

Unit roots and double smooth transitions

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ABSTRACT *Techniques for testing the null hypothesis of difference stationarity against stationarity around some deterministic function have received much attention. In particular, unit root tests where the alternative is stationarity around a smooth transition in a linear trend have recently been proposed to permit the possibility of non-instantaneous structural change. In this paper we develop tests extending such an approach in order to admit more than one structural change. The analysis is motivated by time series that appear to undergo two smooth transitions in the linear trend, and the application of the new tests to two such series (average global temperature and US consumer prices) highlights the benefits of this double transition extension.*

1 Introduction

The issue of characterizing a time series process as (trend) stationary or difference stationary has received much attention in the econometrics and statistics literature and, consequently, following the seminal work of Dickey & Fuller (1979), many unit root tests have been developed. Following Perron (1989, 1990), attention also began to focus on trend functions containing a break, the presence of which complicated the unit root testing procedure, particularly when the break point was unknown (see, for example, Zivot & Andrews, 1992; Vogelsang & Perron, 1998). To provide more flexible trend functions, multiple breaks and higher order polynomials in time have also been considered. Examples of the fitting of linear and quadratic segmented multi-break polynomial trend functions and their testing against a unit root may be found in Mills & Crafts (1996, 2000), while Bai & Perron (1998) consider the possibility of testing for multiple structural changes in a linear regression model. The importance of correctly determining structural

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breaks and the stochastic or deterministic nature of the trend has also recently been emphasized in a forecasting context by Clements & Hendry (1999, 2000).

A drawback of the above examples of trend functions is that the breaks are instantaneous. In a recent paper, Leybourne *et al.* (1998) proposed a set of unit root tests where the process under the alternative hypothesis is stationary around a smooth transition in the linear trend, which is intuitively appealing as it permits structural shifts to occur gradually over time. These Leybourne-Newbold-Vougas tests make use of the logistic smooth transition function, following work by Bacon & Watts (1971), Granger & Teräsvirta (1993) and Lin & Teräsvirta (1994), allowing the speed and midpoint of the transition to be determined endogenously. An application of this technique may be found in Crafts & Mills (1997), where the resulting smooth transition trend is favourably compared to a three break segmented quadratic fitted to the same industrial production series in Mills & Crafts (1996).

In this paper, we consider a further extension to these unit root tests in terms of the specification of the alternative hypothesis as a flexible, but deterministic, trend. Given the above discussion, it is highly plausible that more than one structural change may have occurred during the observation period of the time series being investigated. We therefore consider the case where the unit root null is tested against an alternative of stationarity around two smooth transitions in linear trend.

Our analysis is motivated by visual inspection of time series such as average global temperature, which appears to undergo two structural changes in trend, with the transitions between regimes being relatively smooth. Furthermore, despite the fact that this series does not appear to have a unit root, tests permitting at most one structural change fail to reject the unit root null hypothesis. In contrast, we find that allowing for *two* smooth transitions in the linear trend under the alternative hypothesis does lead to a rejection of the null, and thus to a more appropriate, and reliable, trend specification.

The paper proceeds as follows. Section 2 presents the double smooth transition unit root tests extension that we consider, their simulated critical values, and power comparisons with their single transition counterparts. In Section 3 we conduct empirical applications using the aforementioned average global temperature data, and also the Nelson & Plosser (1982) US consumer price series. Both series illustrate the value of the double transition extension. Conclusions are drawn in Section 4.

2 Unit root tests

Following the precedent of Leybourne *et al.* (1998), we consider three models for the alternative hypothesis, against which the unit root null could be tested. Each model represents a stationary process around two smooth transitions in the linear trend; the differences are in the order of the deterministic. Model A contains no trend and involves transitions in the mean only, Model B has transitions in the intercept only but permits a fixed trend component, while Model C allows for the most generality with transitions in both intercept and trend:

$$\text{Model A: } y_t = \alpha_1 + \alpha_2 S_{1t}(\gamma_1, \tau_1) + \alpha_3 S_{2t}(\gamma_2, \tau_2) + v_t$$

$$\text{Model B: } y_t = \alpha_1 + \beta_1 t + \alpha_2 S_{1t}(\gamma_1, \tau_1) + \alpha_3 S_{2t}(\gamma_2, \tau_2) + v_t$$

$$\text{Model C: } y_t = \alpha_1 + \beta_1 t + \alpha_2 S_{1t}(\gamma_1, \tau_1) + \beta_2 t S_{1t}(\gamma_1, \tau_1) + \alpha_3 S_{2t}(\gamma_2, \tau_2) + \beta_3 t S_{2t}(\gamma_2, \tau_2) + v_t.$$

The disturbance term v_t in each model is a stationary process with zero mean, and the transition functions $S_{it}(\gamma_i, \tau_i)$ are logistic smooth transition functions defined by:

$$S_{it}(\gamma_i, \tau_i) = [1 + \exp\{-\gamma_i(t - \tau_i T)\}]^{-1} \quad i = 1, 2$$

for a sample size T . The midpoints of the two transitions are given by $\tau_1 T$ and $\tau_2 T$ respectively; the transitions speeds are allowed to differ, and are respectively determined by γ_1 and γ_2 .

Tests of a unit root null hypothesis against one of the above models as the alternative can be conducted using the two-step procedure employed by Leybourne *et al.* (1998). The first step involves non-linear estimation of model A, B or C, minimizing the sum of squared residuals (analytically over α_i, β_i , numerically over γ_i, τ_i). The resulting residuals \hat{v}_t are then used to estimate the augmented Dickey-Fuller regression:

$$\Delta \hat{v}_t = \rho \hat{v}_{t-1} + \sum_{i=1}^k \delta_i \Delta \hat{v}_{t-i} + \eta_t$$

where the number of lagged difference terms, k , is determined by some method of order selection. The test statistic is then the t -ratio associated with the ordinary least squares estimate of ρ . Modifying the Leybourne-Newbold-Vougas notation, we denote the test statistics associated with the use of models A, B and C as $s_{2\alpha}$, $s_{2\alpha(\beta)}$ and $s_{2\alpha\beta}$ respectively.

Table 1 presents critical values for these three tests at the 10%, 5% and 1% levels, obtained by Monte Carlo simulation using 10 000 replications. The null hypothesis was generated as a random walk (without drift) with errors drawn from the standard normal distribution. When optimizing numerically in the first step of the test procedure, the Broyden, Fletcher, Goldfarb and Shanno algorithm in the OPTMUM subroutine for GAUSS was employed, and a grid of starting values for the midpoint fractions τ_1, τ_2 was considered each time. In the subsequent augmented Dickey-Fuller regressions, the value of k was set equal to its true value of zero. Critical values for a number of sample sizes are reported, including one large sample to approximate the tests' asymptotic critical values. The critical values are larger in absolute value than those for the single transition tests, as would be expected: for example, with $T = 200$, the 5%-level critical values are $-5.07, -5.53$ and -6.01 for $s_{2\alpha}, s_{2\alpha(\beta)}$ and $s_{2\alpha\beta}$ respectively, compared with $-4.16, -4.63$ and -4.87 for their single transition counterparts.

In an extension to the Leybourne-Newbold-Vougas unit root tests, Sollis *et al.* (1999) considered the further possibility that the transition function under the alternative may often be asymmetric, with the adjustments into and out of the

TABLE 1. Null critical values for unit root tests against stationarity around double smooth transitions in linear trend

T	$s_{2\alpha}$			$s_{2\alpha(\beta)}$			$s_{2\alpha\beta}$		
	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
50	-5.33	-5.73	-6.49	-6.07	-6.48	-7.37	-6.74	-7.16	-8.14
100	-5.04	-5.37	-6.05	-5.64	-5.97	-6.64	-6.20	-6.55	-7.25
150	-4.94	-5.27	-5.84	-5.50	-5.80	-6.39	-6.02	-6.32	-6.90
200	-4.90	-5.20	-5.80	-5.44	-5.74	-6.36	-5.93	-6.21	-6.79
1000	-4.79	-5.07	-5.64	-5.25	-5.53	-6.05	-5.74	-6.01	-6.59

transition phase occurring at different rates. Using this notion, these authors proposed three tests of the unit root null corresponding to those of Leybourne *et al.* (1998), with the transition determined by the generalized logistic function (Nelder, 1961):

$$G_t(\gamma, \tau, \theta) = [1 + \exp\{-\gamma(t - \tau T)/\theta\}]^{-\theta} \quad 0 < \theta \leq 1.$$

The additional parameter θ controls the degree of asymmetry, with $\theta = 1$ corresponding to the symmetric logistic function. Application of this generalization to our framework of double smooth transitions is equally desirable, allowing both transition functions to be potentially asymmetric, and to different degrees. In order to estimate a model with two asymmetric smooth transitions in a linear trend, we must use a grid of possible values for the asymmetry parameters, along the lines of Sollis *et al.* (1999). However, the number of possible pairings of these parameters for an appropriately fine grid is very large. Whilst this is not a problem theoretically, the limits of computing power constrain the number of numerical optimizations that can be performed in a realistic time frame. Unfortunately, these limits currently prevent simulation of critical values for such tests.

In order to investigate the power of the double transition unit root tests, it is useful to conduct comparisons with the Leybourne–Newbold–Vougas single transition tests. While the greater generality of the double transition approach may result in a loss of power for series that have at most one transition, there is potential for power gains to be present when the true data generating process is stationary around two smooth transitions in linear trend. It is instructive therefore to simulate empirical powers for such cases. Focusing on the s_α and $s_{2\alpha}$ tests for purposes of tractability, we generated series from the following model:

$$y_t = 1 + \sqrt{T}S_{1t}(\gamma_1, \tau_1) + \sqrt{T}S_{2t}(\gamma_2, \tau_2) + \mu_t \quad \mu_t = \phi\mu_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim IN(0, \sigma_\varepsilon^2)$, i.e. y_t is stationary around two transitions in the mean with a break size of \sqrt{T} . For a given sample size (and proportional break sizes), the relative powers depend primarily on the speed and timing of the transitions; test power is also determined by the parameters of the underlying stationary process (the autoregressive parameter ϕ and the error standard deviation σ_ε). In order to obtain a range of interesting power comparisons, we set $\phi = 0.8$ and $\sigma_\varepsilon = 0.2$, fixed the timing of the transitions at $\tau_1 = 0.3$ and $\tau_2 = 0.7$, and conducted experiments for different transition speeds, letting $\gamma_1 = \gamma_2$ for simplicity. Table 2 provides test rejection frequencies for experiments based on 5000 replications and sample sizes

TABLE 2. Empirical powers of unit root tests against stationarity around single and double smooth transitions in linear trend ($\tau_1 = 0.3, \tau_2 = 0.7$)

	$T = 100$				$T = 200$			
	s_α		$s_{2\alpha}$		s_α		$s_{2\alpha}$	
	0.10	0.05	0.10	0.05	0.10	0.05	0.10	0.05
$\gamma_1 = \gamma_2$								
0.01	0.560	0.367	0.413	0.255	0.996	0.975	0.931	0.831
0.05	0.619	0.426	0.473	0.307	0.239	0.106	0.913	0.801
0.10	0.123	0.058	0.440	0.287	0.000	0.000	0.910	0.791
0.50	0.000	0.000	0.262	0.146	0.000	0.000	0.813	0.652
1.00	0.000	0.000	0.158	0.081	0.000	0.000	0.757	0.585
5.00	0.000	0.000	0.092	0.043	0.000	0.000	0.716	0.521

$T = 100$ and $T = 200$. As with the simulation of the critical values, the value of k in the Dickey-Fuller regressions was set equal to its correct value of zero in the computation of the test statistics.

Considering the results for $T = 200$, $s_{2\alpha}$ has substantially greater power than s_α for all but the slowest transitions. The difference in the powers is dramatic, with the power of s_α rapidly decreasing to zero as the transition speeds increase, while that of $s_{2\alpha}$ remains much higher. For the very slowest transitions ($\gamma_1 = \gamma_2 = 0.01$), the generated process is very close to stationarity about a linear trend, so it is not surprising that $s_{2\alpha}$ has less power than s_α in this extreme case. The power of $s_{2\alpha}$ decreases steadily as the transitions become more rapid and approximate instantaneous structural breaks, but moderate power is still achieved for the fastest transitions considered, in marked contrast to s_α . For $T = 100$, the power gains of $s_{2\alpha}$ over s_α are less striking. The single transition test now has superior power for the two slowest transition speeds, before again dropping to trivial levels. For $s_{2\alpha}$, a similar pattern is observed to the $T = 200$ case, although now the powers are lower overall and the decrease for near-instantaneous transitions is more severe. In general, the double smooth transition test performs best for larger samples and intermediate speeds for the two transitions.

Other power simulations were also carried out. Altering the midpoints of the smooth transitions τ_1 and τ_2 generally resulted in very similar powers to those reported, although the relative power of s_α improved slightly when the transition midpoints were closer together. Allowing the transition speeds γ_1 and γ_2 to differ produced powers roughly the same as when these parameters were both set at an in-between value; for example, results for $\gamma_1 = 0.01$ and $\gamma_2 = 5$ with $T = 200$ were close to the empirical powers reported for $\gamma_1 = \gamma_2 = 0.5$.

3 Empirical applications

In this section we consider applications of the unit root tests to two interesting time series. Both have been analysed previously using more conventional trend functions but both, from their plots shown in Figures 1 and 2, potentially display non-abrupt shifts of trend. The first is the annual average global temperature data for the period 1856-1998 (143 observations), which was obtained from the Climate Research Unit at the University of East Anglia (www.cru.uea.ac.uk). The series is a combination of land air and sea surface temperatures, expressed as deviations from the average over 1961-90; further details regarding its construction can be found in Jones (1994), Parker *et al.* (1994) and Parker *et al.* (1995). The second time series considered is annual data on the US consumer price index for 1860-1970 (111 observations), as studied by Nelson & Plosser (1982) in their influential work on characterizing economic time series.

In previous work, Galbraith & Green (1992) and Seater (1993) examined global average temperature from the perspective of testing for unit roots using data up to the late 1980s. In this regard, their findings are limited by the fact that the time span did not cover the continued substantial temperature rises in the last decade. Galbraith & Green studied a monthly series for the period 1880-1988, and found sufficient evidence, using an augmented Dickey-Fuller τ_c test, to reject the unit root null in favour of stationarity about a linear trend. However, these authors speculated about the possibility that a longer time series may result in a non-linear trend being a better model for the data. Seater, on the other hand, using annual data for 1854-1989, did not find evidence against the null using the τ_c test, but

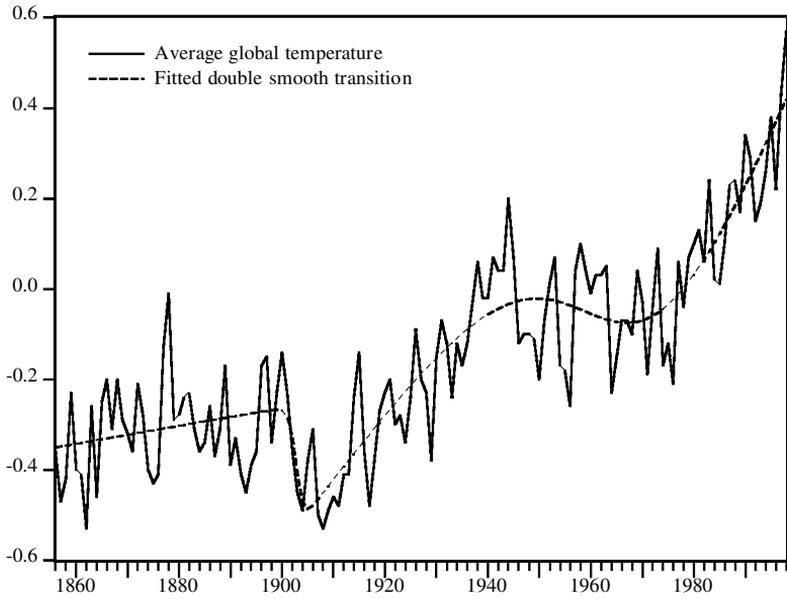


FIG. 1. Average global temperature and fitted double smooth transition function.

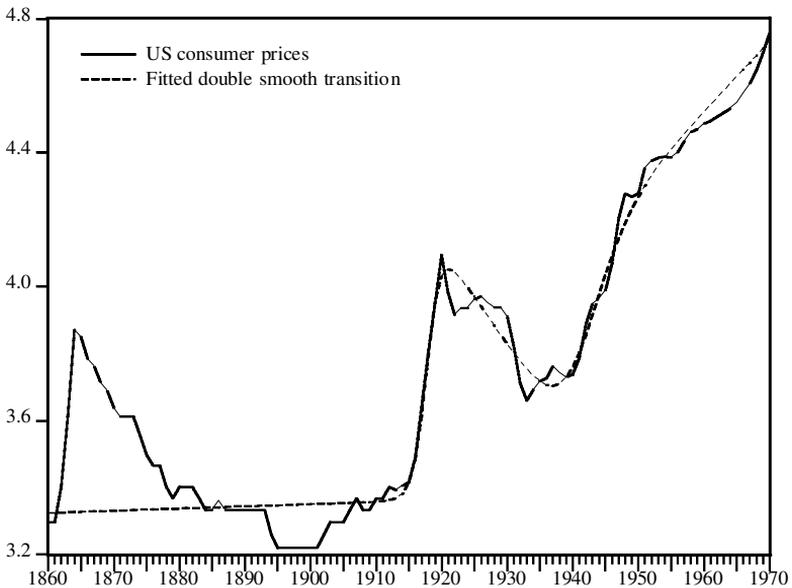


FIG. 2. US consumer prices and fitted double smooth transition function.

argued in favour of trend stationarity on other grounds. Nelson & Plosser (1982) applied the τ_τ test to the US consumer price series, while Leybourne *et al.* (1998) also applied their most general test $s_{\alpha\beta}$. The conclusion of both these analyses was that the series was best characterized as a unit root process, with the tests failing to reject the null.

TABLE 3. Empirical applications of unit root tests

	Average global temperature		US consumer prices	
	$T = 143$		$T = 111$	
	k	Test statistic	k	Test statistic
τ_τ	3	-2.248	2	-1.440
s_α	3	-3.300	2	-2.462
$s_{\alpha(\beta)}$	3	-3.381	5	-3.449
$s_{\alpha\beta}$	3	-3.329	1	-4.756*
$s_{2\alpha}$	0	-8.134***	2	-2.192
$s_{2\alpha(\beta)}$	1	-8.254***	5	-4.243
$s_{2\alpha\beta}$	1	-8.560***	1	-8.172***

Note: * and *** denote significance at the 10%- and 1%-levels respectively.

Table 3 presents results from our application of the augmented Dickey-Fuller unit root test (τ_τ), the Leybourne-Newbold-Vougas tests (s_α , $s_{\alpha(\beta)}$, $s_{\alpha\beta}$) and the double transition variants proposed in this paper ($s_{2\alpha}$, $s_{2\alpha(\beta)}$, $s_{2\alpha\beta}$), to the two time series. In conducting each test, the lag order used in the augmented Dickey-Fuller regressions was determined by sequential downward testing at the 5%-level, starting with $k = 8$. For the tests involving smooth transitions, a grid of starting values for the transition midpoint fractions was considered, as in the Monte Carlo experiments of the previous section.

The unit root null hypothesis is not rejected for either series at the 5%-level when the augmented Dickey-Fuller and Leybourne-Newbold-Vougas tests are employed, although rejection at the 10%-level occurs for the US consumer price series for the $s_{\alpha\beta}$ test. In contrast, rejections are obtained at the 1%-level for all the double transition tests for average global temperature and for the most general $s_{2\alpha\beta}$ test for consumer prices. Thus, when an appropriately general trend function is permitted under the alternative, sufficient evidence exists to reject the unit root null for these data. The series therefore appear to be best characterized as stationary around two smooth transitions in the linear trend, with obvious implications for climatic and economic modelling and forecasting.

In addition to conducting unit root tests, it is interesting to estimate the implied models for the two series. For the average global temperature series it is first necessary to decide which of the three models A, B or C is most appropriate, since rejections were obtained in favour of each alternative hypothesis. We therefore estimated each model with autoregressive errors of a common order, determined by the maximum lag order required in the augmented Dickey-Fuller regressions, i.e. AR(2) errors. Likelihood ratio tests were then performed to compare the models. Testing the restrictions of model A relative to models B and C resulted in probability values of 0.008 and 0.005 respectively, clearly indicating the importance of the trend component. The probability value associated with testing the model B restrictions relative to model C was 0.055; although the restrictions are not quite rejected at the 5%-level, the decision is marginal and our preference is for the more general model with transitions in both intercept and trend. For the US consumer price series, the only rejection in the unit root tests was in favour of model C; thus, the most general model was also adopted for this series, also with AR(2) errors following the results of the augmented Dickey-Fuller regressions.

Estimation of these models resulted in the fitted double smooth transitions that

are also plotted in Figs 1 and 2. The estimated transition midpoint fractions (τ_1, τ_2) were (0.335, 0.756) for average global temperature, corresponding to the years 1903 and 1963, and (0.547, 0.788) for US consumer prices, corresponding to the years 1920 and 1946. The associated transition speeds (γ_1, γ_2) were (1.450, 0.109) and (0.416, 0.126) respectively. The fitted trend lines track the data well in general, clearly picking up the structural changes visible in the time series plots, the only exception being the relatively high prices at the beginning of the US consumer price series. Of particular interest is the clear evidence of increases in trend average global temperature from the early 1900s and again from 1970.

4 Conclusions

Testing for a unit root against an alternative of stationarity around some deterministic function has an important role in time series analysis. In this paper we have broadened the class of trend functions against which the unit root null hypothesis can be tested, allowing for double smooth transitions in a linear trend. Our tests are not as powerful as those having simpler alternatives when the process under consideration has at most one transition. As a result, our tests should not be treated as encompassing Dickey-Fuller and Leybourne-Newbold-Vougas type tests, even though the simpler trend functions involved are special cases of the double transitions we consider. Rather, we recommend use of the new tests in addition to those mentioned, as a further alternative hypothesis to be considered, especially if one suspects two smooth transitions to be present in the series and the unit root null is not rejected by other tests.

If the true generating process is indeed stationary around two smooth transitions in a linear trend, our tests can strongly reject the unit root null hypothesis. This is clearly illustrated by the applications to average global temperature and US consumer price data, where the null is rejected at the 1% significance level once the double transition alternative hypothesis is considered. A more appropriate trend specification for these series then results, which can subsequently be used for further modelling and forecasting.

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