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Serial rank statistics for detection of changes

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Abstract

A class of ranks based test statistics for testing hypothesis of randomness (observations are independent and identically distributed) against the alternative that the observations become dependent at some unknown time point is introduced and its limit properties are studied. The considered problem belongs to the area of the change-point analysis.

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1. Introduction

Let $X_1,\dots,X_n$ be observations obtained at ordered time points $t_1 < \cdots < t_n$. We are interested in testing that the observations are independent identically distributed (iid) random variables (H_0) against the alternative (H_1) that there exists $m \in [1, n-1]$ such that the first m observations are independent identically distributed (iid) random variables and the observations obtained after the mth one are dependent, typically form an AR- or ARMA sequence. In other words we are interested in testing independence against alternative that after some unknown $m(<n)$ the independence of observations changes to a certain dependence.

Most of the test procedures for detection of changes in statistical models was developed for a change in location or regression parameters or in the distribution of single observations (for review see recent books by Csörgő and Horváth, 1997; Brodsky and Darkhovsky, 2000; Chen and Gupta, 2000 among others). The procedures for detection of changes in type of dependence, e.g. independence versus AR-dependence has become of interest mostly during the last 10 years. One of

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the first papers was published by Bagshaw and Johnson (1977) who developed a test for detection of a change in the parameters of ARIMA models based on the sum of the differences of squares of residuals. Tests procedures based on spectral empirical distributions were studied by Picard (1985). Giraitis and Leipus (1990, 1992), Giraitis et al. (1996) generalized her results and also developed and studied procedures based on spectral densities. Bai (1993, 1994) and Horváth (1993) investigated procedures based on residuals. Davis et al. (1995) developed and studied properties of likelihood ratio type test procedures for a change in the parameter value and order of an autoregressive model. Beran and Terrin (1996), Horváth and Kokoszka (1997), Horváth (2001) among others developed and studied procedures for detection of changes in long-memory parameters. Kokoszka and Leipus (2000) proposed and investigated procedures for detection of changes in ARCH models—see the review paper by Kokoszka and Leipus (2002).

In the present paper a class of test statistics based on serial rank statistics is introduced and studied. The procedure is simple and since the proposed test statistic is distribution-free under the null hypothesis the critical values can be calculated relatively easily.

Wilcoxon type serial rank statistics were introduced by Wald and Wolfowitz (1943) for testing randomness against serial dependence. Later on general rank based statistics were introduced and studied not only for testing randomness versus dependence but also for more general problems, e.g. for testing ARMA(p,q) dependence against ARMA$(p+d,q+d)$ dependence. For more information see the survey papers by Hallin and Puri (1992) and Hallin and Werker (1999) and the references there. It appears that rank based procedures are quite useful in a number of problems in time series analysis.

In the present paper we propose serial rank based test procedures for testing hypothesis of independence (H_0) against alternatives where at some unknown time points the observations become serially dependent. The resulting test procedures are distribution-free and are easy to calculate. We focus on the max type statistics. However, the sum type and the MOSUM type test statistics can be introduced along the same line. The limit distributions of the proposed test statistics under the null hypothesis are derived and approximations to the critical values can be obtained either through the limit distributions of the test statistics under the null hypothesis or through simulations.

The main results are contained in Section 2. Namely, a class of test statistics for the considered testing problem is introduced and their limit behavior under the null hypothesis are formulated and some remarks on possible approximations to the desired critical values are made. The proof of the main theorem is postponed to Section 3.

2. Main results

We consider the testing problem

H_0: $X_1,\dots,X_n$ i.i.d. random variables with continuous d.f.

against

H_1 : there is $m<n$ such that

$X_1,\dots,X_m$ are i.i.d. r.v.′s and $X_{m+1},\dots,X_n$ are dependent.

Our test statistics will be based on the rank statistics

$$S_k(\boldsymbol{a}) = \sum_{i=2}^{k} a_n(R_i)a_n(R_{i-1}), \quad k = 2,\dots,n, \tag{2.1}$$

$$S_k^0(\boldsymbol{a}) = \sum_{i=k+1}^{n} a_n(R_i)a_n(R_{i-1}), \quad k = 1,\dots,n, \tag{2.2}$$

where R_i is the rank of X_i among $X_1,\dots,X_n$ and $a_n(1),\dots,a_n(n)$ are scores satisfying

$$\bar{a}_n = \frac{1}{n}\sum_{i=1}^{n} a_n(i) = 0. \tag{2.3}$$

The statistics $S_k(\boldsymbol{a})$ and $S_k^0(\boldsymbol{a})$ can be viewed as serial rank statistics based on the ranks $(R_1,\dots,R_k)$ and $(R_{k+1},\dots,R_n)$, respectively. We set

$$\sigma_n^2(\boldsymbol{a}) = \frac{1}{n}\sum_{i=1}^{n} a_n^2(i). \tag{2.4}$$

At first we recall simple properties of $S_k(\boldsymbol{a})$ and $S_k^0(\boldsymbol{a})$ that provide motivation for definition of the test statistics. Straightforward calculations (for more details see Lemma 3.1) give that under H_0

$$E\left(\frac{1}{k-1}S_k(\boldsymbol{a}) - \frac{1}{n-k}S_k^0(\boldsymbol{a})\right) = 0. \tag{2.5}$$

Tedious but direct calculations (for more details see Section 3) give also that under the null hypothesis, as $n \to \infty$,

$$E\left(\frac{1}{k}S_k(\boldsymbol{a}) - \frac{1}{n-k}S_k^0(\boldsymbol{a})\right)^2 = \frac{n}{k(n-k)}\sigma_n^4(\boldsymbol{a})(1 + \mathrm{O}(n^{-1})) \tag{2.6}$$

uniformly in $1 < k < n$. By a slight modification of the proof of Corollary 1.1 in Hauesler et al. (2000) we observe that under the null hypothesis and the assumptions of Theorem 2.1 below for any $t \in (0,1)$, as $n \to \infty$,

$$\frac{S_{[nt]}(\boldsymbol{a})}{\sqrt{[nt]\sigma_n^2(\boldsymbol{a})}} \to^D N(0,1),$$

$$\frac{S_{[nt]}^0(\boldsymbol{a})}{\sqrt{n-[nt]\sigma_n^2(\boldsymbol{a})}} \to^D N(0,1),$$

where $\to^D$ denotes convergence in distribution, while under alternatives with $X_m, \ldots, X_n$ forming AR- or ARMA sequence

$$\frac{S_{[nt]}(\boldsymbol{a})}{[nt]} \to^P c(t),$$

$$\frac{S^0_{[nt]}(\boldsymbol{a})}{[nt]} \to^P c^0(t),$$

where $c(t) - c^0(t) \neq 0$ for at least some $t \in (0,1)$.

Now, using union-intersection principle (cf. Hawkins, 1989; Csörgő and Horváth, 1997) we introduce our test statistics as follows:

$$T_n(\boldsymbol{a}) = \max_{1<k<n} \sqrt{\frac{k(n-k)}{n}} \frac{1}{\sigma_n^2(\boldsymbol{a})} \left| \frac{1}{k-1} S_k(\boldsymbol{a}) - \frac{1}{n-k} S_k^0(\boldsymbol{a}) \right|. \tag{2.7}$$

This is a so called max-type test statistic. Similarly structured test statistics are used for detecting other type of changes.

Large values of $T_n(\boldsymbol{a})$ indicate that the null hypothesis is violated.

Since the proposed test statistic $T_n(\boldsymbol{a})$ depends on the observations only through the ranks $R_1, \ldots, R_n$ the resulting test is distribution-free under the null hypothesis.

Approximation to the critical values can be obtained either from its limit distribution under the null hypothesis (see Theorem 2.1 below) or by simulations.

We assume that the scores $a_n(1), \ldots, a_n(n)$ in addition to (2.3) satisfy

$$\sigma_n^2(\boldsymbol{a}) \geqslant D_1 \tag{2.8}$$

$$\frac{1}{n}\sum_{i=1}^n a_n^4(i) \leqslant D_2 \tag{2.9}$$

with some positive D_1 and D_2.

Theorem 2.1. *Let $X_1, \ldots, X_n$ be iid random variables with continuous distribution function and let the scores satisfy* (2.3), (2.8)–(2.9). *Then for all y*

$$P(\sqrt{2\log\log n}\, T_n(\boldsymbol{a}) \leqslant 2\log\log n + y + \tfrac{1}{2}\log\log\log n - \tfrac{1}{2}\log \pi) \to \exp\{-2\mathrm{e}^{-y}\}, \tag{2.10}$$

as $n \to \infty$.

The proof is postponed until the next section.

One can introduce other classes of test statistics based on weighted maxima of serial rank statistics for the considered testing problem, e.g.

$$\max_{1<k<n} \frac{1}{\sqrt{n}} \frac{1}{\sigma_n^4(\boldsymbol{a})} \left| \frac{\frac{1}{k-1} S_k(\boldsymbol{a}) - \frac{1}{n-k} S_k^0(\boldsymbol{a})}{w(k/n)} \right|,$$

where w is a weighted function. Eventually, one can consider a MOSUM (moving sum) type tests procedure

$$\max_{G<k<n-G} \frac{1}{\sqrt{2G}} \frac{1}{\sigma_n^4(\boldsymbol{a})} \left| \sum_{i=k-G+1}^{k} a_n(R_i)a_n(R_{i-1}) - \sum_{i=k+1}^{k+G} a_n(R_i)a_n(R_{i-1}) \right|,$$

where $G = G(n)$ satisfies $\lim_{n\to\infty} G(n) = \infty$ and $\lim_{n\to\infty} G(n)/n = 0$.

Motivated by results of Hallin et al. (1987) and the above test statistics one can develop test statistics for more general type of change point alternatives, e.g., one can consider the test statistic

$$T_n(\boldsymbol{a}, s) = \max_{s<k<n-s} \sqrt{\frac{k(n-k)}{n}} \frac{1}{\sigma_n^2(\boldsymbol{a})}$$

$$\times \left| \frac{1}{k-1} \sum_{i=s}^{k} a_n(R_i)a_n(R_{i-s}) - \frac{1}{n-k} \sum_{i=k}^{n-s} a_n(R_i)a_n(R_{i+s}) \right|, \quad s = 1, \dots .$$

3. Proofs

Since the test statistic $T_n(\boldsymbol{a})$ is distribution-free under the null hypothesis we may assume without loss of generality that $R_1, \dots, R_n$ are ranks of $U_1, \dots, U_n$ which is a sample from $(0,1)$-uniform distribution.

We start with a technical lemma on the moments of some rank statistics.

Lemma 3.1. *Let $U_1, \dots, U_n$ be iid random variables with uniform distribution function on* $(0,1)$ *and let $R_1, \dots, R_n$ be corresponding ranks. Let the scores $a_n(.)$ $b_n(.)$ satisfy* (2.3) *and*

$$\sum_{i=1}^{n} b_n(i) = 0, \tag{3.1}$$

respectively. Then, as $n \to \infty$,

$$Ea_n(R_1)b_n(R_2) = -\frac{1}{n(n-1)} \sum_{i=1}^{n} b_n(i)a_n(i) \tag{3.2}$$

and, as $n \to \infty$,

$$E(a_n(R_1)b_n(R_2))^2 = \frac{1}{n^2} \sum_{i=1}^{n} a_n^2(i) \sum_{j=1}^{n} b_n^2(j)(1 + \mathrm{O}(n^{-1})), \tag{3.3}$$

$$Ea_n(R_1)b_n(R_2)a_n(R_3)b_n(R_3) = \mathrm{O}\left\{ \frac{1}{n^3} \left(\sum_{i=1}^{n} a_n^2(i)b_n^2(i) + \left(\sum_{i=1}^{n} b_n(i)a_n(i) \right)^2 \right) \right\} \tag{3.4}$$

$$Ea_n(R_1)a_n(R_2)b_n(R_3)b_n(R_4)$$
$$= \mathrm{O}\left\{\frac{1}{n^4}\left(\sum_{i=1}^n a_n^2(i)b_n^2(i) + \left(\sum_{i=1}^n b_n(i)a_n(i)\right)^2 + \sum_{i=1}^n a_n^2(i)\sum_{j=1}^n b_n^2(j)\right)\right\}. \tag{3.5}$$

Proof. The assertions follow by direct calculations applying (2.3) and (3.1) and

$$P(R_1 = r_1, \dots, R_n = r_n) = \frac{1}{n!}$$

for any permutation $r_1, \dots, r_n$ of $1, \dots, n$.

As easy consequences we obtain

$$ES_k(\boldsymbol{a}) = -\frac{k-1}{n-1}\,\sigma_n^2(\boldsymbol{a})$$

$$ES_k^0(\boldsymbol{a}) = -\frac{n-k}{n-1}\,\sigma_n^2(\boldsymbol{a})$$

hence (2.5) holds true. Again by Lemma 3.1, as $n \to \infty$,

$$\begin{aligned} ES_k^2(\boldsymbol{a}) &= (k-1)Ea_n^2(R_1)a_n^2(R_2) + 2(k-2)Ea_n^2(R_1)a_n(R_2)a_n(R_3) \\ &\quad + (k-2)(k-3)Ea_n(R_1)a_n(R_2)a_n(R_3)a_n(R_3) \\ &= \frac{k}{n^2}\left(\sum_{i=1}^n a_n^2(i)\right)^2\left(1 + \mathrm{O}\left(\frac{1}{n}\right)\right) + \mathrm{O}\left(\frac{k^2}{n^3}\sum_{i=1}^n a_n^4(i)\right) \\ &= k\sigma_n^4(\boldsymbol{a})(1 + \mathrm{O}(n^{-1})) \end{aligned}$$

uniformly in $1 < k < n$. Quite analogously we get

$$ES_k^{02}(\boldsymbol{a}) = (n-k)\sigma_n^4(\boldsymbol{a})(1 + \mathrm{O}(n^{-1})),$$

$$\operatorname{cov}(S_k(\boldsymbol{a}), S_k^0(\boldsymbol{a})) = \frac{k(n-k)}{n^2}\,\sigma_n^4(\boldsymbol{a})(1 + \mathrm{O}(n^{-1}))$$

uniformly in $1 < k < n$ and hence (2.6) holds true.

The idea of the proof of Theorem 2.1 relies on the fact that $S_k(\boldsymbol{a})$ and $S_k^0(\boldsymbol{a})$ are sufficiently close to statistics of a similar structure for which the limit distribution of the corresponding maxima is known. Namely, we show that $S_k(\boldsymbol{a})$ and $S_k^0(\boldsymbol{a})$ are sufficiently close to the partial sums

$$S_k(\boldsymbol{a}, \boldsymbol{U}) = \sum_{i=2}^k (a_n([nU_i]+1) - \bar{a}_n(\boldsymbol{U}))(a_n([nU_{i-1}]+1) - \bar{a}_n(\boldsymbol{U})), \quad k = 2, \dots, n, \tag{3.6}$$

and

$$S_k^0(\boldsymbol{a},\boldsymbol{U})=\sum_{i=k+1}^{n}(a_n([nU_i]+1)-\bar{a}_n(\boldsymbol{U}))(a_n([nU_{i-1}]+1)-\bar{a}_n(\boldsymbol{U})),\quad k=1,\ldots,n-1, \tag{3.7}$$

$$\bar{a}_n(\boldsymbol{U})=\frac{1}{n}\sum_{i=1}^{n}a_n([nU_i]+1),$$

where $[c]$ denotes the integer part of c.

Let $\boldsymbol{U}_{(.)}=(U_{(1)},\ldots,U_{(n)})$ be the ordered sample corresponding to $(U_1,\ldots,U_n)$. It is well-known that $\boldsymbol{U}_{(.)}$ and $(R_1,\ldots,R_n)$ are independent random vectors and we can write

$$S_k(\boldsymbol{a},\boldsymbol{U})=\sum_{i=2}^{k}(a_n([nU_{(R_i)}]+1)-\bar{a}_n(\boldsymbol{U}_{(.)}))(a_n([nU_{(R_{(i-1)})}]+1)-\bar{a}_n(\boldsymbol{U}_{(.)})),\quad k=2,\ldots,n. \tag{3.8}$$

Therefore

$$E\left(\frac{1}{k-1}S_k(\boldsymbol{a},\boldsymbol{U})-\frac{1}{n-k}S_k^0(\boldsymbol{a},\boldsymbol{U})\right)$$
$$=E\left\{E\left(\frac{1}{k-1}S_k(\boldsymbol{a},\boldsymbol{U})-\frac{1}{n-k}S_k^0(\boldsymbol{a},\boldsymbol{U})\right)\Bigg|\,\boldsymbol{U}_{(.)}\right\}=0, \tag{3.9}$$

where we apply (2.5) with $a_n(i)$ replaced by $a_n([nU_{(i)}]+1)$. Proceeding similarly and applying (2.6) instead of (2.5) we obtain

$$E\left(\frac{1}{k-1}S_k(\boldsymbol{a},\boldsymbol{U})-\frac{1}{n-k}S_k^0(\boldsymbol{a},\boldsymbol{U})\right)^2=\frac{n}{k(n-k)}\sigma_n^4(\boldsymbol{a})(1+\mathrm{O}(n^{-1})) \tag{3.10}$$

uniformly in $1<k<n$ under H_0. Clearly,

$$Ea_n(\boldsymbol{U})=\frac{1}{n}\sum_{i=1}^{n}a_n(i). \tag{3.11}$$

The assertion of Theorem 2.1 is a straightforward consequence of the following two theorems.

Theorem 3.1. *Let $U_1,\ldots,U_n$ be iid random variables with uniform distribution function on $(0,1)$ and let $R_1,\ldots,R_n$ be the corresponding ranks. Let the scores $a_n(.)$ satisfy* (2.3), (2.8) *and* (2.9). *Then for all y, as $n\to\infty$,*

$$\max_{1<k<n}\sqrt{\frac{k(n-k)}{n}}\left(\frac{1}{k-1}|S_k(\boldsymbol{a})-S_k(\boldsymbol{a},\boldsymbol{U})|+\frac{1}{n-k}|S_k^0(\boldsymbol{a})-S_k^0(\boldsymbol{a},\boldsymbol{U})|\right)=\mathrm{o}_{\mathrm{P}}(n^{-\nu}) \tag{3.12}$$

with some $\nu>0$.

Theorem 3.2. *For each $n = 1,2\dots$ let $Y_{n1},\dots,Y_{nn}$ be iid random variables with zero mean, unit variance and finite fourth moment, $n = 1,2\dots$. Then for all y*

$$P\left(\sqrt{2\log\log n}\max_{1<k<n}\sqrt{\frac{k(n-k)}{n}}\left|\frac{1}{k-1}\sum_{i=2}^{k}Y_{ni}Y_{n,i-1}-\frac{1}{n-k}\sum_{i=k+1}^{n}Y_{ni}Y_{n,i-1}\right|\right.$$

$$\left.\leqslant 2\log\log n + y + \frac{1}{2}\log\log\log n - \frac{1}{2}\log\pi\right) \to \exp\{-2\mathrm{e}^{-y}\}, \tag{3.13}$$

as $n \to \infty$.

Theorem 3.1 can be useful in deriving the limit behavior of the test statistics discussed at the end of Section 2. Theorem 3.2 can be used in developing test statistics for the considered testing problem using empirical correlation coefficients of the first k and last $n-k$ observations. One should point out that the important issue is that we consider a triangular array of the Y's.

Proof of Theorem 3.1. We employ the martingale property of properly transformed $S_k(\boldsymbol{a})$ and a number of properties of rank statistics including moment inequality for rank statistics proved in Hušková (1997).

Set

$$V_k(\boldsymbol{a},\boldsymbol{b}) = \sum_{i=2}^{k} a_n(R_i)b_n(R_{i-1}), \quad k = 2,\dots,n$$

$$\tilde{V}_k(\boldsymbol{a},\boldsymbol{b}) = V_k(\boldsymbol{a},\boldsymbol{b}) + \sum_{i=2}^{k} b_n(R_{i-1})\frac{1}{n-i+1}\sum_{j=1}^{i-1} a_n(R_j), \quad k = 2,\dots,n-1, \tag{3.14}$$

where $a_n(1),\dots,a_n(n)$ and $b_n(1),\dots,b_n(n)$ are scores satisfying (2.3), (2.8), (2.9) and

$$\frac{1}{n}\sum_{i=2}^{n} b_n^4(i) \leqslant D_3, \quad n \geqslant 3 \tag{3.15}$$

$$\frac{1}{n}\sum_{i=1}^{n} b_n(i) = 0 \tag{3.16}$$

with some $D_3 > 0$. $V_k^0(\boldsymbol{a},\boldsymbol{b})$ and $\tilde{V}_k^0(\boldsymbol{a},\boldsymbol{b})$ are defined accordingly. Clearly,

$$V_k(\boldsymbol{a},\boldsymbol{a}) = S_k(\boldsymbol{a}), \quad V_k^0(\boldsymbol{a},\boldsymbol{a}) = S_k^0(\boldsymbol{a}).$$

Next we prove three auxiliary lemmas.

Lemma 3.2. *Let $U_1,\dots,U_n$ be iid random variables with $(0,1)$-uniform distribution and let $R_1,\dots,R_n$ be the corresponding ranks. Let assumptions (2.3), (2.8), (2.9), (3.15)–(3.16) be satisfied.*

Then $(\tilde V_k(\boldsymbol{a},\boldsymbol{b}),\sigma\{R_1,\dots,R_{k-1}\};k{=}2,\dots,n{-}1)$ *and* $(\tilde V_k^0(\boldsymbol{a},\boldsymbol{b}),\sigma\{R_n,\dots,R_{k+1}\};k{=}n{-}1,\dots,1)$ *form martingales. Here* $\sigma\{R_1,\dots,R_{k-1}\}$ *and* $\sigma\{R_n,\dots,R_{k+1}\}$ *denote a* σ*-field generated by* $R_1,\dots,R_{k-1}$ *and by* $R_n,\dots,R_{k+1}$, *respectively.*

Moreover, as $n\to\infty$,

$$E\tilde V_k(\boldsymbol{a},\boldsymbol{b})=0,\quad k=2,\dots,n, \tag{3.17}$$

$$E\tilde V_k^0(\boldsymbol{a},\boldsymbol{b})=0,\quad k=2,\dots,n, \tag{3.18}$$

$$E\tilde V_k^2(\boldsymbol{a},\boldsymbol{b})=k\sigma_n^2(\boldsymbol{a})\sigma_n^2(\boldsymbol{b})\left(1+\mathrm{O}\left(\frac{k}{n}+\log((n-k)/n)\right)\right), \tag{3.19}$$

$$E(\tilde V_k^0(\boldsymbol{a},\boldsymbol{b}))^2=(n-k)\sigma_n^2(\boldsymbol{a})\sigma_n^2(\boldsymbol{b})\left(1+\mathrm{O}\left(\frac{n-k}{n}+\log((k)/n)\right)\right) \tag{3.20}$$

uniformly in $1<k<n-1$.

Proof. Direct calculations yield (3.17), (3.18) and that

$$\begin{aligned}E(\tilde V_k(\boldsymbol{a},\boldsymbol{b})|R_1,\dots,R_{k-1})&=\tilde V_{k-1}(\boldsymbol{a},\boldsymbol{b})+b_n(R_{k-1})E(a_n(R_k)|R_1,\dots,R_{k-1})\\&\quad+b_n(R_{k-1})\frac{1}{n-k+1}\sum_{j=1}^{k-1}a_n(R_j)\\&=\tilde V_{k-1}(\boldsymbol{a},\boldsymbol{b}),\quad k=3,\dots,n,\end{aligned}$$

where we applied also (2.3). This means that $\tilde V_k(\boldsymbol{a},\boldsymbol{b})$, $k=2,\dots,n$, forms a martingale and also that the expectation of $\tilde V_k(\boldsymbol{a},\boldsymbol{b})$ is zero.

Using Lemma 3.1, (3.17), (2.3), (3.1) and martingale properties of $\tilde V_k$'s we have

$$\begin{aligned}E\tilde V_k^2(\boldsymbol{a},\boldsymbol{b})&=\sum_{i=2}^{k}Eb_n^2(R_{i-1})\left(a_n(R_i)\left(1-\frac{1}{n-i+1}\right)-\frac{1}{n-i+1}\sum_{j=i+1}^{n}a_n(R_j)\right)^2\\&=Eb_n^2(R_1)a_n^2(R_2)\left(\sum_{i=2}^{k}\left(\frac{n-i}{n-i+1}\right)^2+\sum_{i=2}^{k}\frac{n-i}{(n-i+1)^2}\right)\\&\quad Eb_n^2(R_1)a_n(R_2)a_n(R_3)\left(-2\sum_{i=2}^{k}\frac{(n-i)^2}{(n-i+1)^2}+\sum_{i=2}^{k}\frac{(n-i)(n-i+1)}{(n-i+1)^2}\right)\\&=\sigma_n^2(\boldsymbol{a})\sigma_n^2(\boldsymbol{b})(k+\mathrm{O}(\log((n-k)/n)+k/n))\end{aligned}$$

which implies (3.19). The assertions on $\tilde V_k^0$, $k=1,\dots,n$, can be shown in the same way therefore it is omitted. □

Lemma 3.3. *Under the assumptions of Lemma* 3.2 *for any* $C>0$ *there exist* n_C *and* $D_C>0$ *such that for all* $n \geqslant n_C$

$$P\left(\max_{2<k<k_n} \frac{1}{\sqrt{k}}\,|\tilde{V}_k(\boldsymbol{a},\boldsymbol{b})| \geqslant C\right) \leqslant \frac{D_C}{C^2}\,\sigma_n^2(\boldsymbol{b})\sigma_n^2(\boldsymbol{a})(\log k_n+1),$$

$$P\left(\max_{2<n-k<k_n} \frac{1}{\sqrt{n-k}}\,|\tilde{V}_k^0(\boldsymbol{a},\boldsymbol{b})| \geqslant C\right) \leqslant \frac{D_C}{C^2}\,\sigma_n^2(\boldsymbol{b})\sigma_n^2(\boldsymbol{a})(\log k_n+1)$$

for any $1<k_n<n$ *and*

$$P\left(\max_{2<k<n} \frac{1}{\sqrt{n}}\,|\tilde{V}_k(\boldsymbol{a},\boldsymbol{b})| \geqslant C\right) \leqslant \frac{D_C}{C^2}\,\sigma_n^2(\boldsymbol{b})\sigma_n^2(\boldsymbol{a}),$$

$$P\left(\max_{2<k<n} \frac{1}{\sqrt{n}}\,|\tilde{V}_k^0(\boldsymbol{a},\boldsymbol{b})| \geqslant C\right) \leqslant \frac{D_C}{C^2}\,\sigma_n^2(\boldsymbol{b})\sigma_n^2(\boldsymbol{a}).$$

Proof. All inequalities are straightforward consequences of Lemma 3.1 and of the Hájek-Rényi-Chow inequality for martingales (see, e.g. Chow and Teicher, 1988).

Next we investigate

$$\tilde{\tilde{V}}_k(\boldsymbol{a},\boldsymbol{b}) = V_k(\boldsymbol{a},\boldsymbol{b}) - \tilde{V}_k(\boldsymbol{a},\boldsymbol{b}) = -\sum_{i=2}^{k} b_n(R_{i-1})\,\frac{1}{n-i+1}\sum_{j=1}^{i-1} a_n(R_j), \quad k=2,\ldots,n-1 \tag{3.21}$$

and

$$\tilde{\tilde{V}}_k^0(\boldsymbol{a},\boldsymbol{b}) = V_k^0(\boldsymbol{a},\boldsymbol{b}) - \tilde{V}_k^0(\boldsymbol{a},\boldsymbol{b}), \quad k=2,\ldots,n. \tag{3.22}$$

Lemma 3.4. *Let the assumptions of Lemma* 3.2 *be satisfied. Then*

$$\max_{1<k<n} \frac{k(n-k)}{n}\,\frac{1}{k^2}\,(\tilde{\tilde{V}}_k(\boldsymbol{a},\boldsymbol{b})) = \mathrm{O}_{\mathrm{P}}(\sigma_n^2(\boldsymbol{b})\sigma_n^2(\boldsymbol{a})(\log n)^2) \tag{3.23}$$

and

$$\max_{1<k<n} \frac{k(n-k)}{n}\,\frac{1}{(n-k)^2}\,(\tilde{\tilde{V}}_k^0(\boldsymbol{a},\boldsymbol{b}))^2 = \mathrm{O}_{\mathrm{P}}(\sigma_n^2(\boldsymbol{b})\sigma_n^2(\boldsymbol{a})(\log n)^2). \tag{3.24}$$

Proof. By the Markov inequality we have that for any $C>0$

$$P\left(\max_{1<k\leqslant n} \frac{k(n-k)}{n}\,\frac{1}{k^2}|\tilde{\tilde{V}}_k(\boldsymbol{a},\boldsymbol{b})|^2 \geqslant C\right) \leqslant \frac{1}{C}\sum_{k=2}^{n} E(\tilde{\tilde{V}}_k(\boldsymbol{a},\boldsymbol{b}))^2\,\frac{n-k}{kn}. \tag{3.25}$$

and

$$E(\tilde{\tilde{V}}_k(\boldsymbol{a},\boldsymbol{b}))^2 \leqslant 2E\left(\sum_{i=2}^{k} b_n(R_{i-1})a_n(R_{i-1})\,\frac{1}{n-i+1}\right)^2$$

$$+\,2E\left(\sum_{i=2}^{k} b_n(R_{i-1})\,\frac{1}{n-i+1}\sum_{j=1}^{i-2} a_n(R_j)\right)^2.$$

Since $\sum_{i=2}^{k} b_n(R_{i-1})a_n(R_{i-1})1/(n-i+1)$, $k=2,\dots,n$, are is the simple linear rank statistics and regarding the assumptions (2.3), (2.9) we notice that

$$\left(E\sum_{i=2}^{k} b_n(R_{i-1})a_n(R_{i-1})\,\frac{1}{n-i+1}\right)^2 = \left(\frac{1}{n}\sum_{j=1}^{n} a_n(j)b_n(j)\sum_{i=2}^{k}\frac{1}{n-i+1}\right)^2$$

$$= \mathrm{O}(\sigma_n^2(\boldsymbol{a})\sigma_n^2(\boldsymbol{b})(\log((n-k)/n))^2)$$

and

$$\mathrm{var}\left\{\sum_{i=2}^{k} b_n(R_{i-1})a_n(R_{i-1})\,\frac{1}{n-i+1}\right\} = \mathrm{O}\left(\frac{1}{n}\sum_{i=1}^{n} a_n^2(i)b_n^2(i)\sum_{j=2}^{k}\frac{1}{(n-k+1)^2}\right)$$

$$= \mathrm{O}\left(\sigma_n^2(\boldsymbol{b})\sqrt{n}\,\frac{1}{n-k}\right)$$

therefore

$$E\left(\sum_{i=2}^{k} b_n(R_{i-1})a_n(R_{i-1})\,\frac{1}{n-i+1}\right)^2$$

$$= \mathrm{O}\left\{\sigma_n^2(\boldsymbol{b})\sigma_n^2(\boldsymbol{a})\left(\sqrt{n}\,\frac{1}{n-k} + (\log((n-k)/n))^2\right)\right\} \tag{3.26}$$

uniformly in $1<k<n$. Finally, by Lemma 3.1 and regarding (2.3) and (3.1) we get

$$E\left(\sum_{i=2}^{k} b_n(R_{i-1})\,\frac{1}{n-i+1}\sum_{j=1}^{i-2} a_n(R_j)\right)^2$$

$$= E\left(\sum_{v=1}^{k-1} b_n(R_v)\,\frac{1}{n-v}\sum_{j=1}^{v-1} a_n(R_j)\right)^2$$

$$= \mathrm{O}\left(\sigma_n^2(\boldsymbol{b})\sigma_n^2(\boldsymbol{a})\left(\sum_{i=1}^{k}\frac{i}{(n-i)^2} + n^{-3/2}\sum_{i=1}^{k}\frac{i^2}{(n-i)^2} + n^{-5/2}\left(\sum_{i=1}^{k}\frac{i}{n-i}\right)^2\right)\right)$$

uniformly in $1 < k < n$, which together with (3.25) and (3.26) imply (3.21). The proof of (3.22) follows the same line hence it is omitted. □

Now, we finish the proof of Theorem 3.1. We decompose the partial sums $S_k(\boldsymbol{a})$ as follows:

$$S_k(\boldsymbol{a}) = S_k(\boldsymbol{a}, \boldsymbol{U}) + M_k(\boldsymbol{a}) + N_k(\boldsymbol{a}), \quad k = 2, \ldots, n, \tag{3.27}$$

where

$$M_k(\boldsymbol{a}) = \sum_{i=2}^{k} a_n(R_i)(a_n(R_{i-1}) - (a_n([nU_{(R_{i-1})}] + 1) - \bar{a}_n(\boldsymbol{U}_{(.)}))), \tag{3.28}$$

$$N_k(\boldsymbol{a}) = \sum_{i=2}^{k} (a_n(R_i) - (a_n([nU_{(R_i)}] + 1) - \bar{a}_n(\boldsymbol{U}_{(.)})))(a_n([nU_{(R_{i-1})}] + 1) - \bar{a}_n(\boldsymbol{U}_{(.)})). \tag{3.29}$$

We apply Lemmas 3.2–3.4 with

$$b_n(i) = a_n(R_i) - (a_n([nU_{(R_i)}] + 1) - \bar{a}_n(\boldsymbol{U}_{(.)}))$$

taking $\boldsymbol{U}_{(.)}$. Particularly, we find that with this choice of $a_n(i)$'s and $b_n(i)$'s

$$\sigma_n^2(\boldsymbol{b}) = \sigma_n^2(\boldsymbol{a}, \boldsymbol{U}_{(.)}) = \frac{1}{n}\sum_{i=1}^{n} (a_n(i) - (a_n([nU_{(i)}] + 1) - \bar{a}_n(\boldsymbol{U}_{(.)})))^2$$

and by Lemma in Hušková (1997) and (2.9)

$$E(\sigma_n^2(\boldsymbol{a}, \boldsymbol{U}_{(.)}))^2 \leqslant C_2 \frac{1}{n^2}\sum_{i=1}^{n} a_n^4 = \mathrm{O}(n^{-1}).$$

This in combination with Lemmas 3.2–3.4 and (3.27) imply, as $n \to \infty$,

$$\max_{1<k<n} \sqrt{\frac{k(n-k)}{n}}\,\frac{1}{k}\,|S_k(\boldsymbol{a}) - S_k(\boldsymbol{a}, \boldsymbol{U})| = \mathrm{O}_{\mathrm{P}}(n^{-\nu})$$

$$\max_{1<k<n} \sqrt{\frac{k(n-k)}{n}}\,\frac{1}{n-k}\,|S_k(\boldsymbol{a}) - S_k(\boldsymbol{a}, \boldsymbol{U})| = \mathrm{O}_{\mathrm{P}}(n^{-\nu})$$

with some $\nu > 0$. Theorem 3.1 is proved. □

Proof of Theorem 3.2. The proof is based on a classical results of Strassen (1967). We apply Theorem 4.3 in Strassen (1967) to the random variables

$$Z_{ni} = Y_{ni} Y_{n,i-1}, \quad i = 2, \ldots, n,$$

that have the property

$$E(Z_{ni}|Y_{n1},\dots,Y_{n,i-1})=0 \quad \text{a.s. } i=2,\dots,n$$

and

$$E(Z_{ni}^2|Y_{n1},\dots,Y_{n,i-1})=Y_{n,i-1}^2 \quad \text{a.s. } i=2,\dots,n.$$

By Theorem 4.3 of Strassen (1967) there are independent Brownian motions $\{W_{nj}(t);\, t\in[0,\infty)\}$, $j=1,2$ and sequences of nonnegative random variables $V_{n2},\dots,V_{n,[n/2]}$ and $V_{n,[n/2]+1},\dots,V_{nn}$ such that

$$\sum_{i=2}^{k} Z_{ni}=W_n\left(\sum_{i=2}^{k} V_{nk}\right), \quad 2\leqslant k\leqslant n/2 \tag{3.30}$$

and

$$\sum_{i=k+1}^{n} Z_{ni}=W_n\left(\sum_{i=k+1}^{n} V_{nk}\right), \quad n/2<k\leqslant n. \tag{3.31}$$

Moreover, V_{nk} is $\sigma\{Z_{n1},\dots,Z_{nk}\}$ measurable, $W_{n1}(\sum_{i=2}^k V_{ni}+s)-W_{n1}(\sum_{i=2}^k V_{ni})$ is independent of $\sigma\{Z_{n1},\dots,Z_{nk}\}$ for any $s>0$,

$$E(V_{nk}|Z_{n2},\dots,Z_{n,k-1})=E(Z_{nk}^2|Z_{n2},\dots,Z_{n,k-1})=Y_{n,k-1}^2 \quad \text{a.s.} \tag{3.32}$$

and

$$E(V_{nk}^2|Z_{n2},\dots,Z_{n,k-1})\leqslant CE(Z_{nk}^4|Z_{n2},\dots,Z_{n,k-1})=CY_{n,k-1}^4 \quad \text{a.s.} \tag{3.33}$$

$2\leqslant k\leqslant n/2$, with some $C>0$.

Then by the Hájek-Rényi-Chow inequality (cf. Chow and Teicher, 1988) for any $\beta>\frac{1}{2}$ and any $x>0$ we have

$$P\left(\max_{2\leqslant k\leqslant n/2}\frac{1}{k^\beta}\left|\sum_{i=2}^{k}(V_{ni}-E(V_{ni}|Z_{n2},\dots,Z_{n,i-1}))\right|\geqslant x\right)$$

$$\leqslant\frac{1}{x^2}\sum_{i=2}^{n}k^{-2\beta}EZ_{n,k}^4\leqslant Cx^{-2} \tag{3.34}$$

with some $C>0$. Since Y_{nk} are i.i.d. random variables with zero mean, unit variance and finite fourth moment and by (3.32) we also have for any $\beta>\frac{1}{2}$ and any $x>0$

$$P\left(\max_{2\leqslant k\leqslant n/2}\frac{1}{k^\beta}\left|\sum_{i=2}^{k}(E(V_{ni}|Z_{n2},\dots,Z_{n,i-1})-1)\right|\geqslant x\right)\leqslant Cx^{-2} \tag{3.35}$$

with some $C>0$. Lemma 1.2.1 of Csörgő and Révész (1981) yields that

$$\max_{2\leqslant k\leqslant n}\sup_{|s|\leqslant Ck^\beta}(k^\beta\log k)^{-1/2}|W_{n1}(k)-W_{n1}(k+s)|=O_P(1). \tag{3.36}$$

Now, from (3.30) and (3.32)–(3.36) we can infer that

$$\max_{1<k\leqslant n/2}\left|\sum_{i=2}^{k} Y_{ni}Y_{n,i-1}-W_{n1}(k-1)\right|\bigg/k^{1/2-r}$$

$$=O_P\left(\max_{1<k\leqslant n/2} k^{\beta/2-(1/2-r)}(\log k)^2\right)=O_P(1) \tag{3.37}$$

for any given $0<r<\frac{1}{4}$ with properly chosen $\beta=\beta(r)>\frac{1}{2}$

By symmetry we get for any given $r\in(0,\frac{1}{2})$ that

$$\max_{n/2<k\leqslant n}\left|\sum_{i=k+1}^{n} Y_{ni}Y_{n,i-1}-W_{n2}(n-k)\right|\bigg/(n-k)^{1/2-r}=O_P(1). \tag{3.38}$$

Now, the assertion of the theorem can be concluded from (3.37), (3.38) and the Darling-Erdős theorem (e.g., Theorem A.4.1 in Csörgő and Horváth, 1997). The proof is now complete. □

Proof of Theorem 2.1. The assertion follows from Theorems 3.1 and 3.2 with

$$Y_{ni}=(a_n([nU_i]+1)-\bar{a}_n)/\sigma_n(\boldsymbol{a}), \quad i=2,\dots,n$$

and from the observation

$$\bar{a}_n(\boldsymbol{U}_{(.)})=O_P(n^{-1/2}). \qquad \square$$

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