHANGEPOINT PROBLEM

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Abstract: Let $X_1, \ldots, X_{[n\lambda]}, X_{[n\lambda]+1}, \ldots, X_n$ be independent, continuous random variables such that X_i, $i = 1, \ldots, [n\lambda]$, have distribution H, and X_i, $i = [n\lambda]+1, \ldots, n$, have distribution function G. The integer $[n\lambda]$ refers to the changepoint of the random sequence. We propose nonparametric procedures for testing $\lambda = 1$ (no changepoint) versus $\lambda \in (0, 1)$. An asymptotically strongly consistent estimator of λ is also given.

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1. Introduction

Let $X_1, \ldots, X_n$ be independent *continuous* random variables. We want to test the null hypothesis

$$\mathrm{H}_0: \quad X_i,\ 1 \le i \le n, \text{ have the same distribution.}$$

The alternative hypothesis is that there is a changepoint in the sequence $X_1, \ldots, X_n$, namely that we have

$$\begin{aligned} \mathrm{H}_1: \quad & \text{there is a } \lambda \in (0, 1) \text{ such that } P\{X_1 \le t\} = P\{X_2 \le t\} = \cdots \\ & = P\{X_{[n\lambda]} \le t\},\ P\{X_{[n\lambda]+1} \le t\} = \cdots = P\{X_n \le t\},\ -\infty < t < \infty, \\ & \text{and } P\{X_{[n\lambda]} \le t_0\} \ne P\{X_{[n\lambda]+1} \le t_0\} \text{ for some } t_0. \end{aligned}$$

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First we introduce a few notations. Let

$$H(t)=P\{X_{[n\lambda]}\le t\},\qquad G(t)=P\{X_{[n\lambda]+1}\le t\}$$

and

$$F(t)=\lambda H(t)+(1-\lambda)G(t).$$

Under the null hypothesis we have $F=H$. Let $Q(y)=\inf\{t:F(t)\ge y\}$, $0<y<1$, the inverse of F.

Tests for a changepoint of a random sequence have been studied by many authors. Assuming that H and G are normal distribution functions, Chernoff and Zacks (1964) proposed test statistics for detecting shifts in the mean of a normal distribution function. Their results were generalized to exponential family by Kander and Zacks (1966), Gardner (1969) and MacNeill (1974). Recently Matthews, Farewell and Pyke (1985) studied maximal score statistics to test for constant hazard against a changepoint alternative. Haccou and Meelis (1985) proposed likelihood ratio test for a change point in a sequence of independent exponentially distributed random variables. The latter test is shown to be optimal in the sense of Bahadur by Haccou, Meelis and van de Geer (1985). Yao and Davis (1984) considered the asymptotic behaviour of the likelihood ratio statistic for testing a shift in mean in a sequence of independent normal random variables. A common feature of these papers is that they assume the form of H and G to be known.

A review of nonparametric procedures for the changepoint problem is given by Wolfe and Schechtman (1984). In particular, they summarize results of Page (1954, 1955), Bhattacharyya and Johnson (1968), Sen and Srivastava (1975), and Pettitt (1979). In all these nonparametric approaches it is assumed that $H(t)=G(t-\Delta)$, and they use different forms of rank statistics with median scores and Wilcoxon scores for testing this shift. Csörgő and Horváth (1986) obtained invariance principles for tests based on U-statistic processes.

In order to describe our nonparametric approach, let

$$F_n(t)=n^{-1}\#\{1\le i\le n: X_i\le t\}$$

denote the empirical distribution function of $X_1,\dots,X_n$, and let

$$Q_n(y)=\inf\{t:F_n(t)\ge y\},\quad 0<y<1,$$

be the corresponding empirical quantile function. We define

$$\psi_t(x)=\begin{cases}-(1-t) & \text{if } x\le 0,\\ t & \text{if } x>0,\end{cases}$$

and our statistics will be based on the process

$$Y_n(s,t)=n^{-1/2}\sum_{1\le i\le[ns]}\psi_t(X_i-Q_n(t)),\quad 0<s<1,\ 0<t<1,$$

($\sum_\emptyset=0$). In the next section we prove a weak approximation of Y_n in $D([0,1]\times[0,1])$ under H_0. Based on this weak convergence result we propose test statistics

and estimate also the rate of convergence of their distribution functions. In Section 3 we show that the proposed tests are consistent against H_1, and construct a strongly consistent estimator for λ.

2. Asymptotics under H_0

First we state the main result of this paper.

Theorem 2.1. *We assume that* H_0 *holds. One can define a sequence of two-parameter Gaussian processes* $\{\Gamma_n(s,t),\ 0\le s,\ t\le 1\}$ *such that*

$$P\left\{\sup_{0<s<1}\sup_{0<t<1}|Y_n(s,t)-\Gamma_n(s,t)|>A_1n^{-1/4}(\log n)^{3/4}\right\}\le B_1n^{-\varepsilon}$$

for all $\varepsilon>0$, *where* $A_1=A_1(\varepsilon)$ *and* B_1 *are constants, and*

$$E\Gamma_n(s,t)=0,\qquad E\Gamma_n(s,t)\Gamma_n(u,v)=(\min(s,u)-su)(\min(t,v)-tv).$$

Proof. By the continuity of F the $U_i=F(X_i)$ $(1\le i\le n)$ are independent uniform-(0, 1) random variables (r.v.'s). Let

$$E_m(t)=m^{-1}\#\{1\le i\le m: U_i\le t\}$$

and

$$U_m(y)=\inf\{t:E_m(t)\ge y\},\quad 0<y<1,$$

be the empirical distribution and quantile functions of the first m $(1\le m\le n)$ of these uniform-(0, 1) r.v.'s. Using the definition of ψ_t we obtain

$$\sum_{1\le i\le m}\psi_t(X_i-Q_n(t))=m(t-E_m(U_n(t))).\tag{2.1}$$

One can easily verify that

$$\sup_{0\le t\le 1}|E_n(U_n(t))-t|=O(1/n).\tag{2.2}$$

Also, the Dvoretzky, Kiefer and Wolfowitz (1956) inequality yields

$$P\left\{\sup_{0\le t\le 1}|U_n(t)-t|\ge n^{-1/2}x\right\}$$
$$=P\left\{\sup_{0\le t\le 1}|E_n(t)-t|\ge n^{-1/2}x\right\}\le A_{11}\exp(-2x^2),$$

and therefore we have

$$P\left\{\sup_{0\le t\le 1}|U_n(t)-t|\ge A_{12}n^{-1/2}(\log n)^{1/2}\right\}\le B_{12}n^{-\varepsilon},\tag{2.3}$$

where A_{11}, $A_{12}=A_{12}(\varepsilon)$ and B_{12} are constants. Komlós, Major and Tusnády (1975) constructed a Kiefer process $\{K(t,s),\ 0\le t<\infty,\ 0\le s\le 1\}$ (i.e., $K(t,s)$ is a two-parameter Gaussian process with $EK(t,s)=0$ and

$$EK(t,s)K(u,v)=\min(t,u)(\min(s,v)-sv)$$

such that

$$P\left\{\max_{1\le m\le n}\sup_{0\le s\le 1}|m(s-E_m(s))-K(m,s)|\ge A_{13}(\log n)^2\right\}\le B_{13}n^{-\varepsilon}, \tag{2.4}$$

where $A_{13}=A_{13}(\varepsilon)$ and B_{13} are constants. For each $m\ge 1$ the process $\{m^{-1/2}K(m,s),\ 0\le s\le 1\}$ is equal in distribution to a Brownian bridge process. Hence by Lemma 1.1.1 and Lemma 1.4.1 of Csörgő and Révész (1981) we get

$$P\left\{\max_{1\le m\le n}\sup_{0\le t\le 1-h}\sup_{0\le s\le h}|K(m,t)-K(m,t+s)|\ge xh^{1/2}n^{1/2}\right\}$$
$$\le nB_{14}h^{-1}\exp(-\tfrac{1}{3}x^2). \tag{2.5}$$

By (2.2) we have

$$U_n(t)-t=-(E_n(U_n(t))-U_n(t))+O(1/n)$$

uniformly in $t\in(0,1)$. Hence by (2.3), (2.4) and (2.5) with $h=A_{12}n^{-1/2}(\log n)^{1/2}$ and $x=3(1+\varepsilon)(\log n)^{1/2}$ we obtain

$$P\left\{\sup_{0\le t\le 1}|U_n(t)+E_n(t)-2t|\ge A_{15}n^{-3/4}(\log n)^{3/4}\right\}\le B_{15}n^{-\varepsilon}. \tag{2.6}$$

We observe that

$$m(t-E_m(U_n(t)))=m(U_n(t)-E_m(U_n(t)))+m(E_n(t)-t)$$
$$-m(U_n(t)+E_n(t)-2t). \tag{2.7}$$

Applying now (2.3)–(2.7) we conclude that

$$P\left\{\max_{1\le m\le n}\sup_{0\le t\le 1}\left|m(t-E_m(U_n(t)))-\left\{K(m,t)-\frac{m}{n}K(n,t)\right\}\right|\right.$$
$$\left.\ge A_{16}n^{1/4}(\log n)^{3/4}\right\}\le B_{16}n^{-\varepsilon}. \tag{2.8}$$

Let $\{W(x,y),\ 0\le x,\ y<\infty\}$ be a two-parameter Wiener process. Then

$$\left\{n^{-1/2}\left(K(m,t)-\frac{m}{n}K(n,t)\right),\ 1\le m\le n,\ 0\le t\le 1\right\}$$
$$\overset{\mathcal{D}}{=}\left\{K\left(\frac{m}{n},t\right)-\frac{m}{n}K(1,t),\ 1\le m\le n,\ 0\le t\le 1\right\}$$
$$\overset{\mathcal{D}}{=}\left\{W\left(\frac{m}{n},t\right)-tW\left(\frac{m}{n},1\right)-\frac{m}{n}W(1,t)+t\frac{m}{n}W(1,1),\right.$$
$$\left.1\le m\le n,\ 0\le t\le 1\right\}. \tag{2.9}$$

Using now Lemma 1.11.1 of Csörgő and Révész (1981) we get

$$P\left\{\sup_{0\le s\le 1}\sup_{0\le t\le 1}\left|\left(K\left(\frac{[ns]}{n},t\right)-\frac{[ns]}{n}K(1,t)\right)-(K(s,t)-sK(1,t))\right|\right.$$
$$\left.\ge A_{17}n^{-1/2}(\log n)^{1/2}\right\}\le B_{17}n^{-\varepsilon}. \tag{2.10}$$

An elementary calculation shows that $E(K(s,t)-sK(1,t))=0$ and

$$E(K(s,t)-sK(1,t))(K(u,v)-uK(1,v))=(\min(s,u)-su)(\min(t,v)-tv).$$

Now the result follows from (2.8)–(2.10) and Lemma 4.4.4 of Csörgő and Révész (1981).

The next corollaries are immediate consequences of Theorem 2.1.

Corollary 2.1. *We assume that* H_0 *holds. Then, as* $n\to\infty$,

$$\sup_{1/n\le s\le 1-1/n}\sup_{1/n\le t\le 1-1/n}|Y_n(s,t)-\Gamma_n(s,t)|t^{-\nu_1}(1-t)^{-\nu_2}s^{-\nu_3}(1-s)^{-\nu_4}=o_p(1)$$

if $\nu_i\ge 0$, $1\le i\le 4$, *and* $\nu_i+\nu_j<\frac{1}{4}$, $1\le i<j\le 4$.

Let $\Gamma(s,t)$ and $\{B(s),\ 0\le s\le 1\}$ be a Brownian bridge.

Corollary 2.2. *We assume that* H_0 *holds. Then, as* $n\to\infty$,

$$\sup_{0<s<1}\sup_{0<t<1}|Y_n(s,t)|\xrightarrow{\mathcal{D}}\sup_{0\le s\le 1}\sup_{0\le t\le 1}|\Gamma(s,t)|, \tag{2.11}$$

$$\int_0^1\int_0^1 Y_n^2(s,t)\,ds\,dt\xrightarrow{\mathcal{D}}\int_0^1\int_0^1\Gamma^2(s,t)\,ds\,dt, \tag{2.12}$$

$$(s_0(1-s_0))^{-1/2}\sup_{0<t<1}|Y_n(s_0,t)|\xrightarrow{\mathcal{D}}\sup_{0\le t\le 1}|B(t)|, \tag{2.13}$$

$$(t_0(1-t_0))^{-1/2}\sup_{0<s<1}|Y_n(s,t_0)|\xrightarrow{\mathcal{D}}\sup_{0\le s\le 1}|B(s)|, \tag{2.14}$$

$$12\int_0^1\int_0^1 Y_n(s,t)\,ds\,dt\xrightarrow{\mathcal{D}}N(0,1), \tag{2.15}$$

$$\left(\frac{12}{t_0(1-t_0)}\right)^{1/2}\int_0^1 Y_n(s,t_0)\,ds\xrightarrow{\mathcal{D}}N(0,1), \tag{2.16}$$

$$(48)^{1/2}\int_0^1(-Y_n(s,\tfrac{1}{2}))\,ds\xrightarrow{\mathcal{D}}N(0,1) \tag{2.17}$$

and

$$\left(\frac{12}{s_0(1-s_0)}\right)^{1/2}\int_0^1 Y_n(s_0,t)\,\mathrm{d}t \xrightarrow{\mathscr{D}} \mathrm{N}(0,1), \tag{2.18}$$

where $0<t_0, s_0<1$, *and* $\mathrm{N}(0,1)$ *stands for the standard normal r.v.*

By Theorem 2.1 the rate of convergence in (2.12)–(2.15) is at least $\mathrm{O}(n^{-1/4}\times(\log n)^{3/4})$ (cf. Komlós et al. (1975)). The limiting Gaussian process Γ appeared in Blum, Kiefer and Rosenblatt (1961) in the context of testing for independence. The distribution function of the limiting r.v. in (2.12) is tabulated in their paper and also in Cotterill and Csörgő (1985). Naturally the distribution function of the limit in (2.13) and (2.14) has been extensively tabulated. The distribution function of the limiting r.v. in (2.11) is tabulated in Picard (1985). We note also that the statistic in (2.17) is a sum of linear rank statistics with median scores, and it was studied by Wolfe and Schechtman (1984). The statistic in (2.18) is related to a Mann–Whitney type statistic proposed by Sen and Srivastava (1975) for the changepoint problem. For further results on Mann–Whitney type statistics we refer to Csörgő and Horváth (1986).

Using Corollary 2.1 we can have an immediate joint generalization of (2.15) and (2.16). Namely if ϕ is such that

$$\sup_{0<t<1}\sup_{0<s<1}|\phi(t,s)|\,t^{\nu_1}(1-t)^{\nu_2}s^{\nu_3}(1-s)^{\nu_4}<\infty,$$

$\nu_i\geq 0$, $1\leq i\leq 4$, $\nu_i+\nu_j<\frac{1}{4}$, $1\leq i<j\leq 4$, then

$$\int_{1/n}^{1-1/n}\int_{1/n}^{1-1/n}\phi(t,s)Y_n(s,t)\,\mathrm{d}s\,\mathrm{d}t \xrightarrow{\mathscr{D}} \int_0^1\int_0^1\phi(t,s)\Gamma(s,t)\,\mathrm{d}s\,\mathrm{d}t.$$

It is easy to see that $\int_0^1\int_0^1\phi(t,s)\Gamma(s,t)\,\mathrm{d}s\,\mathrm{d}t$ is a normal r.v. with expectation zero and variance

$$E\left(\int_0^1\int_0^1\phi(t,s)\Gamma(s,t)\,\mathrm{d}s\,\mathrm{d}t\right)^2$$
$$=\int_0^1\int_0^1\int_0^1\int_0^1\phi(t,s)\phi(u,v)\{\min(t,u)-tu\}\{\min(s,v)-sv\}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v.$$

3. Asymptotics under H_1

The next theorem implies the consistency of the tests proposed in Corollary 2.2.

Theorem 3.1. *We assume that* H_1 *holds and* Q *is continuous at* $t_0\in(0,1)$. *Then*

$$\lim_{n\to\infty} n^{-1/2}Y_n(s,t_0)\overset{\text{a.s.}}{=} g_\lambda(s,t_0)$$
$$=\begin{cases} s(t_0-H(Q(t_0))) & \text{if } 0<s\leq\lambda,\\ (1-s)\dfrac{\lambda}{1-\lambda}(t_0-H(Q(t_0))) & \text{if } \lambda\leq s<1.\end{cases}$$

Proof. Let

$$H_m(t)=m^{-1}\#\{1\leq i\leq m: X_i\leq t\},\quad 1\leq m\leq[n\lambda],$$

and

$$G_k(t)=k^{-1}\#\{[n\lambda]+1\leq i\leq[n\lambda]+1+k: X_i\leq t\},\quad 1\leq k<n-[n\lambda].$$

Then we have

$$F_n(t)=\frac{[n\lambda]}{n}H_{[n\lambda]}(t)+\frac{n-[n\lambda]}{n}G_{n-[n\lambda]-1}(t).$$

By the Glivenko–Cantelli theorem

$$\sup_{-\infty<t<\infty}|F_n(t)-F(t)|\overset{\text{a.s.}}{=}\mathrm{o}(1).$$

Hence it follows from the proof of Theorem in Section 2.3.1 of Serfling (1980) that

$$|Q_n(t_0)-Q(t_0)|\overset{\text{a.s.}}{=}\mathrm{o}(1). \tag{3.1}$$

Using the definition of $Y_n(s,t)$ we obtain

$$n^{-1/2}Y_n(s,t)=n^{-1}\sum_{1\leq i\leq[ns]}\psi_t(X_i-Q_n(t)) \tag{3.2}$$
$$=\begin{cases}\dfrac{[ns]}{n}(t-H_{[ns]}(Q_n(t))), & 0<s\leq\lambda,\\[2ex] \dfrac{[n\lambda]}{n}(t-H_{[n\lambda]}(Q_n(t)))\\ +\dfrac{[ns]-[n\lambda]}{n}(t-G_{[ns]-[n\lambda]}(Q_n(Q_n(t))), & \lambda\leq s<1.\end{cases}$$

By (3.1) and the Glivenko–Cantelli theorem again we get

$$\lim_{n\to\infty} n^{-1/2}Y_n(s,t_0)\overset{\text{a.s.}}{=}\begin{cases} s(t_0-H(Q(t_0))), & 0<s\leq\lambda,\\ \lambda(t_0-H(Q(t_0)))+(s-\lambda)(t_0-G(Q(t_0))), & \lambda\leq s<1\end{cases}$$

and hence also the theorem.

Applying Theorem 3.1 we can immediately find conditions for the consistency of tests based on Corollary 2.2. For example, if there is a $t_0\in(0,1)$ such that $H(Q(t_0))\neq t_0$ and Q is continuous at t_0, then

$$\lim_{n\to\infty}\sup_{0<s<1}\sup_{0<t<1}|Y_n(s,t)|\overset{\text{a.s.}}{=}\infty.$$

It is also clear that if $H(t)=G(t-\Delta)$ and H is strictly monotone, then tests based on (2.11)–(2.18) are consistent against the alternative $\Delta\neq 0$.

Next we consider the problem of estimating λ. With $0<a\leq b<1$ let

$$M_n(s)=M_n(s;a,b)=\sup_{a\leq t\leq b} n^{-1/2}|Y_n(s,t)|$$
$$=n^{-1}\sup_{a\leq t\leq b}\left|\sum_{1\leq i\leq [ns]}\psi_t(X_i-Q_n(t))\right|,$$

$$M_n^*=\sup_{0<s<1} M_n(s),$$

and

$$\hat{\lambda}_n=\inf\{s: M_n(s)=M_n^*\}.$$

Theorem 3.2. *We assume that* H_1 *holds,* Q *is continuous on* $[a,b]$ *and*

$$\sup_{a\leq t\leq b}|t-H(Q(t))|>0. \tag{3.3}$$

Then

$$\lim_{n\to\infty}\hat{\lambda}_n \overset{\text{a.s.}}{=} \lambda. \tag{3.4}$$

Proof. Using again Serfling (1980) we have, as $n\to\infty$,

$$\sup_{a\leq t\leq b}|Q_n(t)-Q(t)|\overset{\text{a.s.}}{=}\mathrm{o}(1), \tag{3.5}$$

and, therefore, from (3.2) and the Glivenko–Cantelli theorem we obtain

$$\sup_{0<s<1}\sup_{a\leq t\leq b}|n^{-1/2}Y_n(s,t)-g_\lambda(s,t)|\overset{\text{a.s.}}{=}\mathrm{o}(1). \tag{3.6}$$

Thus

$$\lim_{n\to\infty}M_n^*\overset{\text{a.s.}}{=}\lambda\sup_{a\leq t\leq b}|t-H(Q(t))|, \tag{3.7}$$

and by definition

$$M_n(\hat{\lambda}_n)=M_n^*. \tag{3.8}$$

If there is an event A of positive probability such that (3.6) and (3.7) hold on A but $\hat{\lambda}_n(\omega)\to\lambda^*\neq\lambda$ whenever $\omega\in A$, then by (3.6),

$$\lim_{n\to\infty}n^{-1/2}\sup_{a\leq t\leq b}|Y_n(\hat{\lambda}_n,t)|\overset{\text{a.s.}}{=}\sup_{a\leq t\leq b}g_\lambda(\lambda^*,t)$$
$$<\lambda\sup_{a\leq t\leq b}|t-H(Q(t))|,$$

which contradicts (3.8) and (3.7). Hence the result is proven.

Using the Mann–Whitney statistic, Darkhovskh (1976) constructed a weakly consistent estimate of λ which is related to our estimator $\hat{\lambda}_n$.

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