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Testing for changes in the mean or variance of a stochastic process under weak invariance ☆

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Abstract

Asymptotic CUSUM tests are derived for detecting changes in the mean or variance of a stochastic process for which a weak invariance principle is available. Conditions for the consistency of these tests are also discussed. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Testing for a change in the mean and/or the variance of a sequence of observations is one of the most important problems in change-point analysis. For a recent comprehensive survey we refer to Csörgő and Horváth (1997). In particular, cases of dependent observations have drawn increasing attention in the literature (cf. e.g. Bai, 1994; Davis et al., 1995; Giraitis and Leipus, 1992; Horváth, 1993, 1997; Horváth and Kokoszka, 1997; Horváth et al., 1998; Kokoszka and Leipus, 1997; Kulperger, 1985; and Picard, 1985).

One of the key tools in change-point analysis is to make use of a weak (or strong) invariance principle for the observed sequence and to develop an asymptotic test based on certain properties of the approximating process. The main aim of this paper is to pursue the latter idea in the following general model. Assume that we observe a stochastic process $\{Z(t)\colon 0\leqslant t<\infty\}$ having the following structure:

$$Z(t)=\begin{cases} at+bY(t), & 0\leqslant t\leqslant T^*,\\ Z(T^*)+a^*(t-T^*)+b^*Y^*(t-T^*), & T^*< t\leqslant T, \end{cases} \tag{1.1}$$

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where a, b, a^*, b^* and T^* are unknown parameters, $\{Y(t), 0 \leqslant t < \infty\}$ and $\{Y^*(t), 0 \leqslant t < \infty\}$ are unobservable stochastic processes satisfying a weak invariance principle. Namely, we assume that for any $T > 0$ there are independent Wiener processes $\{W_T(t),\ 0 \leqslant t \leqslant T^*\}$ and $\{W_T^*(t),\ 0 \leqslant t \leqslant T - T^*\}$ such that

$$\sup_{0 \leqslant t \leqslant T^*} |Y(t) - W_T(t)| = O_P(T^\alpha) \quad (T \to \infty) \tag{1.2}$$

and

$$\sup_{0 \leqslant t \leqslant T - T^*} |Y^*(t) - W_T^*(t)| = O_P(T^\alpha) \quad (T \to \infty) \tag{1.3}$$

with some $\alpha < 1/2$. On observing $\{Z(t),\ 0 \leqslant t \leqslant T\}$ we wish to test possible changes in the mean drift or variance of the process on $[0, T]$.

First we assume that $a \neq a^*$. In Section 2 we develop a CUSUM procedure for testing the null hypothesis

H_0: $T^* = T$ (no change in the mean over $[0, T]$)

against the alternative

H_A^m: $0 < T^* < T$ and $a \neq a^*$ (change in the mean at $T^* \in (0, T)$).

Theorem 2.1 gives the limit distribution of the CUSUM statistic allowing us to find asymptotic critical values. An estimator for the variance will also be discussed.

In Section 3 we consider the case when $b \neq b^*$. Similar to the detection of a possible change in the drift, an asymptotic test will be derived for testing H_0 against

H_A^v: $0 < T^* < T$ and $b \neq b^*$ (change in the variance at $T^* \in (0, T)$).

It turns out that, for the asymptotic test, it does not make any difference whether the mean drift is assumed to be known or not.

Before we state our main results we discuss some statistical models where conditions (1.1)–(1.3) are satisfied.

Example 1.1 (*Partial sums*). Let $\{X_i,\ 1 \leqslant i < \infty\}$ and $\{X_i^*,\ 1 \leqslant i < \infty\}$ be two independent sequences of independent identically distributed random variables with $EX_1 = \mu$, $\operatorname{var} X_1 = \sigma^2 > 0$, $EX_1^* = \mu^*$ and $\operatorname{var} X_1^* = \sigma^{*2} > 0$. Consider $Z(t) = S_{[t]}$, where $S_0 = 0$ and

$$S_k = \begin{cases} X_1 + X_2 + \cdots + X_k & \text{if } 1 \leqslant k \leqslant T^*, \\ S_{[T^*]} + X_1^* + \cdots + X_{k-[T^*]} & \text{if } T^* < k \leqslant T. \end{cases} \tag{1.4}$$

If $E|X_1|^{2+\delta} < \infty$ and $E|X_1^*|^{2+\delta} < \infty$ with some $\delta > 0$, then Komlós et al. (1975) yields (1.1)–(1.3) with $a = \mu$, $b = \sigma$, $Y(t) = (Z(t) - \mu t)/\sigma$, $a^* = \mu^*$, $b^* = \sigma^*$, $Y^*(t - T^*) = (Z(t) - Z(T^*) - \mu^*(t - T^*))/\sigma^*$. Here the Wiener processes W_T and W_T^* are constructed to the partial sums of $\{X_1, \ldots, X_{[T^*]}\}$ and $\{X_1^*, \ldots, X_{[T]-[T^*]}^*\}$, respectively, each with approximation rate $O_P(T^{1/(2+\delta)})$, i.e. $\alpha = 1/(2 + \delta)$.

Example 1.2 (*Renewal processes*). The random variables $\{X_i,\ 1\leqslant i<\infty\}$ and $\{X_i^*,\ 1\leqslant i<\infty\}$ are defined in Example 1.1. In addition to satisfying the conditions in Example 1.1 we assume that $\mu>0$ and $\mu^*>0$. Let

$$Z(t)=\begin{cases} N_1(t) & \text{if } 0\leqslant t\leqslant T^*,\\ N_1(T^*)+N_2(t-T^*) & \text{if } T^*<t<\infty, \end{cases} \tag{1.5}$$

where

$$N_1(t)=\min\left\{k\geqslant 1:\ \sum_{1\leqslant i\leqslant k} X_i>t\right\}-1,\quad 0\leqslant t<\infty$$

and

$$N_2(t)=\min\left\{k\geqslant 1:\ \sum_{1\leqslant i\leqslant k} X_i^*>t\right\}-1,\quad 0\leqslant t<\infty.$$

If $a=1/\mu$, $b=(\sigma^2/\mu^3)^{1/2}$, $a^*=1/\mu^*$ and $b^*=(\sigma^{*2}/\mu^{*3})^{1/2}$, then by Csörgő et al. (1987) (cf. also Csörgő and Horváth, 1993; Steinebach, 1994) we have the approximations in (1.2) and (1.3) for $Y(t)=(N_1(t)-at)/b$ and $Y^*(t)=(N_2(t)-a^*t)/b^*$ with $\alpha=1/(2+\delta)$ again.

Example 1.3 (*Dependent observations*). Following Example 1.1 we assume that $Z(t)=S_{[t]}$, where S_k is defined in (1.4). However, the independence of $X_1,X_2,\ldots;X_1^*,X_2^*,\ldots$ is not assumed anymore. Namely, $X_i=\mu+\sigma e_i, 1\leqslant i\leqslant T^*$ and $X_i^*=\mu^*+\sigma^* e_{[T^*]+i}$, $1\leqslant i\leqslant T-[T^*]$. We assume only that there is a Wiener process $\{W(t),\ 0\leqslant t<\infty\}$ such that

$$\left|\sum_{1\leqslant i\leqslant k} e_i-\tau W(k)\right| \stackrel{\text{a.s.}}{=} \mathrm{O}(k^\alpha),\quad \text{as } k\to\infty \tag{1.6}$$

with some $\alpha<\frac{1}{2}$ and $\tau>0$. Such approximations have been derived for weak Bernoulli processes (cf. Eberlein, 1983), martingales and their generalizations (cf. Eberlein, 1986a, b), α- and ϕ-mixing sequences, general Gaussian sequences and others (cf. Philipp, 1986 for a comprehensive review). Now (1.6) implies that (1.2) and (1.3) hold for $Y(t)=(Z(t)-\mu t)/b$ and $Y^*(t-T^*)=(Z(t)-Z(T^*)-\mu^*(t-T^*))/b^*$ with $b=\sigma\tau$ and $b^*=\sigma^*\tau$.

Example 1.4 (*Linear processes*). Motivated by change-point detection in time series, the following special case of Example 1.3 received special attention (cf. Antoch et al., 1997; Bai, 1994; Horváth, 1997). We assume that the sequence in Example 1.3 is a linear process, i.e.

$$e_i=\sum_{0\leqslant j<\infty} a_j\varepsilon_{i-j},\quad 1\leqslant i<\infty,$$

where $\{\varepsilon_i, 1\leqslant i<\infty\}$ is a sequence of independent identically distributed random variables with $E\varepsilon_i=0$, $E\varepsilon_i^2=1$ and $E|\varepsilon_i|^{2+\delta}<\infty$ with some $\delta>0$. If ε_1 has a smooth density and $\{a_k,\ 0\leqslant k<\infty\}$ satisfies some regularity conditions, then (1.6) holds. For details and exact conditions we refer to Lemmas 2.1 and 2.2 in Horváth (1997).

Example 1.5 (*Nonlinear time series*). ARCH-type models were introduced by Engle (1982) and they have become one of the most popular and extensively studied financial econometric models. The stationary solutions of the equations defining these models are typically Markovian, aperiodic, ergodic and α-mixing with geometrically decreasing mixing coefficients (cf. Bhattacharya and Lee, 1995; Diebolt and Guégan, 1993, 1994; Diebolt and Laïb, 1995 and Tjøstheim, 1990). For further discussion and examples we refer to Lu and Cheng (1997). These properties are sufficient to have an invariance principle like (1.6) and therefore we also have (1.1)–(1.3) in these models.

2. Testing for a change in the drift

We assume that we have observed $\{Z(t),\ 0\leqslant t\leqslant T\}$ at $t_i=t_{i,N}=iT/N,\ 1\leqslant i\leqslant N$. Let $Z_0=0,\ Z_i=Z(t_i),\ R_i=Z_i-Z_{i-1},\ 1\leqslant i\leqslant N$ and $Z_0^*=0$,

$$Z_k^* = \sum_{1\leqslant i\leqslant k} R_i - \frac{k}{N}\sum_{1\leqslant i\leqslant N} R_i, \quad 1\leqslant k\leqslant N.$$

Note that, in view of (1.1)–(1.3), the R_i roughly behave like independent normal $N(aT/N, b^2T/N),\ 1\leqslant i\leqslant NT^*/T$, respectively $N(a^*T/N,\ b^{*2}T/N),\ NT^*/T<i\leqslant N$, random variables. Taking this into account it will turn out that the change analysis for the mean drift can essentially be pursued under the normal distribution.

First we study the limit properties of

$$M_T=(Tb^2)^{-1/2}\max_{1\leqslant k\leqslant N}|Z_k^*| \tag{2.1}$$

as $T\to\infty$ with b from (1.1).

Theorem 2.1. *We assume that* (1.1)–(1.3) *hold and* $N=N(T)\to\infty$, *as* $T\to\infty$. *Then under* H_0 *we have*

$$M_T \xrightarrow{\mathscr{D}} \sup_{0\leqslant t\leqslant 1}|B(T)|, \quad T\to\infty, \tag{2.2}$$

with $\{B(t),\ 0\leqslant t\leqslant 1\}$ *denoting a Brownian bridge.*

Proof. On observing that under H_0

$$\begin{aligned} Z_k^* &= Z\left(\frac{kT}{N}\right)-\frac{k}{N}Z(T)\\ &= a\frac{kT}{N}+bY\left(\frac{kT}{N}\right)-\frac{k}{N}\{aT+bY(T)\}, \end{aligned}$$

assumption (1.2) yields that

$$\max_{1\leqslant k\leqslant N}\left|Z_k^*-b\left\{W_T\left(\frac{kT}{N}\right)-\frac{k}{N}W_T(T)\right\}\right|=\mathrm{O_P}(T^{\alpha}),$$

as $T \to \infty$. By the scale transformation of the Wiener process we have

$$\left\{ T^{-1/2} \left(W_T \left(\frac{[xN]}{N} T \right) - \frac{[xN]}{N} W_T(T) \right),\ 0 \leqslant x \leqslant 1 \right\}$$
$$\stackrel{\mathscr{D}}{=} \left\{ W_T \left(\frac{[xN]}{N} \right) - \frac{[xN]}{N} W_T(1),\ 0 \leqslant x \leqslant 1 \right\},$$

and therefore the almost sure continuity of $W_T(u)$ gives

$$\begin{aligned}
&\sup_{0 \leqslant x \leqslant 1} \left| (Tb^2)^{-1/2} Z^*_{[Nx]} - T^{-1/2}(W_T(xT) - xW_T(T)) \right| \\
&\quad = \mathrm{O_P}(T^{\alpha - 1/2}) + \sup_{0 \leqslant x \leqslant 1} \left| T^{-1/2} \left(W_T \left(\frac{[xN]}{N} T \right) - \frac{[xN]}{N} W_T(T) \right) \right. \\
&\qquad \left. - T^{-1/2}(W_T(xT) - xW_T(T)) \right| \\
&\quad = \mathrm{o_P}(1),
\end{aligned}$$

which completes the proof of Theorem 2.1. □

For practical use of Theorem 2.1, we have to replace b^2 in (2.1) with a consistent estimator. We choose

$$\hat{b}_T^2 = \frac{1}{T} \sum_{1 \leqslant i \leqslant N} \left(R_i - \frac{T}{N} \hat{a}_T \right)^2 \tag{2.3}$$

to estimate b^2, where

$$\hat{a}_T = \frac{1}{T} \sum_{1 \leqslant i \leqslant N} R_i.$$

Let

$$\hat{M}_T = (T\hat{b}_T^2)^{-1/2} \max_{1 \leqslant k \leqslant N} |Z_k^*|.$$

Theorem 2.2. *We assume that* (1.1)–(1.3) *hold*, $N = N(T) \to \infty$ *and* $N = \mathrm{o}(T^{1-2\alpha})$ *as* $T \to \infty$. *Then under* H_0 *we have*

$$\hat{M}_T \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t \leqslant 1} |B(t)|, \quad T \to \infty, \tag{2.4}$$

with $\{B(t),\ 0 \leqslant t \leqslant 1\}$ *denoting a Brownian bridge.*

Proof. It is enough to show that

$$\hat{b}_T^2 \xrightarrow{P} b^2, \quad T \to \infty. \tag{2.5}$$

First we note that under H_0, assumptions (1.1)–(1.3) together with the normal distribution of $T^{-1/2}W_T(T)$ yield

$$\begin{aligned}\hat{a}_T &= \frac{1}{T}\sum_{1\leqslant i\leqslant N} R_i = \frac{Z(T)}{T} = a + b\frac{Y(T)}{T}\\ &= a + b\frac{W_T(T)}{T} + \mathrm{O_P}(T^{\alpha-1})\\ &= a + \mathrm{O_P}(T^{-1/2}).\end{aligned} \tag{2.6}$$

We can assume, without loss of generality, that $a=0$. Applying (2.6) we get

$$\begin{aligned}\hat{b}_T^2 &= \frac{1}{T}\left(\sum_{1\leqslant i\leqslant N} R_i^2 - \frac{T^2}{N}\hat{a}_T^2\right)\\ &= \frac{1}{T}\sum_{1\leqslant i\leqslant N} R_i^2 + \mathrm{o_P}(1).\end{aligned}$$

Set $W_i = b\{W_T(iT/N) - W_T((i-1)T/N)\}$ and $\eta_i = R_i - W_i$, $1\leqslant i\leqslant N$. Assumption (1.2) implies that

$$\max_{1\leqslant i\leqslant N} |\eta_i| = \mathrm{O_P}(T^\alpha).$$

Hence

$$\begin{aligned}\frac{1}{T}\sum_{1\leqslant i\leqslant N} R_i^2 &= \frac{1}{T}\sum_{1\leqslant i\leqslant N} (W_i+\eta_i)^2\\ &= \frac{1}{T}\sum_{1\leqslant i\leqslant N} W_i^2 + \frac{2}{T}\sum_{1\leqslant i\leqslant N}\eta_i W_i + \frac{1}{T}\sum_{1\leqslant i\leqslant N}\eta_i^2\\ &= \frac{1}{T}\sum_{1\leqslant i\leqslant N} W_i^2 + \frac{2}{T}\sum_{1\leqslant i\leqslant N}\eta_i W_i + \mathrm{O_P}(NT^{2\alpha-1}).\end{aligned}$$

Since $(N/T)^{1/2}W_i$, $1\leqslant i\leqslant N$, are independent, identically distributed normal $\mathrm{N}(0,b^2)$ random variables, by Markov's inequality we have

$$\sum_{1\leqslant i\leqslant N} |W_i| = \mathrm{O_P}((NT)^{1/2})$$

and the central limit theorem gives

$$\frac{1}{T}\sum_{1\leqslant i\leqslant N} W_i^2 - b^2 = \frac{1}{N}\sum_{1\leqslant i\leqslant N}\left(\frac{N}{T}W_i^2 - b^2\right) = \mathrm{O_P}(N^{-1/2}).$$

Thus, we get

$$\begin{aligned}\frac{1}{T}\sum_{1\leqslant i\leqslant N} R_i^2 &= b^2 + \mathrm{O_P}(N^{-1/2}) + \mathrm{O_P}(T^{\alpha-1/2}N^{1/2}) + \mathrm{O_P}(NT^{2\alpha-1})\\ &= b^2 + \mathrm{o_P}(1),\end{aligned}$$

which yields (2.5).

Next, we discuss the behavior of the test statistic $\hat{M}_T$, as $T\to\infty$, under the alternative H_A^m.

Theorem 2.3. *We assume that* (1.1)–(1.3) *hold*, $N = N(T) \to \infty$, $N = \mathrm{o}(T^{1-2\alpha})$, *and*

$$\frac{|a-a^*|N^{1/2}T^*(T-T^*)}{T^2} \to \infty, \tag{2.7}$$

then under H_A^m *we have that*

$$\hat{M}_T \xrightarrow{P} \infty. \tag{2.8}$$

Remark 2.1. We note that no assumption is made on b and b^* in Theorem 2.3. This means that we always have consistency regardless if the variance changes or not. The same observation was made by Gombay et al. (1996) in case of independent observations.

Proof. Setting $k^* = [NT^*/T]$, by (1.2) and (1.3) we have

$$\begin{aligned} Z_{k^*}^* &= Z\left(\frac{k^*T}{N}\right) - \frac{k^*}{N}\{Z(T^*) + (Z(T) - Z(T^*))\} \\ &= (a-a^*)\frac{k^*}{N}(T-T^*) + b\left\{W_T\left(\frac{k^*T}{N}\right) - \frac{k^*}{N}W_T(T^*)\right\} \\ &\quad - b^*\frac{k^*}{N}W_T^*(T-T^*) + \mathrm{O_P}(T^\alpha). \end{aligned}$$

Since the distribution of $T^{-1/2}\sup_{0\leqslant x\leqslant 1}|W_T(Tx)|$ does not depend on T, we get that

$$Z_{k^*}^* = (a-a^*)\frac{k^*}{N}(T-T^*) + \mathrm{O_P}(T^{1/2}). \tag{2.9}$$

If

$$\frac{|a-a^*|T^*(T-T^*)}{T^{3/2}} \to \infty, \quad T \to \infty, \tag{2.10}$$

then by (2.9) we have

$$Z_{[NT^*/T]}^* \Big/ \left\{\frac{(a-a^*)}{T}T^*(T-T^*)\right\} \xrightarrow{P} 1. \tag{2.11}$$

Since $\alpha < \frac{1}{2}$, we obtain

$$\begin{aligned} \hat{a}_T &= \frac{Z(T)}{T} = \frac{Z(T^*)}{T} + \frac{Z(T) - Z(T^*)}{T} \\ &= a\frac{T^*}{T} + a^*\frac{T-T^*}{T} + b\frac{W_T(T^*)}{T} + b^*\frac{W_T^*(T-T^*)}{T} + \mathrm{O_P}(T^{\alpha-1}) \\ &= \mathrm{O}(1) + \mathrm{O_P}(T^{-1/2}) + \mathrm{O_P}(T^{\alpha-1}) \\ &= \mathrm{O_P}(1). \end{aligned} \tag{2.12}$$

Using (2.12) we get

$$\begin{aligned} \hat{b}_T^2 &= \frac{1}{T}\left(\sum_{1\leqslant i\leqslant N} R_i^2 - \frac{T^2}{N}\hat{a}_T^2\right) \\ &= \frac{1}{T}\sum_{1\leqslant i\leqslant N} R_i^2 + \mathrm{O_P}\left(\frac{T}{N}\right). \end{aligned}$$

Next, we write

$$\sum_{1\leqslant i\leqslant N} R_i^2 = \sum_{1\leqslant i\leqslant k^*} R_i^2 + \sum_{k^*<i\leqslant N} R_i^2$$
$$= \sum_{1\leqslant i\leqslant k^*}\left(a\frac{T}{N} + W_i + \eta_i\right)^2 + \sum_{k^*<i\leqslant N}\left(a^*\frac{T}{N} + W_i^* + \eta_i^*\right),$$

where $W_i = b\{W_T(iT/N) - W_T((i-1)T/N)\}$, $\eta_i = R_i - W_i - aT/N$, $1\leqslant i\leqslant k^*$ and $W_i^* = b^*\{W_T^*(iT/N - T^*) - W_T^*((i-1)T/N - T^*)\}$, $\eta_i^* = R_i - W_i^* - a^*T/N$. By (1.2) and (1.3) we have

$$\max_{1\leqslant i\leqslant k^*} |\eta_i| = O_P(T^\alpha) \quad \text{and} \quad \max_{k^*<i\leqslant N} |\eta_i^*| = O_P(T^\alpha)$$

and, therefore,

$$\sum_{1\leqslant i\leqslant k^*} \eta_i^2 = O_P(k^*T^{2\alpha}) \quad \text{and} \quad \sum_{k^*<i\leqslant N} \eta_i^{*2} = O_P((N-k^*)T^{2\alpha}).$$

Since $(N/(b^2T))^{1/2}W_i$ and $(N/(b^{*2}T))^{1/2}W_i^*$ are independent, standard normal random variables we get

$$\sum_{1\leqslant i\leqslant k^*} W_i^2 = O_P(b^2Tk^*/N) \quad \text{and} \quad \sum_{k^*<i\leqslant N} W_i^{*2} = O_P((N-k^*)b^{*2}T/N).$$

Hence,

$$\hat{b}_T^2 = a^2\frac{T^*}{N}(1+o_P(1)) + a^{*2}\frac{T-T^*}{N}(1+o_P(1)) + O_P\left(\frac{T}{N}\right) = O_P\left(\frac{T}{N}\right).$$

Combining (2.11) and condition (2.7) we immediately obtain (2.8). $\square$

For example, if we assume that $T^* = [T\theta]$ with some $0<\theta<1$, then the conditions of Theorem 2.3 are satisfied, if $N\to\infty$, $N = o(T^{1-2\alpha})$ and $|a-a^*|N^{1/2}\to\infty$, as $T\to\infty$.

3. Testing for a change in the variance

Our test is based on the partial sums of $\tilde{R}_i^2 = (Z_i - Z_{i-1} - aT/N)^2$, $1\leqslant i\leqslant N$, where $Z_0, Z_1, \ldots, Z_N$ are defined in Section 2. Let $\tilde{Z}_0 = 0$ and

$$\tilde{Z}_k = \sum_{1\leqslant i\leqslant k} \tilde{R}_i^2 - \frac{k}{N}\sum_{1\leqslant i\leqslant N} \tilde{R}_i^2, \quad 1\leqslant k\leqslant N.$$

Similarly to Chapter 2, the $\tilde{R}_i^2$ here are roughly independent $(b^2T/N)\chi_1^2$, $1\leqslant i\leqslant NT^*/T$, respectively $(b^{*2}T/N)\chi_1^2$, $NT^*/T<i\leqslant N$, random variables, and the change analysis for the variance will essentially be based on this asymptotic chi-square situation.

First, we study the asymptotic properties of

$$\tilde{M}_T = \frac{N^{1/2}}{2^{1/2}b^2T}\max_{1\leqslant k\leqslant N} |\tilde{Z}_k|.$$

We note that a and b are the drift and variance terms in (1.1) under H_0.

Theorem 3.1. *We assume that* (1.1)–(1.3) *hold*, $N = N(T) \to \infty$ *and* $N = \mathrm{o}(T^{1/2-\alpha})$ *as* $T \to \infty$. *Then under* H_0 *we have*

$$\tilde{M}_T \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t \leqslant 1} |B(t)|, \quad T \to \infty, \tag{3.1}$$

with $\{B(t),\ 0 \leqslant t \leqslant 1\}$ *denoting a Brownian bridge.*

Proof. Let $\xi_i = (N/T)^{1/2}(W_T(iT/N) - W_T((i-1)T/N)),\ 1 \leqslant i \leqslant N$. It is easy to see that $\xi_1, \xi_2, \ldots$ are independent, standard normal random variables. Using (1.2) we get that

$$\tilde{R}_i^2 = b^2 \frac{T}{N} \{\xi_i + \tau_i\}^2$$

and

$$\max_{1 \leqslant i \leqslant N} |\tau_i| = \mathrm{O}_{\mathrm{P}}(N^{1/2} T^{\alpha - 1/2}).$$

Hence by the law of large numbers we have

$$\max_{1 \leqslant k \leqslant N} \left| \sum_{1 \leqslant i \leqslant k} \left(\tilde{R}_i^2 - b^2 \frac{T}{N} \right) - b^2 \frac{T}{N} \sum_{1 \leqslant i \leqslant k} (\xi_i^2 - 1) \right|$$
$$= \mathrm{O}_{\mathrm{P}}(N^{1/2} T^{\alpha + 1/2}) + \mathrm{O}_{\mathrm{P}}(T^{2\alpha} N). \tag{3.2}$$

Since $\xi_1^2 - 1, \xi_2^2 - 1, \ldots$ are independent identically distributed random variables with $E(\xi_1^2 - 1) = 0$, $\mathrm{var}(\xi_1^2 - 1) = 2$ and a finite moment generating function, by the Komlós et al. (1976) strong approximation we can define a Wiener process $\{\tilde{W}(t),\ 0 \leqslant t < \infty\}$ such that

$$\max_{1 \leqslant k \leqslant N} \left| \sum_{1 \leqslant i \leqslant k} (\xi_i^2 - 1) - 2^{1/2} \tilde{W}(k) \right| \overset{\text{a.s.}}{=} \mathrm{O}(\log N). \tag{3.3}$$

Putting together (3.2) and (3.3) we conclude

$$\max_{1 \leqslant k \leqslant N} \left| \frac{N^{1/2}}{2^{1/2} b^2 T} \tilde{Z}_k - N^{-1/2} \left(\tilde{W}(k) - \frac{k}{N} \tilde{W}(N) \right) \right|$$
$$= \mathrm{O}_{\mathrm{P}}(N^{-1/2} \log N) + \mathrm{O}_{\mathrm{P}}(N T^{\alpha - 1/2}) + \mathrm{O}_{\mathrm{P}}(N^{-1/2} (N T^{\alpha - 1/2})^2)$$
$$= \mathrm{o}_{\mathrm{P}}(1).$$

It is easy to check that $\tilde{B}_N(t) = N^{-1/2}(\tilde{W}(Nt) - t\tilde{W}(N))$ is a Brownian bridge for each N. By the continuity of $\tilde{B}_N(t)$ we have

$$\max_{1 \leqslant k \leqslant N} \sup_{(k-1)/N \leqslant t \leqslant k/N} \left| \tilde{B}_N(t) - \tilde{B}_N\left(\frac{k}{N}\right) \right| = \mathrm{o}_{\mathrm{P}}(1),$$

which also completes the proof of (3.1). □

For practical use of the statistic $\tilde{M}_T$, we have to replace a and b^2 with suitable estimators again. We recall the estimators

$$\hat{a}_T = \frac{1}{T} \sum_{1 \leqslant i \leqslant N} R_i$$

and

$$\hat{b}_T^2 = \frac{1}{T} \sum_{1 \leqslant i \leqslant N} \left(R_i - \frac{T}{N} \hat{a}_T \right)^2.$$

From Section 2. Let $\hat{R}_i^2 = (Z_i - Z_{i-1} - \hat{a}_T T/N)^2$, $1 \leqslant i \leqslant N$ and

$$\hat{Z}_k = \sum_{1 \leqslant i \leqslant k} \hat{R}_i^2 - \frac{k}{N} \sum_{1 \leqslant i \leqslant N} \hat{R}_i^2, \quad 1 \leqslant k \leqslant N.$$

Next, we obtain the limit distribution of

$$\tilde{M}_T^* = \frac{N^{1/2}}{2^{1/2} T \hat{b}_T^2} \max_{1 \leqslant k \leqslant N} |\hat{Z}_k|$$

under H_0.

Theorem 3.2. *We assume that* (1.1)–(1.3) *hold*, $N = N(T) \to \infty$ *and* $N = \mathrm{o}(T^{1/2-\alpha})$, *as* $T \to \infty$. *Then under* H_0 *we have*

$$\tilde{M}_T^* \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t \leqslant 1} |B(t)|, \quad T \to \infty, \tag{3.4}$$

with $\{B(t),\ 0 \leqslant t \leqslant 1\}$ *denoting a Brownian bridge.*

Proof. By (2.6) we have

$$\hat{a}_T - a = \mathrm{O}_{\mathrm{P}}(T^{-1/2}). \tag{3.5}$$

Next, we note that

$$\begin{aligned}\sum_{1 \leqslant i \leqslant k} (\tilde{R}_i^2 - \hat{R}_i^2) &= \sum_{1 \leqslant i \leqslant k} \frac{T}{N} (\hat{a}_T - a) \left(2(Z_i - Z_{i-1}) + \frac{T}{N} (\hat{a}_T + a) \right) \\ &= 2 \frac{T}{N} (\hat{a}_T - a) Z\left(\frac{kT}{N}\right) - k \left(\frac{T}{N}\right)^2 (\hat{a}_T - a)(\hat{a}_T + a),\end{aligned}$$

and therefore Theorem 2.1 and (3.5) imply that

$$\begin{aligned}\frac{N^{1/2}}{T} \max_{1 \leqslant k \leqslant N} |\tilde{Z}_k - \hat{Z}_k| &\leqslant \frac{2N^{1/2}}{T} \max_{1 \leqslant k \leqslant N} \left| \sum_{1 \leqslant i \leqslant k} (\tilde{R}_i^2 - \hat{R}_i^2) \right| \\ &= \mathrm{O}_{\mathrm{P}}(N^{-1/2}) = \mathrm{o}_{\mathrm{P}}(1).\end{aligned} \tag{3.6}$$

Using (2.5), we immediately obtain (3.4) from (3.6) and Theorem 3.1. □

Finally, we discuss the behavior of $\tilde{M}_T^*$ under H_A^v.

Theorem 3.3. *We assume that* (1.1)–(1.3) *hold*, $a = a^*$, $N = N(T) \to \infty$, $NT^*/T \to \infty$, $N(T - T^*)/T \to \infty$, $N = \mathrm{o}(T^{1/2-\alpha})$, *and*

$$N^{1/2} T^* (T - T^*) |b^2 - b^{*2}| / T^2 \to \infty$$

as $T \to \infty$. *Then under* H_A^v *we have*

$$\tilde{M}_T^* \xrightarrow{P} \infty, \quad T \to \infty. \tag{3.7}$$

Proof. First we note that since $a^*=a$, (3.5) holds under H_A^v. Again, with $k^*=[NT^*/T]$, similarly to the proof of Theorem 3.2 we have

$$\frac{N^{1/2}}{T}|\hat{Z}_{k^*}| = \mathrm{O_P}(1) + \frac{|b^2-b^{*2}|k^*}{N^{3/2}} \sum_{k^*<i\leqslant N} \hat{N}_i^2(1+\mathrm{o_P}(1)),$$

where $\hat{N}_1, \hat{N}_2, \ldots$ are independent, standard normal random variables. Hence by the law of large numbers we conclude

$$\frac{N^{1/2}}{T}|\hat{Z}_{k^*}| = \frac{|b^2-b^{*2}|k^*(N-k^*)}{N^{3/2}}(1+\mathrm{o_P}(1)). \tag{3.8}$$

Moreover,

$$\begin{aligned}\hat{b}_T^2 &= \frac{1}{T}\sum_{1\leqslant i\leqslant N}\left(Z_i - Z_{i-1} - \hat{a}_T\frac{T}{N}\right)^2\\ &= \frac{1}{T}\sum_{1\leqslant i\leqslant N}\left(Z_i - Z_{i-1} - a\frac{T}{N}\right)^2 - \frac{T}{N}(\hat{a}_T - a)^2\\ &= \left(b^2\frac{k^*}{N} + b^{*2}\frac{N-k^*}{N}\right)(1+\mathrm{o_P}(1)),\end{aligned}$$

and therefore (3.7) follows from (3.8).

For example, if we assume that $T^*=[T\theta]$ with some $0<\theta<1$, sufficient conditions for Theorem 3.3 are $N=N(T)\to\infty$, $N=\mathrm{o}(T^{1/2-\alpha})$, and $N^{1/2}|b^2-b^{*2}|\to\infty$.

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