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CHANGE-POINT METHODS AND THEIR APPLICATIONS: CONTRIBUTIONS OF IAN MACNEILL

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SUMMARY

The present paper reviews the important contributions of Ian MacNeill to the theory and methodology of change-point analysis and environmental statistics. The review concentrates on four areas of change-point analysis: sequences of independent random variables; linear regression models with independent as well as serially correlated random errors; regression models with continuity constraints and spatial models of change-points. Copyright © 1999 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Ian MacNeill made important contributions to the development of change-point methods and discussed their applications to many other branches of knowledge, especially to Environmetrics and Biostatistics. His interest in applications led him to play a significant role in the creation of *The International Environmetrics Society* (TIES) and its journal *Environmetrics*. Due to space limitations, this article is not intended to cover all aspects of his extensive work but to focus only on his theoretical contributions to the change point problem. His pursuits for wide ranging applications of this methodology including environmetrics and biostatistics may be found in the selected list of his published works: MacNeill (1980, 1982, 1993, 1995), Tang and MacNeill (1989, 1990, 1992), MacNeill *et al*. (1991, 1994, 1995a, 1995b) and MacNeill and Mao (1993, 1997).

The change-point problem was formally introduced by Page (1955) where the celebrated CUSUM procedure was proposed to test for a change in a parameter occurring at an unknown time-point. While the problem was originally formulated to improve the Shewhart's 3-sigma control chart procedures in quality control, today, the methods have applications in all areas of science and technology.

The approaches adapted for solving change-point problems include maximum likelihood, Bayesian, Bayes-type, non-parametric, as well as decision theoretic procedures. Soon after the problem was introduced by Page (1955), Quandt (1958, 1960) derived the likelihood ratio statistic

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to test for change in the parameters of simple and general linear regression models. The distributions of the test statistics derived by Quandt, either in small samples or asymptotic approximations, were not available.

Chernoff and Zacks (1964) studied the problem of estimating the current mean of a sequence of independent normal random variables whose means are subjected to random amounts of changes at random epochs. As a side problem they studied in this celebrated paper the following testing problem. Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables such that $Y_i \sim N(\mu_i, \sigma^2)$, $i = 1, \ldots, n$. Consider the null hypothesis $H_0 : \mu_1 = \cdots = \mu_n = \mu_0$, $-\infty < \mu_0 < \infty$ versus the composite alternative

$$\begin{aligned} H_a : \mu_1 &= \cdots = \mu_\nu = \mu_0 \\ \mu_{\nu+1} &= \cdots = \mu_n = \mu_0 + \delta, \\ \nu \in \{1, 2, \ldots, n-1\},&\ \ \delta > 0,\ \ -\infty < \mu_0 < \infty. \end{aligned}$$

Using a Bayesian approach and assuming μ_0 to be unknown, Chernoff and Zacks derived the test statistic

$$T_{1n} = \sum_{j=1}^{n-1} p(j) \sum_{i=j+1}^{n} (Y_i - \bar{Y}_n), \tag{1}$$

where $p(j)$ are prior probabilities assigned to the change-point ν, and $\bar{Y}_n = 1/n(\Sigma_{i=1}^n Y_i)$. They also showed that in the simpler case, where $p(j) = 1/(n-1)$ and $\mu_0 = 0$, the test statistic, T_{1n}, reduces to

$$T_{1n} = \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} Y_i = \sum_{j=1}^{n} (j-1) Y_j. \tag{2}$$

It is very simple to derive the null distribution of T_{1n} and its power function under the assumption of normality. Chernoff and Zacks (1964) also showed that T_{1n} compares favorably to Page's CUSUM test, when the changes δ are small. Kander and Zacks (1966) generalized the results of Chernoff and Zacks to the case where the distributions of $Y_i (i = 1, \ldots, n)$ are of the exponential type, with density functions

$$f(x; \theta) = h(x)\exp\{\psi_1(\theta)U(x) + \psi_2(\theta)\}, \tag{3}$$

where $\psi_1(\theta)$ and $\psi_2(\theta)$ have continuous derivatives and $\psi_1'(\theta) > 0$. For testing the simple hypothesis

$$H_0 : \theta_1 = \cdots = \theta_n = \theta_0 \text{ (known)},$$

against

$$\begin{aligned} H_1 : \theta_1 &= \cdots = \theta_\nu = \theta_0 \\ \theta_{\nu+1} &= \cdots = \theta_n = \theta_0 + \delta \\ \nu \in \{1, \ldots, n-1\},&\ \ \delta > 0 \end{aligned}$$

they arrived at the test statistic (2), in which $Y_i = U(X_i)$, $i = 1, \ldots, n$. Kander and Zacks established the asymptotic normality of T_{1n} but showed that the weak convergence to a normal distribution is slow. They suggested an Edgeworth expansion approximation for the distribution of T_{1n}, when n is not very large.

Gardner (1969) studied the testing problem for normal random variables, when the alternative hypothesis is two-sided, i.e. $\delta \neq 0$. Using the Chernoff and Zacks approach he derived the statistic

$$Q_n = \sum_{j=1}^{n-1} p(j) \sum_{i=j+1}^{n} (Y_i - \bar{Y}_n)^2. \tag{4}$$

Gardner has shown that under H_0

$$\frac{6n}{n^2 - 1} Q_n \sim \sum_{k=1}^{n-1} \lambda_k U_k^2,$$

where $U_1, \ldots, U_{n-1}$ are i.i.d. standard normal random variables, and

$$\lambda_k = \frac{6n^2}{\pi^2 (n^2 - 1) k^2} \left[\frac{2n}{k\pi} \cos\left(\frac{k\pi}{2n} \right) \right]^{-2}, \tag{5}$$

$k = 1, \ldots, n - 1$. Thus.

$$\frac{6n}{n^2 - 1} Q_n \xrightarrow{d} \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} U_k^2, \tag{6}$$

as $n \to \infty$. Gardner (1969) did not derive the asymptotic distribution of his test statistic under the alternative hypothesis. Power computations were done by simulations. The first study which derived the asymptotic distribution of a test statistic similar to (4) was that of MacNeill (1974). MacNeill used methods of weak convergence to approximate the distribution of the test statistic, as $n \to \infty$, by the distribution of a functional of a Brownian process. The work of MacNeill will be described in Section 2. Section 3 reviews the contributions of MacNeill to change-points in regression models. These results are of special value for environmetrics and chemometrics. Section 4 reviews MacNeill's studies on change-point analysis for time series with serial correlations. Section 5 is devoted to models with continuity constraints. In Section 6 we give a brief account of the spatial analog. As will be shown, the approach of MacNeill to the change-point analysis is basically a Bayesian approach, as in Chernoff and Zacks (1964). There is an alternative approach, namely the maximum likelihood approach. The asymptotic theory of the maximum likelihood approach is given in the book of Csörgö and Horváth (1997).

2. CHANGE-POINT METHODS FOR SEQUENCES OF RANDOM VARIABLES

In the present section we review the study of MacNeill (1974), which extended that of Kander and Zacks (1966) to two-sided hypotheses, $\delta \neq 0$. Thus, let $\{X_j\}_{j=1}^{n}$ be a sequence of independent random variables having a one-parameter exponential densities (3). Let $Y_i = U(X_i)$. MacNeill obtained the test statistic

$$T_{2n} = \sum_{j=1}^{n-1} p(j)\left[\sum_{i=j+1}^{n} \frac{(\psi_1'(\theta_0)Y_i + \psi_2'(\theta_0))\sqrt{\psi_1'(\theta_0)}}{\psi_1''(\theta_0)\psi_2'(\theta_0) - \psi_1'(\theta_0)\psi_2''(\theta_0)}\right]^2. \tag{7}$$

The mean and variance of a random variable having a density function (3) are

$$\mu(\theta) = -\frac{\psi_2'(\theta)}{\psi_1'(\theta)} \tag{8}$$

and

$$\tau^2(\theta) = \frac{\psi_1''(\theta)\psi_2'(\theta) - \psi_1'(\theta)\psi_2''(\theta)}{(\psi_1'(\theta))^3}.$$

Thus, the test statistic $T_{2,n}$ is given by

$$T_{2n} = \sum_{j=1}^{n-1} p(j)\left[\sum_{i=j+1}^{n} \frac{Y_i - \mu(\theta_0)}{\tau(\theta_0)}\right]^2. \tag{10}$$

If the initial value of θ_0 is unknown, we substitute the MLE (under H_0) i.e..

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

Thus, let

$$T_{3n} = \sum_{j=1}^{n-1} p(j)\left[\sum_{i=j+1}^{n} \frac{Y_i - \bar{Y}_n}{\tau(\hat{\theta}_0)}\right]^2. \tag{11}$$

Let $\{B(t), t \in [0, 1]\}$ denote the standard Brownian motion process (Wiener process) and $\{B_0(t), t \in [0, 1]\}$ denote the standard Brownian bridge (see Karlin and Taylor 1981).

Using weak convergence methods (see Billingsley 1968) MacNeill proved the following theorems.

Theorem 1. Let $\psi(\cdot)$ be a non-negative weight function defined on the unit interval [0, 1] such that $\int_0^1 t\psi(t)\,\mathrm{d}t < \infty$. Let $\{p(j)\}_{j=1}^{n-1}$ be defined by

$$p\left(\frac{n-j}{n}\right) = \int_{j/n}^{(j+1)/n} \psi(t)\,\mathrm{d}t, \quad j = 1, \ldots, n-1.$$

Then.

$$n^{-1}T_{2n} \xrightarrow{d} \int_0^1 \psi(t)B^2(t)\,\mathrm{d}t. \tag{12}$$

Theorem 2. Let $\psi(t)$ be a non-negative weight function defined on [0, 1] such that $\int_0^1 t(1-t)\psi(t)\,\mathrm{d}t < \infty$. If $\{p(j/n)\}_{j=1}^{n-1}$ is defined by

$$p\left(\frac{n-j}{n}\right) = \int_{(2j-1)/2n}^{(2j+1)/2n} \psi(t)\,\mathrm{d}t,$$

and

$$n^{-1}T_{3n} \xrightarrow{d} \int_0^1 \psi(t)B_0^2(t)\,\mathrm{d}t. \tag{13}$$

MacNeill (1974) also studied the asymptotic distribution of T_{2n} under contiguous type alternatives. Under certain regularity conditions, MacNeill (1974) has shown that:

$$n^{-1}T_{2n} \xrightarrow{d} \int_0^1 \psi(t)(H(t)+B(t))^2\,\mathrm{d}t, \tag{14}$$

where

$$H(t) = \int_0^t \{h(1-s) - h_0\}\,\mathrm{d}s,$$

and h is a bounded, Reimann-integrable function on [0, 1].

Next, letting $\psi(t) = a\cdot t^k, k > -2$, MacNeill found quantiles for the distributions of the stochastic integrals in (12) and (13) by applying the methods of Anderson and Darling (1952) to Brownian bridge processes.

3. PARAMETER CHANGES IN LINEAR REGRESSION MODELS

MacNeill's contributions to this area of the change-point problem are quite extensive. Included among them are MacNeill (1978a, 1978b) and Jandhyala and MacNeill (1989, 1991, 1992, 1997).

Consider the linear regression model:

$$\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{15}$$

where $\boldsymbol{\epsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)' \sim N(0, \sigma^2 I)$, $\boldsymbol{\beta}' = (\beta_0, \ldots, \beta_{p-1})$ is the vector of regression parameters, $\mathbf{Y}' = (Y_1, \ldots, Y_n)$ is the dependent observation vector, and X is the design matrix with

$$X = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1\,p-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n1} & \cdots & x_{n\,p-1} \end{bmatrix}.$$

Jandhyala and MacNeill (1989, 1991) derived the one-sided as well as two-sided Bayes-type change detection statistics and established the asymptotic distribution theory of the derived statistics. While the statistics have been derived under the assumption of normality, the asymptotic theory, however, does not require such an assumption. The alternative hypothesis of the detection problem was formulated in a very general framework for the purpose of deriving the statistic and several important special cases follow.

Let $W = (w_{ij})$, $i = 1, \ldots, n-1; j = 0, \ldots, p-1$ be the change matrix. The (i, j)th component, w_{ij} is 1 or 0 according to whether or not there is a change in β_j between the ith and $(i+1)$th time-points. Also, let $\Delta = ((\delta_{ij}))$, $i = 1, \ldots, n-1$; $j = 0, \ldots, p-1$ be a matrix representing the amounts of changes in the parameters. The null and alternative hypotheses for the two-sided testing problem then are:

$$H_0 : \delta_{ij} = 0,\ i = 1, \ldots, n-1;\ j = 0, \ldots, p-1$$

against

$$H_a : \delta_{ij} \neq 0,\ \text{for some } i, j.$$

Let $\boldsymbol{\delta}_j = (\delta_{j0}, \ldots, \delta_{j(p-1)})$ be the vectors of change quantities $(j = 1, \ldots, n-1)$. The derivation of the two-sided Bayes-type statistic proceeds by assuming the following prior distributions on $\boldsymbol{\beta}, \boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_{n-1}$

$$\boldsymbol{\beta} \sim N(0, \tau^2 I),\ \ \boldsymbol{\delta}_j \sim N(0, \theta^2 I),\ j = 1, \ldots, n-1,$$

with $\boldsymbol{\beta}, \boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_{n-1}$ and ϵ all distributed independently. The derived statistic is given by:

$$T_{4n} = \sum_{\{W\}} p(W)\{Y' R \left(\sum_{i=0}^{p-1} C_i C_i' \right) RY], \tag{16}$$

where $p(W)$ is the prior probability mass function on the collection of change matrices $\{W\}$ and where

$$C_i = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & x_{2i}w_{1i} & 0 & \ldots & 0 \\ 0 & x_{3i}w_{1i} & x_{3i}w_{2i} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & x_{ni}w_{1i} & x_{ni}w_{2i} & \ldots & x_{ni}w_{n-1\ i} \end{bmatrix}.$$

Several important special cases of interest follow from the statistic T_{5n}. Suppose only one change takes place occurring between m and $(m+1)$, $m = 1, \ldots, n-1$ and let $p(m)$ denote the prior on this unknown change-point. Then, the corresponding statistic is:

$$T_{5n} = \sum_{m=1}^{n-1} p(m) Y' R H_{(m)} H'_{(m)} RY, \tag{17}$$

where the ith row of $H_{(m)}$ is given by $H_{(m)i} = (x_{m0}w_{m0}, \ldots, x_{p-1}w_{p-1})$, $i = m+1, \ldots, n$ with the first m rows being all identically zero. If one is interested in testing for two-sided changes in all of the parameters at the unknown change-point, then, one has

$$T_{6n} = \sum_{m=1}^{n-1} p(m) Y' R X_{(m)} X'_{(m)} RY, \tag{18}$$

where $X_{(m)}$ is the design matrix X with the first m rows replaced by zeros.

The statistic for testing at most one change in the parameter β_i alone is given by:

$$T_{7n}^{(i)} = \sum_{m=1}^{n-1} p(m) Y' R \mathbf{X}_{mi} \mathbf{X}'_{mi} R Y, \tag{19}$$

where $\mathbf{X}_{mi}$ is the ith column vector of X with the first m elements replaced by zeros. The one-sided Bayes-type statistic was derived to be:

$$T_{8n} = \sum_{m=1}^{n-1} p(m) \left\{ Y' R \sum_{j=0}^{p-1} w_j \mathbf{X}_{mj} \right\}. \tag{20}$$

The one-sided statistic was derived to incorporate multiple change-points.

The one-sided statistic will be distributed normally for finite samples under both null and alternative hypotheses as long as the error variables are normal. This would be true only asymptotically when the error variables are non-normal.

Distribution theory for the two-sided statistics is more complicated. While exact distribution of the two-sided statistic even under the null hypothesis is quite intractable, asymptotic null distribution theory has been derived and is based on distributions of Cramér von-Mises type functionals defined on partial sum residual processes. The distribution theory, however, requires that one considers a linear regression model based on regressor functions $f_i(\cdot)$, $i = 0, \ldots, p-1$ defined on [0, 1] and with equispaced observations. Simple modifications will extend the theory to cases where the regressor functions may be defined on compact subsets of the real line and where observations are not equispaced. The regression model (15) may be written as

$$Y_{nj} = \sum_{i=0}^{p-1} \beta_i f_i(j/n) + \varepsilon_j, \; j = 1, \ldots, n. \tag{21}$$

Under this model, the statistics T_{7n} and $T_{8n}^{(i)}$ may respectively be written as:

$$\begin{aligned} \sigma^{-2} n^{-1} T_{6n} &= \sum_{m=1}^{n-1} p(m) \left\{ \frac{1}{\sigma\sqrt{n}} \sum_{k=m+1}^{n} f_0(k/n)(Y_{nk} - \hat{Y}_{nk}) \right\}^2 + \cdots \\ &\quad + \sum_{m=1}^{n-1} p(m) \left\{ \frac{1}{\sigma\sqrt{n}} \sum_{k=m+1}^{n} f_{p-1}(k/n)(Y_{nk} - \hat{Y}_{nk}) \right\}^2, \end{aligned} \tag{22}$$

$$\sigma^{-2} n^{-1} T_{7n}^{(i)} = \sum_{m=1}^{n-1} p(m) \left\{ \frac{1}{\sigma\sqrt{n}} \sum_{k=m+1}^{n} f_i(k/n)(Y_{ni} - \hat{Y}_{ni}) \right\}^2. \tag{23}$$

Asymptotic distribution theory of both (22) and (23) requires limit processes for sequences of stochastic processes defined on partial sums of weighted regression residuals.

MacNeill (1978a, 1978b) initiated the fundamental work in this direction by considering the statistic for the case $i = 0$. Since $f_0(k/n) \equiv 1$, the case $i = 0$ corresponds to considering partial sums of unweighted regression residuals.

MacNeill (1978a) first derived the limit processes for stochastic processes, defined by partial sums of residuals, when the underlying regression functions are polynomial $f_i(t) = t^i$, $t \in [0, 1]$, $i = 0, \ldots, p-1$, . MacNeill (1978b) extended this result to the case of general linear regression models. We present below the general result of MacNeill (1978b).

Define sequences of partial sums of regression residuals by $\{S_{nj}, 1 \leqslant j \leqslant n, \; n \geqslant 1\}$ where

$$S_{nj} = \sum_{i=1}^{j} (Y_{ni} - \hat{Y}_{ni}).$$

Then define sequences of stochastic processes possessing continuous sample paths by

$$\sigma n^{1/2} \theta_{fn}(t) = S_{n[nt]} + (nt - [nt])(Y_{n[nt]+1} - \hat{Y}_{n[nt]+1}). \tag{24}$$

Further, let $X_n = ((f_i(j/n)), \; i = 0, \ldots, p-1, \; j = 1, \ldots, n$ be the design matrix and let $\lim n^{-1} X_n' X_n \equiv F$ and also define a bi-linear function $g(s, t)$ by

$$g(s, t) = f^t(s) F^{-1} f(t) \text{ where } f^t(s) = (f_0(s), \ldots, f_{p-1}(s)).$$

Then, the limit process for the sequence of stochastic processes $\{\theta_{fn}(t), t \in [0, 1]\}, n \geqslant 1$ is given by the following theorem.

Theorem 3. Let $f_i(t), t \in [0, 1], \; i = 0, \ldots, p-1$ be continuously differentiable on [0, 1]. Then $\{\theta_{f_n}(t), t \in [0, 1]\}$ converges weakly to the Gaussian process $\{B_f(t), t \in [0, 1]\}$ defined by

$$B_f(t) = B(t) - \int_0^t \left\{ \int_0^1 g(x, y) \, \mathrm{d}B(y) \right\} \mathrm{d}x. \tag{25}$$

Theorem 3 was subsequently extended by Jandhyala and MacNeill (1989) to partial sums of weighted regression residuals. Let $r(t), t \in [0, 1]$ be a continuous function and define sequences of stochastic processes $\{\theta_{fn}^{(r)}(t), t \in [0, 1]\}, n = 1, \ldots$ by

$$\sigma n^{1/2} \theta_{f_n}^{(r)}(t) = S_{n[n,t]}^{(r)} + (nt - [nt]) r\left(\frac{[nt]+1}{n}\right)(Y_{n[nt]+1} - \hat{Y}_{n[nt]+1}). \tag{26}$$

Then, the following generalizes Theorem 3.

Theorem 4. Let the regressor functions $f_i(t), t \in [0, 1], \; i = 0, \ldots, p-1$ be continuously differentiable on [0, 1]. Then, for $r(t)$ also continuously differentiable on [0, 1], the sequences of stochastic processes $\{\theta_{fn}^{(r)}(t), t \in [0, 1]\}, n = 1, 2, \ldots$ converges weakly to the Gaussian process $\{B_p^{(r)}(t), t \in [0, 1]\}$ defined by

$$B_p^{(r)}(t) = \int_0^t r(x) \, \mathrm{d}B(x) - \int_0^1 \left\{ \int_0^1 r(y) g(x, y) \, \mathrm{d}B(y) \right\} \mathrm{d}x. \tag{27}$$

Further, the covariance kernel $K_p^{(r)}(s, t)$ of the limit process is given by

$$K_p^{(r)}(s, t) = \int_0^{\min(s,t)} r^2(x) \, \mathrm{d}x - \int_0^s \int_0^t r(x) r(y) g(x, y) \, \mathrm{d}x \, \mathrm{d}y. \tag{28}$$

The following theorem of Jandhyala and MacNeill (1989) then establishes the asymptotic null distributions of T_{6n} and $T_{7n}^{(i)}$, respectively. In the theorem below, $p(m)$ represents a weight sequence.

Theorem 5. Let $\psi(\cdot)$ be a non-negative weight function such that $\int_0^1 t(1-t)\psi(t)\,\mathrm{d}t < \infty$. Define the weight sequence $\{p(m)\}_{m=1}^{n-1}$ as

$$p(n-m) = \int_{(2m-1)/2n}^{(2m+1)/2n} \psi(t)\,\mathrm{d}t.$$

Then, under the regularity conditions of Theorem 4.

$$\sigma^{-2}n^{-1}T_{6n} \xrightarrow{d} \int_0^1 \psi(t)\{B_p^{(f_0)}(t)\}^2\,\mathrm{d}t + \cdots + \int_0^1 \psi(t)\{B_p^{(f_{p-1})}(t)^2\,\mathrm{d}t. \tag{29}$$

As a special case.

$$\sigma^{-2}n^{-1}T_{7n}^{(i)} \xrightarrow{d} \int_0^1 \psi(t)\{B_p^{(f_i)}(t)\}^2\,\mathrm{d}t. \tag{30}$$

Computing quantiles for the stochastic integrals involved in (29) and (30) can pose analytic difficulties. Anderson and Darling (1952) developed a methodology for computing quantiles for the Cramér von-Mises type stochastic integrals in (29) and (30) when the underlying stochastic process is a Brownian Bridge. The methodology involves identifying the sequence of eigenvalues and the associated orthonormal functions $\{\lambda_{pn}, \phi_{pn}^{(t)}\}_{n=1}^{\infty}$ satisfying the Fredholm equation

$$\int_0^1 \{\psi(t)\psi(s)\}^{1/2} K_p^{(f)}(s,t)\phi_{pn}(s)\,\mathrm{d}s = \lambda_{pn}\phi_{pn}(t),\ n = 1, 2, \ldots \tag{31}$$

Then, the characteristic function of the stochastic integral $\int_0^1 \psi(t)\{B_p^{(f)}(t)\}^2\,\mathrm{d}t$ is given by:

$$\Phi_p^{(f)}(s) = \prod_{n=1}^{\infty}\{1 - 2is\lambda_{pn}\}^{-1/2}. \tag{32}$$

Numerical inversion of this characteristic function will then provide the necessary quantiles.

Assuming a Uniform prior ($\psi(t) \equiv 1$), MacNeill (1978a) computed the quantiles for $\int_0^1 \{B_p^{(f_0)}(t)\}^2\,\mathrm{d}t$ in the case of a pth order polynomial regression by analytically solving the Fredholm equation (31). Jandhyala and MacNeill (1989) developed a general method of analytically solving Fredholm equations and then computed quantiles for $\int_0^1 \{B_p^{(f_0)}(t)\}^2\,\mathrm{d}t$ in the case of a pth order harmonic regression. These analytical solutions are important contributions to the development of change-point methods for regression models and the two analytical solutions are briefly stated below.

In the case of a pth order polynomial regression, the eigenvalues satisfying the Fredholm equation (31) are found to be

$$\begin{aligned} \lambda_{p,2n-1} &= \frac{1}{4Z_{p-1,n}^2} \\ \lambda_{p,2n} &= \frac{1}{4Z_{p,n}^2} \end{aligned} \qquad n = 1, 2, \ldots,$$

where $Z_{p,n}$ is the nth positive zero of the pth order Spherical Bessel function of the first kind.

Now consider the case of fitting a harmonic regression model of degree p to a set of data such that

$$Y_{nj} = \beta_0 + \sum_{i=0}^{p} \{\beta_i \cos 2\pi i(j/n) + \beta_{p+i} 2\pi i(j/n)\} + \varepsilon_j, \; j = 1, \ldots, n. \tag{33}$$

Under the model (33), assuming $\psi(t) \equiv 1$, Jandhyala and MacNeill (1989) found $\{\lambda_{pn}\}_{n=1}^{\infty}$ that satisfy the Fredholm equation (31) to be:

$$\lambda_{pn} = 1/4\pi^2 n^2, \; n = p+1, p+2, \ldots$$

and those satisfying the equation

$$\tan\left(\frac{1}{2\sqrt{\lambda_{pn}}}\right) = \frac{1}{4\sqrt{\lambda_{pn}}} \left\{ \left(\sum_{j=1}^{p} \frac{1}{1 - 4\pi^2 j^2 \lambda_{pn}} \right)^{-1} \right\}, \quad n = 1, 2, \ldots.$$

Once the eigenvalues are found, quantiles for the stochastic integral are obtained by numerically inverting the characteristic function applying Gaussian quadrature. For the case of testing for a change in the intercept alone ($i = 0$), MacNeill (1978a) computed the quantiles for selected values of p for a pth order polynomial regression and Jandhyala and MacNeill (1989) computed the quantiles for selected values of p for a pth order harmonic regression.

Econometricians formulate the problem of dynamic stability of regression parameters in one of the two following ways:

(i) Random Coefficient Regression (RCR) models; and
(ii) Sequential Variation Regression (SVR) models.

Random coefficient regression models are obtained by treating the vector of regression parameters as random. The SVR models are more general formulations of the RCR models in the sense that RCR models are relevant only when data follow a strictly stationary time series, whereas the time series associated with SVR models may also be non-stationary.

Nabeya and Tanaka (1988) addressed the problem of testing for the constancy of regression coefficients in the context of random walk alternatives. The random walk model considered by Nabeya and Tanaka (1988) is given by

$$y_t = x_t\beta_t + \mathbf{z}_t'\mathbf{r} + \varepsilon_t,$$
$$\beta_t = \beta_{t-1} + u_t, \;\; t = 1, 2, \ldots, \tag{34}$$

where $\{y_t\}$ is a sequence of scalar observations, $\{x_t\}$ and $\{\mathbf{z}_t\}$ are scalar and $p \times 1$ non-stochastic, fixed sequences respectively, $\{\varepsilon_t\}$ and $\{u_t\}$ are independent of each other and are i.i.d. with $E(\varepsilon_t) = 0$, $E(\varepsilon_t^2) = \sigma_\varepsilon^2 > 0$, $E(u_t) = 0$ and $E(u_t^2) = \sigma_u^2 \geqslant 0$, β_t starts with β_0 which is assumed to be either a known or unknown constant, and $\mathbf{r}$ is a $p \times 1$ vector of unknown parameters. They have then derived a locally best invariant (LBI) test statistic for testing the hypotheses:

$$H_0 : \delta = \frac{\sigma_u^2}{\sigma_\varepsilon^2} = 0 \text{ against } H_a : \delta > 0, \tag{35}$$

under the normality assumption on $\{y_t\}$.

The statistic derived by Nabeya and Tanaka (1988) was a quadratic form in regression residuals. Jandhyala and MacNeill (1992) showed that the two-sided Bayes-type static derived under change-point alternative and the LBI statistic derived under random walk alternative were equivalent. Thus, they have established a dual relationship between change-point and random walk formulations.

4. CHANGE-POINTS AT UNKNOWN TIMES UNDER SERIAL CORRELATIONS

The test statistics and their distribution theory derived thus far were under model formulations that assume independence among observations. Such an assumption for time-series data might be violated quite frequently. When independence is violated, the statistics derived thus far cannot be applied until correlation structures have been accounted for in the derivations of the change-detection statistics and their distribution theory. Tang and MacNeill (1993) tackled the problem of testing for change-points when data are serially correlated. We first consider their basic model. Let $\{Y_n(j)\}_{j=1}^n$, $n \geqslant 1$ be a triangular array of dependent variables satisfying the regression model

$$Y_n(j) = \sum_{i=0}^{p} \beta_i f_i(j/n) + X(j), \tag{36}$$

where $X(j)$, $j = 0, \pm 1, \ldots$ is a zero mean, discrete time, stationary time series with covariance function given by

$$R(\nu) = E\{X(n)X(n+\nu)\}, |\nu| < \infty.$$

If the covariance function is absolutely summable, i.e..

$$\sum_{\nu=-\infty}^{\infty} |R(\nu)| < \infty, \tag{37}$$

then, the spectral density function.

$$f(\lambda) = \frac{1}{2\pi} \sum_{|\nu| < \infty} \mathrm{e}^{-i\lambda\nu} R(\nu), \;\; \lambda \in [-\pi, \pi] \tag{38}$$

exists. The error process $\{X(j)\}_{j=1}^{n}, n \geqslant 1$ is invertible if the spectral density is positive, i.e..

$$f(\lambda) \geqslant a > 0, \;\; \lambda \in (-\pi, \pi). \tag{39}$$

The matrix formulation of model (36) is given by

$$Y_n = A_n \beta_p + X_n,$$

where $X_n' = \{X(1), X(2), \ldots, X(n)\}$ is a portion of a realization of the stationary time series and where the jth component of the design matrix is $f_i(j/n)$. Then, Tang and MacNeill (1993) first established limit processes for sequences of stochastic processes defined by a stationary error process. To state the result, we first need some preliminary notation.

Let

$$S_{X_j} = \sum_{i=1}^{j} X(i)$$

and define another sequence of stochastic processes $\{\theta_{X_n}(t), t \in [0, 1]\}, n \geqslant 1$ possessing continuous sample paths by

$$n^{1/2} \theta_{X_n}(t) = S_{X_{[nt]}} + (nt - [nt]) X([nt] + 1). \tag{40}$$

We also need the following Brillinger condition on the covariants of the process $\{X(n)\}$. Let

$$C_{k+1}(\nu_1, \ldots, \nu_k) = \mathrm{Cum}\{X(n + \nu_1), X(n + \nu_2), \ldots, X(n + \nu_k), X(n)\}.$$

Then, the Brillinger condition is given by:

$$|C_{k+1}(\nu_1, \nu_2, \ldots, \nu_k)| < \frac{L_k}{\prod_{j=1}^{k} (1 + \nu_j^2)} \tag{41}$$

for some finite L_k, $k = 1, 2, \ldots$.

Then, Tang and MacNeill (1993) proved the following theorem.

Theorem 6. Let assumptions (39) and (41) hold. Then the sequence of stochastic processes $\{\theta_{X_n}(t), t \in [0, 1]\}, n \geqslant 1$ converges weakly to the Gaussian process $\{B_X(t), t \in [0, 1]\}$ given by:

$$B_X(t) = \{2\pi f(0)\}^{1/2} B(t), \tag{42}$$

where $B(t)$ denotes the standard Brownian motion.

One may now derive the residual process for regression residuals with stationary error structure. For the model (36), define partial sums of regression residuals $\{S_{fnj}\}_{j=1}^{n}, n \geqslant 1$ where

$$S_{fnj} = \sum_{i=1}^{j} \{Y_n(i) - \hat{Y}_n(i)\}.$$

Define sequences of stochastic processes $\{\theta_{fX_n}(t), t \in [0, 1]\}$, possessing continuous sample paths by

$$n^{1/2}\theta_{fX_n}(t) = S_{fn[nt]} + (nt - [nt])\{Y([nt] + 1) - \hat{Y}([nt] + 1)\}. \tag{43}$$

Define bi-linear function $g(s, t) = g'(s)G^{-1}g(t)$ where

$$G = \lim n^{-1}A_n'A_n.$$

Then, Tang and MacNeill (1993) derived the limit processes for $\{\theta_{fX_n}(t), t \in [0, 1]\}, n \geqslant 1$ by the following theorem.

Theorem 7. Assume conditions (39) and (41) hold. Further, assume regressor functions $f_i(t), i = 0, \ldots, p$ are continuously differentiable and linearly independent. Then, the sequence of Stochastic processes $\{\theta_{fX_n}(t), t \in [0, 1]\}, n \geqslant 1$ converges weakly to the Gaussian processes $\{B_{fX}(t), t \in [0, 1]\}$ given by

$$B_{fX}(t) = B_X(t) - \int_0^t \left\{ \int_0^1 g(x, y)\, \mathrm{d}B_X(y) \right\} \mathrm{d}x. \tag{44}$$

Furthermore, the covariance kernel $k_{fX}(s, t)$ is given by

$$k_{fX}(s, t) = 2\pi f(0) \left\{ \min(s, t) - \int_0^s \int_0^t g(x, y)\, \mathrm{d}x\, \mathrm{d}y \right\}. \tag{45}$$

Tang and MacNeill (1993) then suggested how Theorem 7 could be used to adjust large sample distributional results for statistics which are defined in terms of partial sums of residuals so as to account for serially correlated errors. Let $F(\cdot)$ be a continuous functional defined on $C[a, b]$, the space of continuous functions on the interval $[a, b]$. Furthermore, assume $F(\cdot)$ to be homogeneous of degree d; that is, if $f \in C[a, b]$ and k is a constant, then

$$F(kf) = k^d F(f). \tag{46}$$

Also, let $F_n(\cdot), n = 1, 2, \ldots$ be a sequence of continuous functionals defined on $C[a, b]$ such that $F_n(\cdot) \to F(\cdot)$. Then, for functions f_n, $n = 1, 2, \ldots$ and f elements of $C[a, b]$ such that $f_n \to f$ uniformly on $[a, b]$, one obtains

$$k^d F_n(f_n) \to F(kf) = k^d F(f). \tag{47}$$

The following theorem then follows.

Theorem 8. If $F_n(\cdot)$, $n = 1, 2, \ldots$ and $F(\cdot)$ are continuous functions on $C[a, b]$ satisfying (47), then for $\alpha \in (0, 1)$.

$$P\left\{\frac{F_n\{\theta_{fX_n}(\cdot)\}}{\{\hat{R}_n(0)\}^{d/2}} > z_\alpha \left\{\frac{2\pi f(0)}{\displaystyle\int_{-\pi}^{\pi} f(\lambda)\,\mathrm{d}\lambda}\right\}^{d/2}\right\} \to P[F\{B_f(\cdot)\} > z_\alpha] = \alpha. \tag{48}$$

The implication of Theorem 8 is that if distribution theory is available for the case of white noise error structure, then (48) gives simple precise large sample adjustments to account for serial correlation in the noise process. Tang and MacNeill (1993) discussed such adjustments for Bayes-type change-detection statistics with serial correlations in the error structure. For example, the statistic to test for change in the ith regression parameter β_i is given earlier to be:

$$\sigma^{-2} n^{-1} T_{8n}^{(i)} = \sum_{m=1}^{n-1} p(m) \left\{\frac{1}{\alpha\sqrt{n}} \sum_{k=m+1}^{n} f_i(k/n)(Y_n(i) - \hat{Y}_n(i))\right\}^2. \tag{49}$$

The above statistic has been derived under the white noise error structure. Now suppose the error process is not white noise and that $R(0)$ is estimated consistently by $\hat{R}(0)$ which might be used in place of σ^2. Then it follows from Theorem 8 that

$$\sigma^{-2} n^{-1} T_{8n}^{(i)} \xrightarrow{d} \frac{2\pi f(0)}{\displaystyle\int_{-\pi}^{\pi} f(\lambda)\,\mathrm{d}\lambda} \int_0^1 \psi(t)\{B^{(f_i)}(t)\}^2\,\mathrm{d}t \tag{50}$$

where $\psi(t)$ is a non-negative weight function such that $\int_0^1 t(1-t)\psi(t) < \infty$.

Thus, the large sample effects of serial correlations on change-detection statistics can be adjusted for precisely by multiplying the quantiles of distributions for the white noise by

$$\left\{\frac{2\pi f(0)}{\displaystyle\int_{-\pi}^{\pi} f(\lambda)\,\mathrm{d}\lambda}\right\}.$$

Tang and MacNeill (1993) further show that similar adjustments are applicable for maximum likelihood based statistics also.

5. CHANGE-POINT MODELS WITH CONTINUITY CONSTRAINTS

The regression change-point models considered thus far have been unconstrained in that no particular constraints about continuity or smoothness have been imposed on the nature of the regression regimes at the change-point. However, imposing such constraints on regression regimes may be called for in modeling certain data sets. Segmented polynomial models are obtained by imposing the constraint that successive polynomials from one regime to another be continuous at the change-points or joint points. Piecewise simple linear models are particular cases of the more general segmented polynomial models. For example, Gallant and Fuller (1973) discussed fitting polynomial regression regimes constrained to continuity to data on boys' height/weight ratio. Jandhyala and MacNeill (1997) derived Bayes-type statistics for testing one-sided changes in regression parameters under continuity constraints. They have also derived asymptotic distribution theory for the derived statistics. The asymptotic theory involved deriving limit processes defined by iterated partial sums of regression residuals. We shall first state their results on iterated partial sum residual processes and their properties and then discuss Bayes-type statistics for models with continuity constraints.

Consider the standard regression model with regression functions $f_j(t)$, $t \in [0,1], i = 0, 1, \ldots, p$, by

$$Y_{ni} = \sum_{j=0}^{p} \beta_j f_j(i/n) + \varepsilon_i, \quad i = 1, \ldots, n. \tag{51}$$

Let $\mathbf{r}' = (r_1, \ldots, r_n)$ be the vector of least squares regression residuals. For notational convenience, let $\{S_{pk}^{(-1)}\}^n$ be the sequence of residuals such that $S_{pk}^{(-1)} = r_k, k = 1, \ldots, n$. One then defines the ℓth order iterated partial sum sequence $\{S_{pk}\}_{k=0}^n$ by $S_{p0}^{(\ell)}) \equiv 0$ and

$$S_{pk}^{(\ell)} = \sum_{i=1}^{k} S_{pi}^{(\ell-1)}, \quad k = 1, \ldots, n, \tag{52}$$

where ℓ may take one of the values $\{0, 1, \ldots\}$. Then, for any fixed ℓ, define sequences of stochastic processes $\{\theta_{pn}^{(\ell)}(t), t \in [0, 1]\}, n \geqslant 1$, having continuous sample paths by

$$\theta_{pn}^{(\ell)}(t) = \{\sigma n^{\ell+1/2}\}^{-1}\{S_{p[nt]}^{(\ell)} + (nt - [nt])S_{p[nt]+1}^{(\ell-1)}\}, \quad \ell = 0, 1, \ldots \tag{53}$$

Jandhyala and MacNeill (1997) derived the limit processes for $\{\theta_{pn}^{(\ell)}(t), t \in [0, 1]\}, n \geqslant 1$, by the following theorem.

Theorem 9. Let the regressor functions $f_j(t)$, $t \in [0, 1]$, $j = 0, \ldots, p$ be continuously differentiable on [0, 1]. Then, for any fixed ℓ, $\ell = 0, 1, \ldots$, the sequence of stochastic processes $\{\theta_{pn}^{(\ell)}(t), t \in [0, 1]\}, n \geqslant 1$ converges weakly to the Gaussian process $\{B_p^{(\ell)}(t), t \in [0, 1]\}$ defined by

$$B_p^{(\ell)}(t) = \frac{1}{\ell'}\left[\int_0^t (t-x)^\ell \, \mathrm{d}B(x) - \int_0^t \left\{\int_0^1 g(x, y)(t-y)^\ell \, \mathrm{d}B(y)\right\}\mathrm{d}x\right], \tag{54}$$

where $g(x, y)$ is the bi-linear function defined earlier. Furthermore, the covariance kernel $K_p^{(\ell)}(s, t), s, t \in [0, 1]$ is given by:

$$K_p^{(\ell)}(s, t) = \frac{1}{(\ell!)^2}\left\{\int_0^{\min(s,t)} (t-x)^l (s-x)^\ell \, \mathrm{d}x - \int_0^s \int_0^t (t-x)^\ell (s-y)^\ell g(x, y) \, \mathrm{d}y \, \mathrm{d}x\right\}. \tag{55}$$

The following theorem shows bridge type properties for $\{B_p^{(\ell)}(t), t \in [0, 1]\}$ for all $p \geqslant \ell, \ell = 1, 2, \ldots$ for the case of polynomial regression models.

Theorem 10. Consider a pth order polynomial model such that $f_j(i/n) = (i/n)^j, j = 0, \ldots, p$. Then,

$$B_p^{(\ell)}(1) = \int_0^1 B_p^{(\ell-1)}(x) \, \mathrm{d}x \overset{a \cdot e}{=} 0, \quad \text{for } p \geqslant \ell, \ell = 1, 2, \ldots. \tag{56}$$

It turns out that the above bridge-type property holds for finite samples also. Let $S_p^{(\ell)}$ denote the sum of the $(\ell-1)$th order iterated partial sum sequence such that $S_p^{(\ell)} = S_{p_n}^{(\ell)} = \Sigma_{i=1}^n S_{p_i}^{(\ell-1)}$, $\ell = 0, 1, 2, \ldots$. Then, the following theorem characterizes the properties of $S_p^{(\ell)}$, $\ell = 0, 1, \ldots$.

Theorem 11. For a pth order polynomial regression model, let $S_p^{(\ell)}$, $\ell = 0, 1, \ldots$ denote the sums of iterated partial sum sequences $\{S_{pk}^{(\ell)}\}$. Then, for any fixed p.

$$S_p^{(\ell)} \equiv 0, \quad \ell = 0, 1, \ldots, p. \tag{57}$$

In order to discuss applications of the above results, consider a two-regime change-point polynomial model constrained to continuity at the unknown change-point by:

$$\mathbf{Y} = X\boldsymbol{\beta} + \sum_{j=1}^{p} \delta_j \mathbf{g}_{mj} + \boldsymbol{\epsilon}, \tag{58}$$

where m is the unknown change-point, $\delta_1, \ldots, \delta_p$ represent the amounts of change in $\beta_1, \ldots, \beta_p$ and

$$\mathbf{g}'_{mj} = \left(0, \ldots, 0, \left\{\frac{m+1-m)}{n}\right\}^j, \ldots, \left\{\frac{(n-m)}{n}\right\}^j\right).$$

The hypotheses considered for the one-sided test are:

$$H_0 : \delta_1 = 0, \ldots, \delta_p = 0 \text{ against } H_a : \delta_1 > 0, \ldots, \delta_p > 0.$$

Jandhyala and MacNeill (1997) proved the following theorem which shows that the Bayes-type statistic to test for H_0 against H_A is defined in terms of iterated partial sums of regression residuals.

Theorem 12. Let $T_{9n}(p)$ be the Bayes-type statistic to test for the hypotheses in (59). Then, under a uniform prior on the unknown change-point m, the Bayes-type statistic $T_{9n}(p)$ is defined in terms of the iterated partial sums and is given by

$$(n-1)T_{9n}(p) = S_p^{(p+1)}. \tag{60}$$

The asymptotic distribution of the above statistic may be obtained as a special case of Theorem 9 and is given by the following theorem.

Theorem 13. Let $V_p^{(p+1)}$ be the variance of $B_p^{(p+1)}(1)$ and let $\hat{\sigma}^2$ be a consistent estimator for the unknown error variance σ^2. Then, asymptotically, the null distribution of the statistic $S_p^{(p+1)}$ has the distribution given by:

$$\left\{\hat{\sigma} n^{p+3/2} \sqrt{V_p^{(p+1)}(1)}\right\}^{-1} S_p^{(p+1)} \xrightarrow{d} N(0, 1). \tag{61}$$

6. SPATIAL ANALOG OF THE CHANGE-POINT PROBLEM

Analogues of the change-point problem exist in spatial and in spatio-temporal contexts. A model characterized by a single set of parameters may be suitable for describing an entire set of spatial data. On the other hand it may be that boundaries separate the region under consideration into sub-regions with data for each sub-region characterized by its own parameters. Furthermore, the location of possible boundaries may not be specified. Hence, appropriate statistical methods are required to test for the presence of such boundaries. If the presence of a boundary is detected, then appropriate statistical methods are required to identify its location. Examples of change-detection for spatial data include, among others, areas such as remote sensing, environmental monitoring where boundaries need to be identified for snow cover during winter, and the boundary of the site to be cleared of toxic waste sites.

Carlstein and Krishnamoorthy (1992) considered the problem of estimating the location of an unknown boundary given that the boundary is present. MacNeill and Jandhyala (1993) formulated the problem of testing for boundaries in spatial data. Extensions of these formulations to include heteroscedasticity in spatial data have been considered by MacNeill *et al.* (1994). Here, we shall briefly describe the modeling as carried out in MacNeill and Jandhyala (1993). Consider a rectangular region R on the unit square and assume that there are n^2 gauge points (i, j) in R. Let Y_{ij} denote the observation taken at the gauge located at (i, j) and assume that m_ℓ gauges are located in A_ℓ. For simplicity, we assume that the observations are independent. Let $f_k(\cdot, \cdot), k = 0, \ldots, p$ be a set of bivariate regressor functions, and let $\varepsilon_{ij}, i, j = 1, \ldots, n$ be a set of independent $N(0, \sigma^2)$ variables. Let Y_{ij} satisfy the spatial regression model given by:

$$Y_{ij} = \sum_{k=0}^{p} \beta_k f_k(t_{1i}, t_{2j}) + \varepsilon_{ij}. \tag{62}$$

The model (62) may be written in matrix form as

$$\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon} \quad \text{(stacked)},$$

where $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_p)'$ and X the design matrix. As $n \to \infty, 1/\sigma^2(X'X) \to G$, with the (i, j)th component given by

$$\int_0^1 \int_0^1 f_i(t_1, t_2) f_j(t_1, t_2) \, \mathrm{d}t_2, \mathrm{d}t_2.$$

A Bayes-type statistic for detecting boundaries in the spatial data may be derived to be

$$T_{10n} = \sum_{\ell=1}^{n} \sum_{k=1}^{n} \left(\sum_{\ell=1}^{k} \sum_{j=1}^{\ell} r_{ij} \right)^2 \tag{63}$$

when r_{ij} is the (ij)th least squares residual. Then, it may be shown that under H_0

$$\frac{1}{n^4 \sigma^2} T_{10n} \to \int_0^1 \int_0^1 B_f^2(t_1, t_2) \, \mathrm{d}t_1 \, \mathrm{d}t_2, \tag{64}$$

where $\{B_f(t_1, t_2), (t_1, t_2) \in [0, 1] \times [0, 1]\}$ is the limit process for partial sums on spatial residuals and is given by:

$$B_f(t_1, t_2) = B(t_1, t_2) - \int_0^{t_1} \int_0^{t_2} \int_0^1 \int_0^1 f'(s_1, s_2) G^{-1} f(s_1', s_2') \, \mathrm{d}B(s_1', s_2') \, \mathrm{d}s_1 \, \mathrm{d}s_2. \tag{65}$$

The covariance kernel of $B_f(t_1, t_2)$ is:

$$k_f(t_1, t_2, t_1', t_2') = \min(t_1, t_1')\min(t_2, t_2') - \int_0^{t_1} \int_0^{t_2} \int_0^{t_1'} \int_0^{t_2'} g(s_1, s_2, u, u_2) \, \mathrm{d}u_2 \, \mathrm{d}u_1 \, \mathrm{d}s_2 \, \mathrm{d}s_1. \tag{66}$$

Computing quantiles for the stochastic integral in (64) is quite complicated. The expected value and variance can, however, be easily obtained from the covariance kernel in (66).

When the existence of a boundary is identified, the next problem to be considered is estimation of the boundary location. MacNeill and Jandhyala (1993) provide a method for estimating the boundary location. Let B be a collection of points $(i/n, j/n)$, on the grid and let $\bar{B}$ be the complimentary set. Let r_{ij} be the residual associated with $(i/n, j/n)$. Then, let

$$S_B^2 = \sum_{(i,j) \in B} r_{ij}^2,$$

$$S_{\bar{B}}^2 = \sum_{(i,j) \in \bar{B}} r_{ij}^2.$$

The estimated boundary B^* is obtained as that boundary B, that minimizes

$$S_{B\bar{B}}^2 = S_B^2 + S_{\bar{B}}^2. \tag{67}$$

That is:

$$S_{B^*\bar{B}^*}^2 = \min_B S_{B\bar{B}}^2. \tag{68}$$

For even moderate sized n, the number of possible boundaries becomes unmanageably large. To reduce the number of boundaries to be considered in the above minimization, one makes appropriate assumptions regarding the smoothness and continuity of the boundary.

Substantial extensions of these spatial methods may be found in L. Xie's Ph.D. thesis written under the supervision of Ian MacNeill.

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