

## Approximations for the time of change and the power function in change-point models

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### Abstract

Assuming that the observations are from an exponential family we obtain the asymptotic distribution of the maximum likelihood estimator of the time of change. We also prove that the maximum likelihood ratio test is asymptotically normal, if there is a change in the parameters at an unknown time.

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### 1. Introduction and results

Let  $X_1, X_2, \dots, X_n$  be independent random vectors in  $\mathbb{R}^m$ . We assume that  $X_1, X_2, \dots, X_n$  have probability densities with respect to  $\nu$ , a  $\sigma$ -finite measure and the density of  $X_i$  is in the exponential form

$$f(\mathbf{x}; \boldsymbol{\theta}_i) = \exp(\mathbf{T}(\mathbf{x})\boldsymbol{\theta}_i^\top + S(\mathbf{x}) - A(\boldsymbol{\theta}_i))I\{\mathbf{x} \in C\},$$

where  $\mathbf{x} = (x_1, \dots, x_m)$ ,  $\boldsymbol{\theta}_i = (\theta_{i,1}, \dots, \theta_{i,d}) \in \Theta \subseteq \mathbb{R}^d$ ,  $\mathbf{T} = (T_1, \dots, T_d)$  and  $C \subseteq \mathbb{R}^m$ . Several authors studied the detection of changes in the parameters of random sequences (cf. the review papers of Csörgő and Horváth (1988) and Zacks (1991)). In the simplest

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case we want to test the null hypothesis of ‘no change’, i.e.,

$$H_0 : \theta_1 = \theta_2 = \dots = \theta_n$$

against a ‘one change’ alternative, i.e.,

$$H_A : \text{there is an integer } k^*, 1 \leq k^* < n \text{ such that } \theta_1 = \dots = \theta_{k^*} \neq \theta_{k^*+1} = \dots = \theta_n.$$

We can use the maximum likelihood method to test  $H_0$  against  $H_A$ . It is easy to see that the generalized likelihood ratio is

$$A_k = \frac{\sup_{\theta} \prod_{1 \leq i \leq k} f(X_i; \theta) \sup_{\theta} \prod_{k+1 \leq i \leq n} f(X_i; \theta)}{\sup_{\theta} \prod_{1 \leq i \leq n} f(X_i; \theta)}, \tag{1.1}$$

if  $k = k^*$  is known. Since  $k^*$  is unknown, we reject  $H_0$  for large values of

$$Q_n = \max_{1 \leq k < n} (2 \log A_k). \tag{1.2}$$

Assuming some regularity conditions on the function  $A(\theta)$ , we get a simpler expression for  $A_k$ . Let  $A'(\theta) = (\frac{\partial}{\partial \theta_1} A(\theta), \dots, \frac{\partial}{\partial \theta_d} A(\theta))$ ,  $\theta = (\theta_1, \dots, \theta_d)$ . We assume that

C.1.  $\text{inv} A'(\theta)$ , the unique inverse of  $A'(\theta)$  exists for all  $\theta \in \Theta$ .

If C.1 holds, then the log likelihood ratio can be written as

$$\log A_k = kH(\mathbf{B}_k) + (n - k)H(\mathbf{B}_k^*) - nH(\mathbf{B}_n), \tag{1.3}$$

where

$$H(\mathbf{x}) = (\text{inv} A'(\mathbf{x}))\mathbf{x}^T - A(\text{inv} A'(\mathbf{x})), \tag{1.4}$$

$$\mathbf{B}_k = \frac{1}{k} \sum_{1 \leq i \leq k} \mathbf{T}(X_i) \tag{1.5}$$

and

$$\mathbf{B}_k^* = \frac{1}{n - k} \sum_{k+1 \leq i \leq n} \mathbf{T}(X_i). \tag{1.6}$$

Restricting the more general case in Gombay and Horváth (1994) to the exponential family, we get the limit distribution of  $Q_n$  under  $H_0$ . Let  $a(t) = (2 \log t)^{1/2}$  and  $b_d(t) = 2 \log t + \frac{d}{2} \log \log t - \log \Gamma(d/2)$ , where  $\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy$  is the Gamma function.

**Theorem 1.1.** *Assume that  $H_0$  holds, and the true value of the parameter is  $\theta_o$ . In addition to C.1, assume that there is an open interval  $\Theta_o \subseteq \Theta \subseteq \mathbb{R}^d$  containing  $\theta_o$  such that  $A(\theta)$  has continuous derivatives up to the third order, if  $\theta \in \Theta_o$  and*

$A''(\boldsymbol{\theta}) = \left\{ \frac{\partial^2}{\partial \theta_i \partial \theta_j} A(\boldsymbol{\theta}), 1 \leq i, j \leq d \right\}$  is a positive definite matrix, if  $\boldsymbol{\theta} \in \Theta_0$ . Then,

$$\lim_{n \rightarrow \infty} P\{a(\log n)Q_n^{1/2} \leq t + b_d(\log n)\} = \exp(-2e^{-t})$$

for all  $t$ .

In this paper we are interested in the distribution of  $Q_n$  under the alternative. So far only some special cases of  $Q_n$  have been considered under  $H_A$  in the parametric framework (cf. Yao and Davis, 1984; Haccou et al., 1988). Ferger (1994a, b, c, d) studied the behaviour of some tests derived from  $U$ -statistics under the null as well as the alternative hypothesis without assuming the form of the underlying densities. Szyszkowicz (1991a, b, c) obtained the weak convergence of empirical and related processes under contiguous alternatives which include the ‘small disorder’ as a possible change-point alternative.

If  $H_0$  does not hold, we may want to estimate the time of change. The maximum likelihood estimator of  $k^*$ , the time of change, is defined by

$$\hat{k} = \min\{k : Q_n = 2 \log A_k\}. \tag{1.7}$$

Bhattacharya (1987) defines a similar estimator which maximizes the quadratic functional of the partial sums  $B_k - B_k^*$  over a restricted range. Estimators which are defined as the time when a random process reaches its maximum have also been studied by Yao (1987), Csörgő and Horváth (1987), Dümbgen (1991), Ferger and Stute (1992), Ferger (1994c, e) and Antoch et al. (1995). Consider first the behavior of  $\hat{k}$  under  $H_0$ .

**Theorem 1.2.** *If the conditions of Theorem 1.1 hold, then*

$$\hat{k}/n \xrightarrow{\mathcal{D}} \xi_0,$$

where  $P\{\xi_0 = 0\} = P\{\xi_0 = 1\} = \frac{1}{2}$ .

**Proof.** Gombay and Horváth (1994) showed that

$$\lim_{n \rightarrow \infty} P\{\hat{k} \leq n/\log n \text{ or } \hat{k} \geq n - n/\log n\} = 1$$

and since  $\{A_k, 1 \leq k < n\} \stackrel{\mathcal{D}}{=} \{A_{n-k+1}, 1 \leq k < n\}$  under  $H_0$ , we get immediately Theorem 1.2.  $\square$

Now we consider the behaviour of  $Q_n$  and  $\hat{k}$  under the alternative. We say that the change occurs early, if  $k^*/n \rightarrow 0$  and the change is small, if the difference between the parameters before and after the change goes to zero, as  $n \rightarrow \infty$ . We have different types of limit results, depending on whether we have small changes or the change occurs immediately after the first few observations. The results will be given in Theorems 1.3–1.6. Let  $\boldsymbol{\theta}_A^{(1)}$  and  $\boldsymbol{\theta}_A^{(2)}$  be the values of the parameter before and after the change.

The size of the change is  $\Delta$ , where  $\Delta^2 = (\theta_A^{(1)} - \theta_A^{(2)})(\theta_A^{(1)} - \theta_A^{(2)})^T$ . We consider the following four cases:

**F.1.**  $k^* = k^*(n)$ ,  $0 < \lim_{n \rightarrow \infty} k^*/n = \lambda < 1$  and  $\theta_A^{(1)}, \theta_A^{(2)}$  are fixed elements of the interior of  $\Theta$ .

**F.2.**  $k^* = k^*(n)$ ,  $0 < \lim_{n \rightarrow \infty} k^*(n)/n = \lambda < 1$ ,  $\theta_A^{(1)} = \theta_A^{(1)}(n) \rightarrow \theta_A, \theta_A^{(2)} = \theta_A^{(2)}(n) \rightarrow \theta_A$ , as  $n \rightarrow \infty$ , where  $\theta_A$  is in the interior of  $\Theta$  and

$$\lim_{n \rightarrow \infty} \frac{n\Delta^2(n)}{\log \log n} = \infty. \tag{1.8}$$

**F.3.**  $k^* = k^*(n)$ ,  $\lim_{n \rightarrow \infty} k^*(n)/n = 0$ ,  $\theta_A^{(1)}, \theta_A^{(2)}$  are fixed elements of the interior of  $\Theta$  and

$$\lim_{n \rightarrow \infty} \frac{k^*(n)}{\log \log n} = \infty. \tag{1.9}$$

**F.4.**  $k^* = k^*(n)$ ,  $\lim_{n \rightarrow \infty} k^*(n)/n = 0$ ,  $\theta_A^{(1)} = \theta_A^{(1)}(n) \rightarrow \theta_A, \theta_A^{(2)} = \theta_A^{(2)}(n) \rightarrow \theta_A$ , as  $n \rightarrow \infty$ , where  $\theta_A$  is in the interior of  $\Theta$  and

$$\lim_{n \rightarrow \infty} \frac{k^*(n)\Delta^2}{\log \log n} = \infty. \tag{1.10}$$

Next we define  $\tau_1 = A'(\theta_A^{(1)})$ ,  $\tau_2 = A'(\theta_A^{(2)})$  and  $\tau_A = A'(\theta_A)$ . Let  $\{Y_i, i < 0\}$  be independent, identically distributed random vectors (i.i.d.r.v.'s) with density function  $f(x; \theta_A^{(1)})$  (with respect to  $\nu$ ) and similarly,  $\{Y_i, i > 0\}$  are i.i.d.r.v.'s with density function  $f(x; \theta_A^{(2)})$ . We assume also, that the two sequences  $\{Y_i, i < 0\}$  and  $\{Y_i, i > 0\}$  are independent. Now we define

$$Z_k = \begin{cases} \left( H'(\tau_2) - H'(\tau_1) \right) \sum_{k \leq i \leq -1} \left( T(Y_i) - \tau_1 \right)^T \\ \quad -k \{ H(\tau_2) - H(\tau_1) + H'(\tau_2)(\tau_1 - \tau_2)^T \} & \text{if } k < 0, \\ 0 & \text{if } k = 0, \\ \left( H'(\tau_1) - H'(\tau_2) \right) \sum_{1 \leq i \leq k} \left( T(Y_i) - \tau_2 \right) \\ \quad +k \{ H(\tau_1) - H(\tau_2) + H'(\tau_1)(\tau_2 - \tau_1)^T \}, & \text{if } k > 0. \end{cases} \tag{1.11}$$

Let

$$\mu^* = k^*H(\tau_1) + (n - k^*)H(\tau_2) - nH\left(\frac{k^*}{n}\tau_1 + \frac{n - k^*}{n}\tau_2\right) \tag{1.12}$$

and

$$\begin{aligned} \sigma_1^2 = & \lambda(H'(\boldsymbol{\tau}_1) - H'(\lambda\boldsymbol{\tau}_1 + (1 - \lambda)\boldsymbol{\tau}_2))A''(\boldsymbol{\theta}_A^{(1)})(H'(\boldsymbol{\tau}_1) \\ & - H'(\lambda\boldsymbol{\tau}_1 + (1 - \lambda)\boldsymbol{\tau}_2))^T + (1 - \lambda)(H'(\boldsymbol{\tau}_2) - H'(\lambda\boldsymbol{\tau}_1 + (1 - \lambda)\boldsymbol{\tau}_2)) \\ & \times A''(\boldsymbol{\theta}_A^{(2)})(H'(\boldsymbol{\tau}_2) - H'(\lambda\boldsymbol{\tau}_1 + (1 - \lambda)\boldsymbol{\tau}_2))^T. \end{aligned} \tag{1.13}$$

As usual,  $N(\mu, \sigma^2)$  denotes a normal r.v. with parameters  $\mu$  and  $\sigma^2$ . Let

$$\mathcal{F} = \{\boldsymbol{\tau} : \boldsymbol{\tau} = A'(\boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta\}$$

and

$$\mathcal{F}^*(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2; \varepsilon) = \{\boldsymbol{\tau} : \|\boldsymbol{\tau} - (s\boldsymbol{\tau}_1 + (1 - s)\boldsymbol{\tau}_2)\| \leq \varepsilon \text{ for some } 0 \leq s \leq 1\},$$

where  $\|\mathbf{x}\|^2 = \mathbf{x}\mathbf{x}^T$ .

**Theorem 1.3.** Assume that  $H_A$ , F.1, C.1 hold, and

C.2. there is  $\varepsilon > 0$  such that  $H'''$  exists and  $H''$  is positive definite on  $\mathcal{F}^*(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2; \varepsilon)$ ,

C.3.  $H'(\boldsymbol{\tau}_2)(\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)^T + H(\boldsymbol{\tau}_2) - H(\boldsymbol{\tau}_1) < 0$ ,

C.4.  $H'(\boldsymbol{\tau}_1)(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)^T + H(\boldsymbol{\tau}_1) - H(\boldsymbol{\tau}_2) < 0$ ,

and

C.5.  $\sup_{\varepsilon \leq s \leq 1 - \varepsilon} (H(s\boldsymbol{\tau}_1 + (1 - s)\boldsymbol{\tau}_2) - sH(\boldsymbol{\tau}_1) - (1 - s)H(\boldsymbol{\tau}_2)) < 0$   
for all  $0 < \varepsilon < \frac{1}{2}$ .

Then,

$$\widehat{k} - k^* \xrightarrow{\mathcal{L}} \xi, \tag{1.14}$$

where

$$\xi = \inf \left\{ k : Z_k = \sup_{-\infty < i < \infty} Z_i \right\} \tag{1.15}$$

and

$$n^{-1/2}(Q_n - 2\mu^*) \xrightarrow{\mathcal{L}} N(0, 4\sigma_1^2). \tag{1.16}$$

Next we consider the case when F.2 holds.

Let  $\delta^2 = (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)(\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)^T$ ,

$$\sigma_A^2 = \lim_{n \rightarrow \infty} \frac{(\boldsymbol{\tau}_1(n) - \boldsymbol{\tau}_2(n))H''(\boldsymbol{\tau}_A)(\boldsymbol{\tau}_1(n) - \boldsymbol{\tau}_2(n))^T}{(\boldsymbol{\tau}_1(n) - \boldsymbol{\tau}_2(n))(\boldsymbol{\tau}_1(n) - \boldsymbol{\tau}_2(n))^T} \tag{1.17}$$

and

$$\sigma_2^2 = \lambda(1 - \lambda)\sigma_A^2. \tag{1.18}$$

We also define

$$W^*(t) = \begin{cases} \sigma_A W_1(-t) - \frac{1}{2} \sigma_A^2 |t| & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \sigma_A W_2(t) - \frac{1}{2} \sigma_A^2 |t| & \text{if } t > 0, \end{cases} \quad (1.19)$$

where  $\{W_1(t), t \geq 0\}$  and  $\{W_2(t), t \geq 0\}$  are independent Wiener processes.

**Theorem 1.4.** Assume that  $H_A$ , F.2, C.1 hold, and

C.6.  $H'''$  exists and  $H''(\tau)$  is positive definite in a neighbourhood of  $\tau_A = H'(\theta_A)$ .

Then,

$$\delta^2(\widehat{k} - k^*) \xrightarrow{\mathcal{D}} \eta, \quad (1.20)$$

where

$$\eta = \inf\{t : W^*(t) = \sup_{-\infty < s < \infty} W^*(s)\} \quad (1.21)$$

and

$$(n\delta^2)^{-1/2}(Q_n - 2\mu^*) \xrightarrow{\mathcal{D}} N(0, 4\sigma_2^2). \quad (1.22)$$

Let

$$\sigma_3^2 = (H'(\tau_1) - H'(\tau_2))A''(\theta_A^{(1)})(H'(\tau_1) - H'(\tau_2))^T. \quad (1.23)$$

**Theorem 1.5.** Assume that  $H_A$ , F.3, C.1–C.5 hold. Then,

$$\widehat{k} - k^* \xrightarrow{\mathcal{D}} \xi, \quad (1.24)$$

where  $\xi$  is defined by (1.15) and

$$(k^*)^{-1/2}(Q_n - 2\mu^*) \xrightarrow{\mathcal{D}} N(0, 4\sigma_3^2). \quad (1.25)$$

Our last theorem considers the case when we have a small and early change.

**Theorem 1.6.** Assume that  $H_A$ , F.3, C.1 and C.6 hold. Then,

$$\delta^2(\widehat{k} - k^*) \xrightarrow{\mathcal{D}} \eta,$$

where  $\eta$  is defined by (1.21) and

$$(k^* \delta^2)^{-1/2} (Q_n - 2\mu^*) \xrightarrow{\mathcal{L}} N(0, 4\sigma_4^2),$$

with  $\sigma_4^2 = \sigma_A^2$ .

It is clear from Theorem 1 and Theorems 1.2–1.6, that the likelihood ratio test is consistent against the alternative if one of the conditions F.1–F.4 hold. The proofs of Theorems 1.2–1.6 are given in Section 3. In the following section we consider some examples.

## 2. Some examples

Let

$$\widehat{W}(t) = \begin{cases} W_1(-t) - \frac{1}{2}|t| & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ W_2(t) - \frac{1}{2}|t| & \text{if } t > 0, \end{cases}$$

where  $W_1$  and  $W_2$  are independent Wiener processes. We also define

$$\widehat{\eta} = \inf \left\{ t : \widehat{W}(t) = \sup_{-\infty < s < \infty} \widehat{W}(s) \right\}. \tag{2.1}$$

Bhattacharya and Brockwell (1976) (cf. also Yao, 1987 and Ferger, 1994c) proved that the density function of  $\widehat{\eta}$  is

$$g(x) = \frac{3}{2} \exp(|x|) \{1 - \Phi(\frac{3}{2}|x|^{1/2})\} - \frac{1}{2}(1 - \Phi(\frac{1}{2}|x|^{1/2})), \quad -\infty < x < \infty, \tag{2.2}$$

where  $\Phi$  denotes the standard normal distribution function and the distribution function is given by

$$G(x) = 1 + (2\pi)^{-1/2} x^{1/2} e^{-x/8} - \frac{1}{2}(x + 5)\Phi(-\frac{1}{2}x^{1/2}) + \frac{3}{2}e^x \Phi(-\frac{3}{2}x^{1/2}),$$

if  $x > 0$  and  $G(x) = 1 - G(-x)$  if  $x \leq 0$ . Using Theorems 1.4 and 1.6 we can get distribution free limit distributions.

**Corollary 2.1.** *If the conditions of Theorem 1.4 or Theorem 1.6 are satisfied, then we have*

$$(\tau_1 - \tau_2)H''(\tau_A)(\tau_1 - \tau_2)^T(\widehat{k} - k^*) \xrightarrow{\mathcal{L}} \widehat{\eta}, \tag{2.3}$$

where  $\widehat{\eta}$  is defined in (2.1) and its density is given in (2.2).

**Proof.** It follows immediately from Theorems 1.4 and 1.6 and from the scale transformation of Wiener processes.  $\square$

Of course, we cannot use (2.3) to construct confidence intervals for  $k^*$ , since  $(\tau_1 - \tau_2)H''(\tau_A)(\tau_1 - \tau_2)^T$ , the size of the jump is unknown. However, it can be estimated from the observations.

**Corollary 2.2.** *If the conditions of Theorem 1.4 or Theorem 1.6 are satisfied, then*

$$(\mathbf{B}_{\hat{k}} - \mathbf{B}_{\hat{k}}^*)H''(\mathbf{B}_n)(\mathbf{B}_{\hat{k}} - \mathbf{B}_{\hat{k}}^*)^T(\hat{k} - k^*) \xrightarrow{\mathcal{L}} \hat{\eta}, \tag{2.4}$$

where  $\hat{\eta}$  is defined in (2.1) and its density is given in (2.2).

**Proof.** It follows from the law of large numbers that

$$\mathbf{B}_n \xrightarrow{P} \tau_A, \tag{2.5}$$

if the conditions of Theorem 1.4 or Theorem 1.6 hold. By Slutsky’s lemma and (2.3) it is enough to show that

$$\frac{(\mathbf{B}_{\hat{k}} - \mathbf{B}_{\hat{k}}^*)H''(\mathbf{B}_n)(\mathbf{B}_{\hat{k}} - \mathbf{B}_{\hat{k}}^*)^T}{(\tau_1 - \tau_2)H''(\tau_A)(\tau_1 - \tau_2)^T} \xrightarrow{P} 1. \tag{2.6}$$

By the law of the iterated logarithm we have

$$\|\mathbf{B}_{\hat{k}} - \tau_1\| = O_P((\log \log k^*/k^*)^{1/2}) \tag{2.7}$$

and

$$\|\mathbf{B}_{\hat{k}}^* - \tau_2\| = O_P((\log \log(n - k^*)/(n - k^*))^{1/2}). \tag{2.8}$$

Now the conditions of Theorems 1.4 and 1.6 imply that

$$\lim_{n \rightarrow \infty} \frac{\log \log k^*}{k^*(\tau_1 - \tau_2)H''(\tau_A)(\tau_1 - \tau_2)^T} = 0 \tag{2.9}$$

and similarly

$$\lim_{n \rightarrow \infty} \frac{\log \log(n - k^*)}{(n - k^*)(\tau_1 - \tau_2)H''(\tau_A)(\tau_1 - \tau_2)^T} = 0 \tag{2.10}$$

It is clear that (2.6) follows from (2.7)–(2.10).  $\square$

Next we consider a few special cases of the results in Section 1.

**Example 2.1** (Normal observations, change in the mean with a known, constant variance). The density function is

$$f(t; \theta) = \exp \left( \frac{t}{\sigma^2} \theta - \frac{\theta^2}{2\sigma^2} - \frac{t^2}{2\sigma^2} - \frac{1}{2} \log(2\pi\sigma^2) \right),$$

and therefore  $T(t) = t/\sigma^2, H(t) = t^2\sigma^2/2$  and  $A(\theta) = \theta^2/(2\sigma^2)$ . Elementary calculations yield that  $A''(\theta) = 1/\sigma^2, H''(t) = \sigma^2, \sigma_1^2 = \lambda(1 - \lambda)(\theta_A^{(1)} - \theta_A^{(2)})^2/\sigma^2, \sigma_2^2 = \lambda(1 - \lambda)\sigma^2,$

$\sigma_3^2 = (\theta_A^{(1)} - \theta_A^{(2)})^2/\sigma^2$  and  $\sigma_A^2 = \sigma_4^2 = \sigma^2$ . Also,  $\mu^* = \frac{1}{2\sigma^2} \frac{k^*(n-k^*)}{n} (\theta_A^{(1)} - \theta_A^{(2)})^2$ . If  $\{N_i, -\infty < i < \infty\}$  is a sequence of i.i.d. standard normal r.v.'s, then

$$\{Z_k, -\infty < k < \infty\} \stackrel{\mathcal{L}}{=} \{\widehat{Z}_k^{(1)}, -\infty < k < \infty\},$$

where

$$\widehat{Z}_k^{(1)} = \begin{cases} \frac{\theta_A^{(2)} - \theta_A^{(1)}}{\sigma} \sum_{k \leq i \leq -1} N_i + \frac{k}{2\sigma^2} (\theta_A^{(1)} - \theta_A^{(2)})^2 & \text{if } k < 0, \\ 0 & \text{if } k = 0, \\ \frac{\theta_A^{(1)} - \theta_A^{(2)}}{\sigma} \sum_{1 \leq i \leq k} N_i - \frac{k}{2\sigma^2} (\theta_A^{(1)} - \theta_A^{(2)})^2 & \text{if } k > 0. \end{cases}$$

Hence the limit distribution of  $\widehat{k}$  depends on  $\theta_A^{(1)} - \theta_A^{(2)}$  only.

**Example 2.2.** (*Exponential observations*). The density is given by

$$f(t; \theta) = \exp(-t\theta + \log \theta) I\{t \geq 0\},$$

and therefore  $T(t) = -t (t \geq 0)$ ,  $H(x) = -1 - \log(-x) (x < 0)$  and  $A(\theta) = -\log \theta$ . Thus we get  $\sigma_1^2 = \lambda(1 - \lambda)(\theta_A^{(1)} - \theta_A^{(2)})^2 / (\lambda\theta_A^{(2)} + (1 - \lambda)\theta_A^{(1)})^2$ ,  $\sigma_2^2 = \lambda(1 - \lambda)\theta_A^2$ ,  $\sigma_3^2 = (1 - \theta_A^{(2)}/\theta_A^{(1)})^2$ ,  $\sigma_4^2 = \sigma_A^2 = \theta_A^2$  and  $\mu^* = k^* \log \theta_A^{(1)} + (n - k^*) \log \theta_A^{(2)} + n \log(\lambda/\theta_A^{(1)} + (1 - \lambda)/\theta_A^{(2)})$ . If  $\{Y_i^*, -\infty < i < \infty\}$  is a sequence of i.i.d. exponential r.v.'s with  $EY_i^* = 1$  and

$$\widehat{Z}_k^{(2)} = \begin{cases} \left(1 - \frac{\theta_A^{(2)}}{\theta_A^{(1)}}\right) \sum_{k \leq i \leq -1} (Y_i^* - 1) - k \left(1 + \log \frac{\theta_A^{(2)}}{\theta_A^{(1)}} - \frac{\theta_A^{(2)}}{\theta_A^{(1)}}\right) & \text{if } k < 0, \\ 0 & \text{if } k = 0, \\ \left(1 - \frac{\theta_A^{(1)}}{\theta_A^{(2)}}\right) \sum_{1 \leq i \leq -k} (Y_i^* - 1) + k \left(1 + \log \frac{\theta_A^{(1)}}{\theta_A^{(2)}} - \frac{\theta_A^{(1)}}{\theta_A^{(2)}}\right) & \text{if } k > 0, \end{cases}$$

then we have

$$\{Z_k, -\infty < k < \infty\} \stackrel{\mathcal{L}}{=} \{\widehat{Z}_k^{(2)}, -\infty < k < \infty\}.$$

The limit distribution of  $Q_n$  and  $\widehat{k}$  depend on  $\theta_A^{(1)}/\theta_A^{(2)}$  only.

**Example 2.3.** (*Poisson observations*). The probability mass function in the natural form is

$$f(t; \theta) = \exp(t\theta - e^\theta - \log t!) I\{t \text{ is non-negative integer}\}$$

and therefore  $T(t) = t, A(\theta) = e^\theta$  and  $H(x) = x(\log x - 1)$ . Some calculations give

$$\begin{aligned} \sigma_1^2 &= \lambda\{\theta_A^{(1)} - \log(\lambda \exp(\theta_A^{(1)}) + (1 - \lambda) \exp(\theta_A^{(2)}))\} \exp(\theta_A^{(1)}) \\ &\quad + (1 - \lambda)\{\theta_A^{(2)} - \log(\lambda \exp(\theta_A^{(1)}) + (1 - \lambda) \exp(\theta_A^{(2)}))\} \exp(\theta_A^{(2)}), \\ \sigma_2^2 &= \lambda(1 - \lambda) \exp(-\theta_A), \quad \sigma_3^2 = (\theta_A^{(1)} - \theta_A^{(2)})^2 \exp(\theta_A^{(1)}), \\ \sigma_A^2 &= \sigma_A^2 = \exp(-\theta_A) \\ \mu^* &= (k^* \theta_A^{(1)} \exp(\theta_A^{(1)}) + (n - k^*) \theta_A^{(2)} \exp(\theta_A^{(2)}) - (k^* \exp(\theta_A^{(1)})) \\ &\quad + (n - k^*) \exp(\theta_A^{(2)})) \log \left( \frac{k^*}{n} \exp(\theta_A^{(1)}) + \frac{n - k^*}{n} \exp(\theta_A^{(2)}) \right). \end{aligned}$$

**Example 2.4** (Normal random vectors, change in the mean vector with a known, constant covariance matrix). In this case  $m = d$  and the density is

$$f(\mathbf{x}; \theta) = \exp(\mathbf{x}\Sigma^{-1}\theta^T - \frac{1}{2}\theta\Sigma^{-1}\theta^T - \frac{1}{2}\mathbf{x}\Sigma^{-1}\mathbf{x}^T - \log((2\pi)^{m/2} \det \Sigma)),$$

where  $\Sigma$  is the covariance matrix. Hence  $T(\mathbf{x}) = \mathbf{x}\Sigma^{-1}$ ,  $H(\mathbf{x}) = \frac{1}{2}\mathbf{x}\Sigma\mathbf{x}^T$  and  $A(\theta) = \frac{1}{2}\theta\Sigma^{-1}\theta^T$ , and we get  $\sigma_1^2 = \lambda(1 - \lambda)(\theta_A^{(1)} - \theta_A^{(2)})\Sigma^{-1}(\theta_A^{(1)} - \theta_A^{(2)})^T$ ,

$$\begin{aligned} \sigma_A^2 &= \lim_{n \rightarrow \infty} \frac{(\theta_A^{(1)} - \theta_A^{(2)})\Sigma^{-1}(\theta_A^{(1)} - \theta_A^{(2)})^T}{\|(\theta_A^{(1)} - \theta_A^{(2)})\Sigma^{-1}\|^2}, \quad \sigma_2^2 = \lambda(1 - \lambda)\sigma_A^2, \\ \sigma_3^2 &= (\theta_A^{(1)} - \theta_A^{(2)})\Sigma^{-1}(\theta_A^{(1)} - \theta_A^{(2)})^T, \\ \sigma_4^2 &= \sigma_A^2, \quad \mu^* = \frac{1}{2} \frac{k^*(n - k^*)}{n} (\theta_A^{(1)} - \theta_A^{(2)})\Sigma^{-1}(\theta_A^{(1)} - \theta_A^{(2)})^T. \end{aligned}$$

Similarly to Example 2.1, the power function depends on  $\theta_A^{(1)} - \theta_A^{(2)}$ .

We note that Hinkley (1970) and Hinkley and Hinkley (1970) suggested the maximum likelihood estimator for  $k^*$ . They also obtained a recursive numerical method to approximate the distribution of  $\hat{k}$ . However, the computations are too involved to compute the distribution of  $\hat{k}$  for large sample sizes.

We checked the accuracy of the limit theorems in Theorems 1.3–1.6 by Monte Carlo simulations for the cases of Examples 2.1–2.4. All simulations were run 2000 times. We used the sample sizes  $n = 50, 100$  and  $500$  and changes at  $k^* = n\lambda$ ,  $\lambda = 0.1, 0.2, \dots, 0.5$  for various values of the parameters before and after the changes. It turned out that the results in (1.16), (1.22), (1.25) and (1.27) were very accurate if  $\mu^*$  of (1.12) was large enough and in this case the limit theorems and the Monte Carlo simulations gave very close values, and the power was above 0.9. If the limit theorems gave different values from the Monte Carlo simulations for the power functions, the limits always underestimated the true power. Serious difference between the true and the asymptotic powers occurred only if  $\mu^*$  is small and the power was less than 0.7.

We also checked the accuracy of the confidence intervals. We observed that the location of the time of change does not have a great effect on the confidence interval and the size of the jump was the important factor. It is interesting to note that Corollary 2.1 gave shorter confidence intervals than the results in (1.15) and (1.24).

### 3. Proofs of Theorems 1.3–1.6

First we obtain bounds for the difference between  $\widehat{k}$  and  $k^*$ . Then we study the asymptotics of  $\log A_k$  in a neighbourhood of  $\widehat{k}$  which yield the limit distributions of  $\widehat{k} - k^*$  and  $\log A_{\widehat{k}}$ . Let

$$\mu_k = \begin{cases} kH(\tau_1) + (n - k)H\left(\frac{k^* - k}{n - k}\tau_1 + \frac{n - k^*}{n - k}\tau_2\right) - k^*H(\tau_1) - (n - k^*)H(\tau_2) & \text{if } 1 \leq k \leq k^*, \\ kH\left(\frac{k^*}{k}\tau_1 + \frac{k - k^*}{k}\tau_2\right) + (n - k)H(\tau_2) - k^*H(\tau_1) - (n - k^*)H(\tau_2) & \text{if } k^* < k \leq n, \end{cases}$$

and

$$V_k = \log A_k - \log A_{k^*}, \quad 1 \leq k < n.$$

**Lemma 3.1.** *If the conditions of Theorem 1.3 are satisfied, then*

$$|\widehat{k} - k^*| = O_P(1). \tag{3.1}$$

**Proof.** Let  $1 \leq k \leq k^*$ . The Taylor formula gives

$$\begin{aligned} \mu_k &= (k - k^*)H(\tau_1) + (k^* - k)H(\tau_2) + (k^* - k)H'(\tau_2)(\tau_1 - \tau_2)^T \\ &\quad + \frac{1}{2} \frac{(k^* - k)^2}{n - k} (\tau_1 - \tau_2)H''(\tau^*)(\tau_1 - \tau_2)^T, \end{aligned}$$

where  $\tau^*$  is on the interval connecting  $\frac{k^* - k}{n - k}\tau_1 + \frac{n - k^*}{n - k}\tau_2$  and  $\tau_2$ . By the continuity of  $H''$  we can find a constant  $C_1 > 0$  such that

$$\frac{1}{2} \frac{(k^* - k)^2}{n - k} |(\tau_1 - \tau_2)H''(\tau^*)(\tau_1 - \tau_2)^T| \leq C_1 \frac{(k^* - k)^2}{n - k} \tag{3.2}$$

for all  $1 \leq k \leq k^*$ . We can choose a small enough  $\alpha$  such that

$$\mu_k \leq \frac{1}{2}(k^* - k)\{H'(\tau_2)(\tau_1 - \tau_2)^\top + H(\tau_2) - H(\tau_1)\}, \tag{3.3}$$

if  $n\alpha \leq k \leq k^*$ . It is easy to see that

$$\begin{aligned} \mu_k &= (n - k) \left\{ H \left( \frac{k^* - k}{n - k} \tau_1 + \frac{n - k^*}{n - k} \tau_2 \right) - \frac{k^* - k}{n - k} H(\tau_1) - \frac{n - k^*}{n - k} H(\tau_2) \right\} \\ &\leq (n - k) \sup_{(k^* - n\alpha)/(n - n\alpha) \leq t \leq k^*/n} \{H(t\tau_1 + (1 - t)\tau_2) - tH(\tau_1) - (1 - t)H(\tau_2)\}, \end{aligned} \tag{3.4}$$

if  $1 \leq k \leq n\alpha$ . By (3.3) and (3.4) we can find a constant  $C_2 < 0$  such that

$$\mu_k \leq C_2(k^* - k), \quad \text{if } 1 \leq k \leq k^*. \tag{3.5}$$

A three-term Taylor expansion gives

$$\begin{aligned} kH(\mathbf{B}_k) - kH(\tau_1) - (k^*H(\mathbf{B}_{k^*}) - k^*H(\tau_1)) \\ &= kH'(\tau_1)(\mathbf{B}_k - \tau_1)^\top - k^*H'(\tau_1)(\mathbf{B}_{k^*} - \tau_1)^\top \\ &\quad + \frac{k}{2}(\mathbf{B}_k - \tau_1)H''(\tau_1)(\mathbf{B}_k - \tau_1)^\top - \frac{k^*}{2}(\mathbf{B}_{k^*} - \tau_1)H''(\tau_1)(\mathbf{B}_{k^*} - \tau_1)^\top \\ &\quad + R_{k,1}, \end{aligned} \tag{3.6}$$

and by the law of the iterated logarithm we have

$$\max_{1 \leq k \leq k^*} |R_{k,1}| = O_P(1). \tag{3.7}$$

Let  $\frac{1}{2} < \alpha < 1$ . Using again the law of the iterated logarithm we get

$$\max_{1 \leq k \leq k^*} |kH'(\tau_1)(\mathbf{B}_k - \tau_1)^\top - k^*H'(\tau_1)(\mathbf{B}_{k^*} - \tau_1)^\top| / (k^* - k)^\alpha = O_P(1) \tag{3.8}$$

and similar arguments yield

$$\begin{aligned} &\max_{1 \leq k \leq k^*} |k(\mathbf{B}_k - \tau_1)H''(\tau_1)(\mathbf{B}_k - \tau_1)^\top \\ &\quad - k^*(\mathbf{B}_{k^*} - \tau_1)H''(\tau_1)(\mathbf{B}_{k^*} - \tau_1)^\top| / (k^* - k)^\alpha \\ &= O_P(1). \end{aligned} \tag{3.9}$$

As in (3.6) we have

$$\begin{aligned}
 & (n-k) \left\{ H(\mathbf{B}_k^*) - H\left(\frac{k^*-k}{n-k}\boldsymbol{\tau}_1 + \frac{n-k^*}{n-k}\boldsymbol{\tau}_2\right) \right\} - (n-k^*) \{ H(\mathbf{B}_{k^*}^*) - H(\boldsymbol{\tau}_2) \} \\
 &= (n-k) H' \left( \frac{k^*-k}{n-k}\boldsymbol{\tau}_1 + \frac{n-k^*}{n-k}\boldsymbol{\tau}_2 \right) \left( \mathbf{B}_k^* - \frac{k^*-k}{n-k}\boldsymbol{\tau}_1 - \frac{n-k^*}{n-k}\boldsymbol{\tau}_2 \right)^\top \\
 & \quad + \frac{n-k}{2} \left( \mathbf{B}_k^* - \frac{k^*-k}{n-k}\boldsymbol{\tau}_1 - \frac{n-k^*}{n-k}\boldsymbol{\tau}_2 \right) H'' \left( \frac{k^*-k}{n-k}\boldsymbol{\tau}_1 + \frac{n-k^*}{n-k}\boldsymbol{\tau}_2 \right) \\
 & \quad \times \left( \mathbf{B}_k^* - \frac{k^*-k}{n-k}\boldsymbol{\tau}_1 - \frac{n-k^*}{n-k}\boldsymbol{\tau}_2 \right)^\top - (n-k^*) H'(\boldsymbol{\tau}_2) (\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_2)^\top \\
 & \quad - \frac{n-k^*}{2} (\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_2) H''(\boldsymbol{\tau}_2) (\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_2)^\top + R_{k,2} \tag{3.10}
 \end{aligned}$$

and

$$\max_{1 \leq k \leq k^*} |R_{k,2}| = O_p(1). \tag{3.11}$$

Next we write

$$\begin{aligned}
 R_{k,3} &= (n-k) H' \left( \frac{k^*-k}{n-k}\boldsymbol{\tau}_1 + \frac{n-k^*}{n-k}\boldsymbol{\tau}_2 \right) \left( \mathbf{B}_k^* - \frac{k^*-k}{n-k}\boldsymbol{\tau}_1 - \frac{n-k^*}{n-k}\boldsymbol{\tau}_2 \right)^\top \\
 & \quad - (n-k^*) H'(\boldsymbol{\tau}_2) (\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_2)^\top \\
 &= \left\{ H' \left( \frac{k^*-k}{n-k}\boldsymbol{\tau}_1 + \frac{n-k^*}{n-k}\boldsymbol{\tau}_2 \right) - H'(\boldsymbol{\tau}_2) \right\} \sum_{k^* < i \leq n} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2)^\top \\
 & \quad + H' \left( \frac{k^*-k}{n-k}\boldsymbol{\tau}_1 + \frac{n-k^*}{n-k}\boldsymbol{\tau}_2 \right) \sum_{k < i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1)^\top \\
 &= \frac{k^*-k}{n-k} (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2) H''(\boldsymbol{\tau}^*) \sum_{k^* < i \leq n} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2)^\top \\
 & \quad + H' \left( \frac{k^*-k}{n-k}\boldsymbol{\tau}_1 + \frac{n-k^*}{n-k}\boldsymbol{\tau}_2 \right) \sum_{k < i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1)^\top. \tag{3.12}
 \end{aligned}$$

Using again the law of the iterated logarithm we obtain for all  $\frac{1}{2} < \alpha < 1$  that

$$\max_{1 \leq k \leq k^*} |R_{k,3}| / (k^* - k)^\alpha = O_p(1). \tag{3.13}$$

Similar arguments yield

$$\begin{aligned} & \max_{1 \leq k \leq k^*} \left| (n-k) \left( \mathbf{B}_k^* - \frac{k^* - k}{n-k} \boldsymbol{\tau}_1 - \frac{n-k^*}{n-k} \boldsymbol{\tau}_2 \right) H'' \left( \frac{k^* - k}{n-k} \boldsymbol{\tau}_1 + \frac{n-k^*}{n-k} \boldsymbol{\tau}_2 \right) \right. \\ & \quad \times \left( \mathbf{B}_k^* - \frac{k^* - k}{n-k} \boldsymbol{\tau}_1 - \frac{n-k^*}{n-k} \boldsymbol{\tau}_2 \right)^\top - (n-k^*) (\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_2) H''(\boldsymbol{\tau}_2) \\ & \quad \left. \times (\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_2)^\top \right| / (k^* - k)^\alpha \\ & = O_p(1) \end{aligned} \tag{3.14}$$

if  $\frac{1}{2} < \alpha < 1$ .

It follows from (3.5) and (3.6)–(3.14) that

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left\{ \max_{1 \leq k \leq k^* - K} V_k > -M \right\} = 0 \tag{3.15}$$

for all  $M > 0$ . By (3.15) we get

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \{ \widehat{k} < k^* - K \} = 0, \tag{3.16}$$

and similarly one can obtain

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \{ \widehat{k} > k^* + K \} = 0. \tag{3.17}$$

Now Lemma 3.1 follows from (3.16) and (3.17).  $\square$

**Lemma 3.2.** *If the conditions of Theorem 1.4 are satisfied, then*

$$\delta^2 |\widehat{k} - k^*| = O_p(1).$$

**Proof.** Let  $1 \leq k \leq k^*$ . A three-term Taylor expansion gives

$$\begin{aligned} \mu_k &= (k^* - k) \left\{ H \left( \frac{k^* - k}{n-k} \boldsymbol{\tau}_1 + \frac{n-k^*}{n-k} \boldsymbol{\tau}_2 \right) - H(\boldsymbol{\tau}_1) \right\} \\ & \quad + (n-k^*) \left\{ H \left( \frac{k^* - k}{n-k} \boldsymbol{\tau}_1 + \frac{n-k^*}{n-k} \boldsymbol{\tau}_2 \right) - H(\boldsymbol{\tau}_2) \right\} \\ &= (k^* - k) H'(\boldsymbol{\tau}_1) \frac{n-k^*}{n-k} (\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)^\top \\ & \quad + \frac{k^* - k}{2} \left( \frac{n-k^*}{n-k} \right)^2 (\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1) H''(\boldsymbol{\tau}_1) (\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)^\top \end{aligned}$$

$$\begin{aligned}
 & + (n - k^*)H'(\boldsymbol{\tau}_2) \frac{k^* - k}{n - k} (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)^\top \\
 & + \frac{n - k^*}{2} \left( \frac{k^* - k}{n - k} \right)^2 (\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)H''(\boldsymbol{\tau}_2)(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)^\top + R_{k,4}
 \end{aligned} \tag{3.18}$$

and

$$\max_{1 \leq k \leq k^*} |R_{k,4}| / \left( \frac{(k^* - k)(n - k^*)}{n - k} \delta^2 \right) = o(1). \tag{3.19}$$

By the mean value theorem, (3.18) and (3.19) we can find a constant  $C_3 < 0$  such that

$$\mu_k \leq C_3 \frac{(k^* - k)(n - k^*)}{n - k} \delta^2 \quad \text{if } 1 \leq k \leq k^*. \tag{3.20}$$

It is easy to see that

$$\begin{aligned}
 V_k - \mu_k & = kH'(\boldsymbol{\tau}_1)(\mathbf{B}_k - \boldsymbol{\tau}_1)^\top - k^*H'(\boldsymbol{\tau}_1)(\mathbf{B}_{k^*} - \boldsymbol{\tau}_1)^\top \\
 & + (n - k)H' \left( \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 + \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right) \left( \mathbf{B}_k^* - \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 - \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right)^\top \\
 & - (n - k^*)H'(\boldsymbol{\tau}_2)(\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_2)^\top + \frac{k}{2}(\mathbf{B}_k - \boldsymbol{\tau}_1)H''(\boldsymbol{\tau}_1)(\mathbf{B}_k - \boldsymbol{\tau}_1)^\top \\
 & - \frac{k^*}{2}(\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_1)H''(\boldsymbol{\tau}_1)(\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_1)^\top \\
 & - \frac{n - k^*}{2}(\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_2)H''(\boldsymbol{\tau}_2)(\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_2)^\top \\
 & + \frac{n - k}{2} \left( \mathbf{B}_k^* - \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 - \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right) H'' \left( \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 + \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right) \\
 & \times \left( \mathbf{B}_k^* - \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 - \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right)^\top + R_{k,5}
 \end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
 |R_{k,5}| & \leq c \left\{ k \|\mathbf{B}_k - \boldsymbol{\tau}_1\|^3 + k^* \|\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_1\|^3 + (n - k^*) \|\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_2\|^3 \right. \\
 & \left. + (n - k) \left\| \mathbf{B}_k^* - \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 - \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right\|^3 \right\}.
 \end{aligned} \tag{3.22}$$

The law of the iterated logarithm yields

$$\max_{1 \leq k \leq k^*} k^{3/2} \|\mathbf{B}_k - \boldsymbol{\tau}_1\|^3 / (\log \log k)^{3/2} = O_P(1), \tag{3.23}$$

and therefore by (3.20) we have

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left\{ \max_{1 \leq k \leq k^* - K/\delta^2} k \|\mathbf{B}_k - \boldsymbol{\tau}_1\|^3 / |\mu_k| > \varepsilon \right\} = 0 \tag{3.24}$$

for all  $\varepsilon > 0$ . Using similar arguments we can establish that for all  $\varepsilon > 0$

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left\{ \max_{1 \leq k \leq k^* - K/\delta^2} |R_{k,5}| / |\mu_k| > \varepsilon \right\} = 0. \tag{3.25}$$

Writing

$$\begin{aligned} & kH'(\boldsymbol{\tau}_1)(\mathbf{B}_k - \boldsymbol{\tau}_1)^\top - k^*H'(\boldsymbol{\tau}_1)(\mathbf{B}_{k^*} - \boldsymbol{\tau}_1)^\top \\ & + (n - k)H' \left( \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 + \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right) \left( \mathbf{B}_k^* - \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 - \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right)^\top \\ & - (n - k^*)H'(\boldsymbol{\tau}_2)(\mathbf{B}_{k^*} - \boldsymbol{\tau}_2)^\top \\ & = \left\{ H' \left( \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 + \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right) - H'(\boldsymbol{\tau}_1) \right\} \sum_{k < i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1)^\top \\ & + \left\{ H' \left( \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 + \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right) - H'(\boldsymbol{\tau}_2) \right\} \sum_{k^* < i \leq n} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2)^\top \\ & = R_{k,6} + R_{k,7}, \end{aligned} \tag{3.26}$$

we get

$$\max_{1 \leq k \leq k^*} |R_{k,6}| / \left( \frac{n - k^*}{n - k} \delta((k^* - k) \log \log(k^* - k))^{1/2} \right) = O_P(1) \tag{3.27}$$

and by the central limit theorem we have

$$\max_{1 \leq k \leq k^*} |R_{k,7}| / \left( \frac{k^* - k}{n - k} \delta(n - k^*)^{1/2} \right) = O_P(1). \tag{3.28}$$

Hence by (1.8) and (3.20) we have

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left\{ \max_{1 \leq k \leq k^* - K/\delta^2} |R_{k,6} + R_{k,7}| / |\mu_k| > \varepsilon \right\} = 0 \tag{3.29}$$

for all  $\varepsilon > 0$ .

By the law of the iterated logarithm we have

$$\max_{1 \leq k \leq k^*} |k(\mathbf{B}_k - \boldsymbol{\tau}_1)H''(\boldsymbol{\tau}_1)(\mathbf{B}_k - \boldsymbol{\tau}_1)^\top| = O_P(\log \log k^*) \tag{3.30}$$

and (1.8) implies

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq \alpha n} \frac{(n - k) \log \log n}{(k^* - k)(n - k^*)\delta^2} = 0 \tag{3.31}$$

for all  $0 < \alpha < \lambda$ . Thus by (3.20) we have

$$\max_{1 \leq k \leq \alpha n} |k(\mathbf{B}_k - \boldsymbol{\tau}_1)H''(\boldsymbol{\tau}_1)(\mathbf{B}_k - \boldsymbol{\tau}_1)^T|/|\mu_k| = O_P(1), \tag{3.32}$$

if  $0 < \alpha < \lambda$ . Similarly, the central limit theorem, (1.8) and (3.20) yield that for all  $0 < \alpha < \lambda$

$$\max_{1 \leq k \leq \alpha n} |k^*(\mathbf{B}_{k^*} - \boldsymbol{\tau}_1)H''(\boldsymbol{\tau}_1)(\mathbf{B}_{k^*} - \boldsymbol{\tau}_1)^T|/|\mu_k| = O_P(1). \tag{3.33}$$

Using again the law of the iterated logarithm and the central limit theorem we obtain

$$\begin{aligned} & \max_{\alpha n \leq k \leq k^* - K/\delta^2} \left| \{k^{1/2}(\mathbf{B}_k - \boldsymbol{\tau}_1) - k^{*1/2}(\mathbf{B}_{k^*} - \boldsymbol{\tau}_1)\}H''(\boldsymbol{\tau}_1)k^{1/2}(\mathbf{B}_k - \boldsymbol{\tau}_1)^T|/|\mu_k| \right. \\ &= O_P(1) \max_{\alpha n \leq k \leq k^* - K/\delta^2} \frac{(n-k)(\log \log k)^{1/2}}{(k^* - k)(n - k^*)\delta^2} \left\{ k^{-1/2} \left\| \sum_{k < i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1) \right\| \right. \\ & \quad \left. + \frac{(k^* - k)}{k^{1/2}(k^* + k)^{1/2}} \left( \frac{1}{k^*} \right)^{1/2} \left\| \sum_{1 \leq i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1) \right\| \right\} \\ &= O_P(1) \max_{\alpha n \leq k \leq k^* - K/\delta^2} \frac{(n-k)(\log \log k)^{1/2}}{(k^* - k)(n - k^*)\delta^2} \\ & \quad \times \left\{ \left( \frac{(k^* - k) \log \log(k^* - k)}{k} \right)^{1/2} + \frac{k^* - k}{(kk^*)^{1/2}} \right\} \\ &= O_P(1) \left( \frac{\log \log K}{K} \right)^{1/2}, \tag{3.34} \end{aligned}$$

where  $O_P(1)$  does not depend on  $K$ . Putting together (3.32), (3.33) and (3.34) we obtain

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left\{ \max_{1 \leq k \leq k^* - K/\delta^2} |k(\mathbf{B}_k - \boldsymbol{\tau}_1)H''(\boldsymbol{\tau}_1)(\mathbf{B}_k - \boldsymbol{\tau}_1)^T - k^*(\mathbf{B}_{k^*} - \boldsymbol{\tau}_1)H''(\boldsymbol{\tau}_1)(\mathbf{B}_{k^*} - \boldsymbol{\tau}_1)^T|/|\mu_k| > \varepsilon \right\} = 0 \tag{3.35}$$

for all  $\varepsilon > 0$ . Similarly to (3.35) one can prove that

$$\begin{aligned} & \lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left\{ \max_{1 \leq k \leq k^* - K/\delta^2} \left| (n - k) \left( \mathbf{B}_k^* - \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 \right. \right. \right. \\ & \quad \left. \left. - \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right) H'' \left( \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 + \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right) \left( \mathbf{B}_k^* - \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 - \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right)^\top \right. \\ & \quad \left. \left. - (n - k^*) (\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_2) H''(\boldsymbol{\tau}_2) (\mathbf{B}_{k^*}^* - \boldsymbol{\tau}_2)^\top \right| / |\mu_k| > \varepsilon \right\} \\ & = 0 \end{aligned} \tag{3.36}$$

for all  $\varepsilon > 0$ . By (3.21), (3.25), (3.29), (3.35) and (3.36) we have

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left\{ \max_{1 \leq k \leq k^* - K/\delta^2} |V_k - \mu_k| / |\mu_k| > \varepsilon \right\} = 0 \tag{3.37}$$

for all  $\varepsilon > 0$ , which by (3.20) immediately implies

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \{ \widehat{k} < k^* - K/\delta^2 \} = 0. \tag{3.38}$$

Similar arguments yield

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \{ \widehat{k} > k^* + K/\delta^2 \} = 0,$$

which also completes the proof of Lemma 3.2.  $\square$

**Lemma 3.3.** *If the conditions of Theorem 1.5 are satisfied, then*

$$|\widehat{k} - k^*| = O_P(1). \tag{3.39}$$

**Proof.** First we assume that  $1 \leq k \leq k^*$ . The Taylor formula yields

$$\begin{aligned} & (n - k) \left\{ H \left( \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 + \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right) - H(\boldsymbol{\tau}_2) \right\} \\ & = (k^* - k) H'(\boldsymbol{\tau}_2) (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)^\top + \frac{1}{2} \frac{(k^* - k)^2}{n - k} (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2) H''(\boldsymbol{\tau}^*) (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)^\top. \end{aligned}$$

Since  $\max_{1 \leq k \leq k^*} \frac{k^* - k}{n - k} \rightarrow 0$ , as  $n \rightarrow \infty$ , by C.3 we can find a constant  $C_4 < 0$  such that

$$\mu_k \leq C_4 (k^* - k) \quad \text{if } 1 \leq k \leq k^*. \tag{3.40}$$

As in the proof of Lemma 3.1, (3.40) implies that

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left\{ \max_{1 \leq k \leq k^* - K} V_k > -M \right\} = 0 \tag{3.41}$$

for all  $M > 0$ .

Since  $k^*/n \rightarrow 0$ , we need different estimates for  $\max_{k^* + K \leq k < n} V_k$ . We show that

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left\{ \max_{k^* + K \leq k < n} |V_k - \mu_k| / |\mu_k| > \varepsilon \right\} = 0 \tag{3.42}$$

for all  $\varepsilon > 0$ . Let  $k^* \leq k < n$ . Using again the Taylor formula we get

$$\begin{aligned} \mu_k &= k \left\{ H \left( \frac{k^*}{k} \tau_1 + \frac{k - k^*}{k} \tau_2 \right) - H(\tau_1) \right\} + (k^* - k) \{ H(\tau_2) - H(\tau_1) \} \\ &= (k^* - k) \{ H'(\tau_1)(\tau_1 - \tau_2)^T + H(\tau_2) - H(\tau_1) \} \\ &\quad + \frac{k}{2} \left( \frac{k - k^*}{k} \right)^2 (\tau_1 - \tau_2) H''(\tau^*)(\tau_1 - \tau_2)^T, \end{aligned}$$

and therefore we can find two constants  $\beta > 1$  and  $C_5 < 0$  such that

$$\mu_k \leq C_5(k - k^*) \quad \text{if } k^* \leq k \leq \beta k^*. \tag{3.43}$$

Similarly to (3.6)–(3.14) one can show that

$$\max_{k^* \leq k \leq \beta k^*} |V_k - \mu_k| / (k - k^*)^\alpha = O_P(1) \tag{3.44}$$

for all  $\frac{1}{2} < \alpha < 1$ . Now (3.43) and (3.44) yield

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left\{ \max_{k^* + K \leq k \leq \beta k^*} |V_k - \mu_k| / |\mu_k| > \varepsilon \right\} = 0 \tag{3.45}$$

for all  $\varepsilon > 0$ . By condition C.4, for each  $1 < \beta < \gamma < \infty$  we can find  $C_6 = C_6(\beta, \gamma) < 0$  such that

$$\mu_k \leq C_6 k^* \quad \text{if } \beta k^* \leq k \leq \gamma k^*. \tag{3.46}$$

Noting that

$$\begin{aligned} \mu_k &= k^* \{ H(\tau_2) - H(\tau_1) + H'(\tau_2)(\tau_1 - \tau_2)^T \} \\ &\quad + \frac{k}{2} \left( \frac{k^*}{k} \right)^2 (\tau_1 - \tau_2) H''(\tau^*)(\tau_1 - \tau_2)^T, \end{aligned}$$

we can choose for each large enough  $\gamma$  a constant  $C_7 = C_7(\gamma) < 0$  such that

$$\mu_k \leq C_7 k^* \quad \text{if } \gamma k^* \leq k < n. \tag{3.47}$$

Putting together (3.46) and (3.47) we get

$$\mu_k \leq C_8 k^* \quad \text{if } \beta k^* \leq k < n \tag{3.48}$$

with some constant  $C_8 < 0$ . By the law of the iterated logarithm and the central limit theorem we have

$$\begin{aligned} & \max_{\beta k^* \leq k < n} \left| k \left( H(\mathbf{B}_k) - H \left( \frac{k^*}{k} \boldsymbol{\tau}_1 + \frac{k - k^*}{k} \boldsymbol{\tau}_2 \right) \right) - k^* (H(\mathbf{B}_{k^*}) - H(\boldsymbol{\tau}_1)) \right. \\ & \quad \left. + (n - k)(H(\mathbf{B}_k^*) - H(\boldsymbol{\tau}_2)) - (n - k^*)(H(\mathbf{B}_{k^*}^*) - H(\boldsymbol{\tau}_2)) \right| \\ &= \max_{\beta k^* \leq k < n} \left| H' \left( \frac{k^*}{k} \boldsymbol{\tau}_1 + \frac{k - k^*}{k} \boldsymbol{\tau}_2 \right) \left\{ \sum_{1 \leq i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1)^\top \right. \right. \\ & \quad \left. \left. + \sum_{k^* < i \leq k} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2)^\top \right\} - H'(\boldsymbol{\tau}_1) \sum_{1 \leq i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1)^\top \right. \\ & \quad \left. + H'(\boldsymbol{\tau}_2) \sum_{k < i < n} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2)^\top - H'(\boldsymbol{\tau}_2) \sum_{k^* < i < n} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2)^\top \right| + O_p(\log \log n) \\ & \leq \max_{\beta k^* \leq k < n} \left| \left( H' \left( \frac{k^*}{k} \boldsymbol{\tau}_1 + \frac{k - k^*}{k} \boldsymbol{\tau}_2 \right) - H'(\boldsymbol{\tau}_1) \right) \sum_{1 \leq i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1)^\top \right| \\ & \quad + \max_{\beta k^* \leq k < n} \left| \left( H' \left( \frac{k^*}{k} \boldsymbol{\tau}_1 + \frac{k - k^*}{k} \boldsymbol{\tau}_2 \right) - H'(\boldsymbol{\tau}_2) \right) \sum_{k^* < i \leq k} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2)^\top \right| \\ & \quad + O_p(\log \log n) \\ &= O_p(1) \max_{\beta k^* \leq k < n} \frac{k - k^*}{k} \sqrt{k^*} + O_p(1) \max_{\beta k^* \leq k < n} \frac{k^*}{k} (k - k^*)^{1/2} (\log \log n)^{1/2} \\ & \quad + O_p(\log \log n) \\ &= o(k^*). \end{aligned} \tag{3.49}$$

By (3.45), (3.48) and (3.49) we have

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left\{ \max_{k^* + K \leq k < n} |V_k - \mu_k| / |\mu_k| > \varepsilon \right\} = 0 \tag{3.50}$$

for all  $\varepsilon > 0$ . Lemma 3.3 follows from (3.41) and (3.50).  $\square$

**Lemma 3.4.** *If the conditions of Theorem 1.6 are satisfied, then*

$$\delta^2 |\widehat{k} - k^*| = O_p(1).$$

**Proof.** The proof is a combination of the proofs of Lemmas 3.2 and 3.3 and therefore it is omitted.  $\square$

In Lemmas 3.1 and 3.3 we showed that the difference between  $k^*$  and  $\widehat{k}$  is bounded. This shows that the distribution is determined by those values of  $V_k$  when  $k - k^*$  is bounded. Next we consider the weak convergence of  $V_{k^*+k}$ .

**Lemma 3.5.** *If the conditions of Theorem 1.3 or 1.5 are satisfied, then for each positive integer  $N$  we have*

$$\{V_{k^*+k}, k = 0, \pm 1, \pm 2, \dots, \pm N\} \xrightarrow{\mathcal{L}} \{Z_k, k = 0, \pm 1, \pm 2, \dots, \pm N\},$$

where  $Z_k$  is defined in (1.11).

**Proof.** Let  $1 \leq k \leq k^*$ . Taylor expansion gives

$$\max_{k^*-N \leq k \leq k^*} |V_k - \mu_k - V_{k,1} - V_{k,2} - V_{k,3} - V_{k,4}| = O_P(n^{-1/2}), \tag{3.51}$$

where

$$\begin{aligned} V_{k,1} &= H'(\boldsymbol{\tau}_1) \sum_{1 \leq i \leq k} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1)^\top - H'(\boldsymbol{\tau}_1) \sum_{1 \leq i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1)^\top, \\ V_{k,2} &= \frac{1}{2k} \left\{ \sum_{1 \leq i \leq k} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1) \right\} H''(\boldsymbol{\tau}_1) \left\{ \sum_{1 \leq i \leq k} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1) \right\}^\top \\ &\quad - \frac{1}{2k^*} \left\{ \sum_{1 \leq i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1) \right\} H''(\boldsymbol{\tau}_1) \left\{ \sum_{1 \leq i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1) \right\}^\top, \\ V_{k,3} &= H' \left( \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 + \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right) \\ &\quad \times \left\{ \sum_{k < i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1)^\top + \sum_{k^* < i \leq n} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2)^\top \right\} \\ &\quad - H'(\boldsymbol{\tau}_2) \sum_{k^* < i \leq n} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2)^\top \end{aligned}$$

and

$$\begin{aligned} V_{k,4} &= \frac{1}{2(n - k)} \left\{ \sum_{k < i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1) + \sum_{k^* < i \leq n} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2) \right\} H'' \left( \frac{k^* - k}{n - k} \boldsymbol{\tau}_1 \right. \\ &\quad \left. + \frac{n - k^*}{n - k} \boldsymbol{\tau}_2 \right) \left\{ \sum_{k < i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1) + \sum_{k^* < i \leq n} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2) \right\}^\top \\ &\quad - \frac{1}{2(n - k^*)} \left\{ \sum_{k^* < i \leq n} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2) \right\} H''(\boldsymbol{\tau}_2) \left\{ \sum_{k^* < i \leq n} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2) \right\}^\top. \end{aligned}$$

The central limit theorem yields that

$$\max_{k^* - N \leq k \leq k^*} |V_{k,2} + V_{k,4}| = o_P(1) \tag{3.52}$$

and

$$\max_{k^* - N \leq k \leq k^*} |V_{k,1} + V_{k,3} - (H'(\tau_2) - H'(\tau_1)) \sum_{k < i \leq k^*} (T(X_i) - \tau_1)^T| = o_P(1). \tag{3.53}$$

It is easy to see that

$$\max_{k^* - N \leq k \leq k^*} \left| \mu_k - (k^* - k) \{ H(\tau_2) - H(\tau_1) + H'(\tau_2)(\tau_1 - \tau_2)^T \} \right| = o(1). \tag{3.54}$$

Putting together (3.51)–(3.54) we get

$$\begin{aligned} \max_{k^* - N \leq k \leq k^*} & \left| V_k - (H'(\tau_2) - H'(\tau_1)) \sum_{k < i \leq k^*} (T(X_i) - \tau_1)^T \right. \\ & \left. + (k^* - k) \{ H(\tau_2) - H(\tau_1) + H'(\tau_2)(\tau_1 - \tau_2)^T \} \right| = o_P(1). \end{aligned} \tag{3.55}$$

Similar arguments give

$$\begin{aligned} \max_{k^* \leq k \leq k^* + N} & \left| V_k - (H'(\tau_1) - H'(\tau_2)) \sum_{k^* < i \leq k} (T(X_i) - \tau_2)^T \right. \\ & \left. + (k^* - k) \{ H(\tau_1) - H(\tau_2) + H'(\tau_1)(\tau_2 - \tau_1)^T \} \right| = o_P(1), \end{aligned} \tag{3.56}$$

and therefore Lemma 3.5 follows from (3.55) and (3.56).  $\square$

Next we consider the problem of Lemma 3.5 under the conditions of Theorems 1.4 and 1.6.

**Lemma 3.6.** *If the conditions of Theorem 1.4 or 1.6 hold, then for each  $K > 0$  we have*

$$\{V_{k^* + t/\delta^2}, -K \leq t \leq K\} \xrightarrow{\mathcal{L}} \{W^*(t), -K \leq t \leq K\},$$

where  $W^*$  is defined in (1.19).

**Proof.** Using Taylor expansion, as in the proof of Lemma 3.6, with the applications of the law of the iterated logarithm and the central limit theorem one can show that

$$\max_{k^* - K/\delta^2 \leq k \leq k^*} \left| V_k - \mu_k - (H'(\tau_2) - H'(\tau_1)) \sum_{k < i \leq k^*} (T(X_i) - \tau_1)^T \right| = o_P(1). \tag{3.57}$$

Similarly,

$$\max_{k^* \leq k \leq k^* + K/\delta^2} \left| V_k - \mu_k - (H'(\boldsymbol{\tau}_1) - H'(\boldsymbol{\tau}_2)) \sum_{k < i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2)^\top \right| = o_P(1), \quad (3.58)$$

and elementary calculations yield

$$\sup_{-K \leq t \leq K} \left| \mu_{k^* + t/\delta^2} - \frac{1}{2} |t| (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2) H''(\boldsymbol{\tau}_A) (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)^\top / \delta^2 \right| = o(1).$$

Now the weak convergence of partial sums of i.i.d. random vectors gives the result.  $\square$

**Lemma 3.7.** (i) *If the conditions of Theorem 1.3 hold, then*

$$n^{-1/2} \{ \log A_{k^*} - \mu^* \} \xrightarrow{\mathcal{D}} N(0, \sigma_1^2),$$

where  $\sigma_1^2$  is defined in (1.13).

(ii) *If the conditions of Theorem 1.4 hold, then we have*

$$(n\delta^2)^{-1/2} \{ \log A_{k^*} - \mu^* \} \xrightarrow{\mathcal{D}} N(0, \sigma_2^2),$$

where  $\sigma_2^2$  is defined in (1.18).

(iii) *If the conditions of Theorem 1.5 hold, then we have*

$$(k^*)^{-1/2} \{ \log A_{k^*} - \mu^* \} \xrightarrow{\mathcal{D}} N(0, \sigma_3^2)$$

where  $\sigma_3^2$  is defined in (1.23).

(iv) *If the conditions of Theorem 1.6 hold, then we have*

$$(k^* \delta^2)^{-1/2} \{ \log A_{k^*} - \mu^* \} \xrightarrow{\mathcal{D}} N(0, \sigma_4^2)$$

where  $\sigma_4^2 = \sigma_A^2$  is defined in (1.17).

**Proof.** First we note that

$$\begin{aligned} \log A_{k^*} - \mu^* &= k^*(H(\mathbf{B}_{k^*}) - H(\boldsymbol{\tau}_1)) + (n - k^*)(H(\mathbf{B}_{k^*}^*) - H(\boldsymbol{\tau}_2)) \\ &\quad - n \left( H(\mathbf{B}_n) - H \left( \frac{k^*}{n} \boldsymbol{\tau}_1 + \frac{n - k^*}{n} \boldsymbol{\tau}_2 \right) \right) \\ &= \left\{ H'(\boldsymbol{\tau}_1) - H' \left( \frac{k^*}{n} \boldsymbol{\tau}_1 + \frac{n - k^*}{n} \boldsymbol{\tau}_2 \right) \right\} \sum_{1 \leq i \leq k^*} (\mathbf{T}(X_i) - \boldsymbol{\tau}_1)^\top \\ &\quad + \left\{ H'(\boldsymbol{\tau}_2) - H' \left( \frac{k^*}{n} \boldsymbol{\tau}_1 + \frac{n - k^*}{n} \boldsymbol{\tau}_2 \right) \right\} \sum_{k^* < i \leq n} (\mathbf{T}(X_i) - \boldsymbol{\tau}_2)^\top \\ &\quad + O_P(1). \end{aligned}$$

Hence the central limit theorem implies Lemma 3.7.  $\square$

**Proofs of Theorems 1.3–1.7.** Lemmas 3.1 and 3.5 imply (1.14) and (1.16) follows from Lemmas 3.1, 3.5 and 3.7. Similarly, combining Lemmas 3.1–3.7 we get the proofs of Theorems 1.3–1.6.

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