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JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Journal of Computational and Applied Mathematics 186 (2006) 64–88

www.elsevier.com/locate/cam

# Permutation principles for the change analysis of stochastic processes under strong invariance

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Received 19 September 2004; received in revised form 30 January 2005

Dedicated to Jef Teugels on the occasion of his 65th anniversary

**Abstract**

Approximations of the critical values for change-point tests are obtained through permutation methods. Both, abrupt and gradual changes are studied in models of possibly dependent observations satisfying a strong invariance principle, as well as gradual changes in an i.i.d. model. The theoretical results show that the original test statistics and their corresponding permutation counterparts follow the same distributional asymptotics. Some simulation studies illustrate that the permutation tests behave better than the original tests if performance is measured by the $\alpha$- and $\beta$-error, respectively.


*Keywords:* Permutation principle; Bootstrap; Change-point; Invariance principle; Abrupt change; Gradual change; Rank statistic; Limiting extreme value distribution

## 1. Introduction

A series of papers has been published on the use of permutation principles for obtaining reasonable approximations to the critical values of change-point tests. This approach was first suggested by Antoch and Hušková [1] and later pursued by other authors (cf. Hušková [7] for a recent survey). But, so far, it

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0377-0427/$ - see front matter 
doi:10.1016/j.cam.2005.03.065

has mostly been dealt with abrupt changes and independent observations. In many practical applications, however, smooth (gradual) changes are more realistic, so are dependent observations.

In this paper, we shall discuss the use of permutation principles in the three models of a gradual change in the mean of i.i.d. observations, an abrupt change in the mean or variance of a stochastic process resp. a gradual change in the mean of a stochastic process under strong invariance.

### 1.1. Gradual change in the mean of independent, identically distributed (i.i.d.) observations

Hušková and Steinebach [7] investigated the following model:

$$X_i = \mu + d\left(\frac{i-m}{n}\right)_+^{\gamma} + e_i, \quad i = 1, \ldots, n, \tag{1}$$

where $x_+ = \max(0, x)$; $\mu$, $d = d_n$, and $m = m_n \leqslant n$ are unknown parameters, and $e_1, \ldots, e_n$ are i.i.d. random variables with

$$Ee_i = 0, \quad 0 < \operatorname{var} e_i = \sigma^2 (< \infty), \quad E|e_i|^{2+\delta} < \infty \quad \text{for some } \delta > 0. \tag{2}$$

The parameter $\gamma$ is supposed to be known.

Note that—in contrast to abrupt changes—the biggest difference in the mean here is not $d$, but $d\left(\frac{n-m}{n}\right)^{\gamma}$, and thus depends on $n$, $m$ and $\gamma$.

One is interested in testing the hypotheses

$$H_0 : m = n \quad \text{vs.} \quad H_1 : m < n, \quad d \neq 0.$$

The following test statistic, which is based on the likelihood ratio approach in case of normal errors $\{e_i\}$, has been used:

$$T_n^{(1)} = \frac{1}{\hat{\sigma}_n} \max_{1 \leqslant k < n} \frac{\left|\sum_{i=1}^{n} (i-k)_+^{\gamma} (X_i - \overline{X}_n)\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2\right)^{1/2}},$$

where $\hat{\sigma}_n$ denotes a suitable estimator of $\sigma$. Asymptotic critical values for the corresponding test can be chosen according to the following null asymptotics (cf. Hušková and Steinebach [8]):

**Theorem 1.** *Let $X_1, X_2, \ldots$ be i.i.d. r.v.'s with $\operatorname{var} X_1 = \sigma^2 > 0$, and $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$. Then, for all $x \in \mathbb{R}$, as $n \to \infty$,*

$$P\left(\alpha_n T_n^{(1)} - \beta_n \leqslant x\right) \to \exp\left(-2e^{-x}\right),$$

*where $\alpha_n = \sqrt{2 \log \log n}$ and $\beta_n = \beta_n(\gamma)$ is as follows*:

(1) *for $\gamma > \frac{1}{2}$*:

$$\beta_n = 2 \log \log n + \log\left(\frac{1}{4\pi}\left(\frac{2\gamma+1}{2\gamma-1}\right)^{1/2}\right);$$

(2) *for* $\gamma = \frac{1}{2}$:

$$\beta_n = 2 \log\log n + \frac{1}{2} \log\log\log\log n - \log(4\pi);$$

(3) *for* $0 < \gamma < \frac{1}{2}$:

$$\beta_n = 2 \log\log n + \frac{1-2\gamma}{2(2\gamma+1)} \log\log\log n + \log\left(\frac{C_\gamma^{1/(2\gamma+1)} H_{2\gamma+1}}{\sqrt{\pi} 2^{2\gamma/(2\gamma+1)}}\right),$$

*with* $H_\gamma$ *as in Remark* 12.2.10 *of Leadbetter et al.* [11] (*e.g.* $H_1 = 1$, $H_2 = 1/\sqrt{\pi}$), *and*

$$C_\gamma = -(2\gamma+1) \int_0^\infty x^\gamma((x+1)^\gamma - x^\gamma - \gamma x^{\gamma-1})\, \mathrm{d}x.$$

*Moreover,* $\hat{\sigma}_n$ *is assumed to be an estimator of* $\sigma$ *satisfying* $\hat{\sigma}_n - \sigma = o_P((\log\log n)^{-1})$ *as* $n \to \infty$.

### 1.2. Abrupt change in the mean or variance of a stochastic process under strong invariance

This model has been considered by Horváth and Steinebach [6]. Suppose one observes a stochastic process $\{Z(t) : 0 \leqslant t < \infty\}$ having the following structure:

$$Z(t) = \begin{cases} at + bY(t), & 0 \leqslant t \leqslant T^*, \\ Z(T^*) + a^*(t - T^*) + b^* Y^*(t - T^*), & T^* < t \leqslant T, \end{cases} \tag{3}$$

where $a, b, a^*, b^*$ are unknown parameters, and $\{Y(t) : 0 \leqslant t < \infty\}$ resp. $\{Y^*(t) : 0 \leqslant t < \infty\}$ are (unobserved) stochastic processes satisfying the following strong invariance principles:

For every $T > 0$, there exist two independent Wiener processes $\{W_T(t) : 0 \leqslant t \leqslant T^*\}$ and $\{W_T^*(t) : 0 \leqslant t \leqslant T - T^*\}$, and some $\delta > 0$, such that, for $T \to \infty$,

$$\sup_{0 \leqslant t \leqslant T^*} |Y(t) - W_T(t)| = \mathrm{O}(T^{1/(2+\delta)}) \quad \text{a.s.} \tag{4}$$

and

$$\sup_{0 \leqslant t \leqslant T - T^*} |Y^*(t) - W_T^*(t)| = \mathrm{O}(T^{1/(2+\delta)}) \quad \text{a.s.} \tag{5}$$

Moreover, we assume $Y(0) = 0$ and $Y^*(0) = 0$. It should be noted that only weak invariance has been assumed in Horváth and Steinebach [6], instead of the strong rates of (4) and (5), which are required for later use here. Moreover, the processes $\{Z(t)\}$, $\{Y(t)\}$, and $\{Y^*(t)\}$ could be replaced by a family of processes $\{Z_T(t)\}$, $\{Y_T(t)\}$, and $\{Y_T^*(t)\}$, $T > 0$, since the asymptotic analysis is merely based on the approximating family of Wiener processes $\{W_T(t)\}$ and $\{W_T^*(t)\}$, respectively.

One is interested in testing the hypothesis of "no change", i.e.

$$H_0 : T^* = T,$$

against the alternative of "a change in the mean at $T^* \in (0, T)$", i.e.

$$H_1^{(1)} : 0 < T^* < T \quad \text{and} \quad a \neq a^*,$$

resp. “a change in the variance at $T^* \in (0, T)$”, i.e.

$$H_1^{(2)} : 0 < T^* < T \quad \text{and} \quad b \neq b^*, \quad \text{but } a = a^*.$$

Basic examples satisfying conditions (3)–(5) are partial sums of i.i.d. random variables and renewal processes based on i.i.d. waiting times, but also sums of dependent observations (for details we refer to Horváth and Steinebach [6]).

It is assumed, that the process $\{Z(t) : t \geqslant 0\}$ has been observed at discrete time points $t_i = t_{i,N} = i\frac{T}{N}$, $1 \leqslant i \leqslant N = N(T)$. Let $\Delta Z_{i,T} = Z(t_i) - Z(t_{i-1})$ and $\widetilde{\Delta Z}_{i,T} = Z(t_i) - Z(t_{i-1}) - \overline{\Delta Z}_T$. The following statistics will be used:

$$M_T = \max_{1 \leqslant k \leqslant N} \left\{ \frac{1}{\sqrt{T}} \frac{1}{\widehat{b}_T} \left| \sum_{i=1}^{k} (\Delta Z_{i,T} - \overline{\Delta Z}_T) \right| \right\}, \tag{6}$$

where $\overline{\Delta Z}_T = \frac{1}{N} \sum_{i=1}^{N} \Delta Z_{i,T}$, and

$$\widehat{b}_T^2 = \frac{1}{T} \sum_{i=1}^{N} (\Delta Z_{i,T} - \overline{\Delta Z}_T)^2,$$

resp.

$$\widetilde{M}_T = \max_{1 \leqslant k \leqslant N} \left\{ \frac{1}{\sqrt{T}} \frac{1}{\widehat{c}_T} \left| \sum_{i=1}^{k} (\widetilde{\Delta Z}_{i,T}^2 - \overline{\widetilde{\Delta Z}_T^2}) \right| \right\}, \tag{7}$$

where $\overline{\widetilde{\Delta Z}_T^2} = \frac{1}{N} \sum_{i=1}^{N} \widetilde{\Delta Z}_{i,T}^2$, and

$$\widehat{c}_T^2 := \frac{1}{T} \sum_{i=1}^{N} \left( (\Delta Z_{i,T} - \overline{\Delta Z}_T)^2 - \frac{1}{N} \sum_{l=1}^{N} (\Delta Z_{l,T} - \overline{\Delta Z}_T)^2 \right)^2.$$

**Remark 2.** The statistic $\widetilde{M}_T$ uses a slightly different variance estimator $\widehat{c}_T^2$ than the one given in Horváth and Steinebach [6]. It possesses, however, the same asymptotic behavior, since the ratio of the two normalizations converges in probability to 1 under the null hypothesis, and to some positive constant under the alternative (cf. Theorem 4.5.2 in Kirch [9]). This modification is necessary for applying the permutation method below, since, under the alternative, the permutation statistic (corresponding to the statistic used in Horváth and Steinebach [6]) does not converge to $\sup_{0 \leqslant t \leqslant 1} |B(t)|$, but to $c \sup_{0 \leqslant t \leqslant 1} |B(t)|$, $c > 0$, $c \neq 1$ in general, where $c$ is the asymptotic ratio of the two variance estimators. Here $\{B(t) : 0 \leqslant t \leqslant 1\}$ denotes a Brownian bridge.

The following null asymptotics hold under the above conditions (cf. Horváth and Steinebach [6]):

**Theorem 3.** *If $N = N(T) \to \infty$ and $N = \mathrm{o}(T^{1-2/(2+\delta)})$ as $T \to \infty$, then, under $H_0$,*

$$M_T \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t \leqslant 1} |B(t)|,$$

*where $\{B(t) : 0 \leqslant t \leqslant 1\}$ is a Brownian bridge.*

**Theorem 4.** *If $N = N(T) \to \infty$ and $N = \mathrm{o}(T^{1/2-1/(2+\delta)})$ as $T \to \infty$, then, under $H_0$,*

$$\widetilde{M}_T \xrightarrow{\mathscr{D}} \sup_{0 \leqslant t \leqslant 1} |B(t)|,$$

*where $\{B(t) : 0 \leqslant t \leqslant 1\}$ is a Brownian bridge.*

### 1.3. Gradual change in the mean of a stochastic process under strong invariance

This model has been considered by Steinebach [14]. Suppose one observes a stochastic process $\{S(t) : 0 \leqslant t < \infty\}$ having the following structure:

$$S(t) := \begin{cases} at + bY(t), & 0 \leqslant t \leqslant T^*, \\ S(T^*) + a^*(t - T^*) + b^* Y^*(t - T^*), & T^* < t \leqslant T, \end{cases} \tag{8}$$

where $a, b, b^*$ and $\{Y(t)\}, \{Y^*(t)\}$ are as in model 1.2 above, $a^*(t - T^*) = a(t - T^*) + \tilde{d}(t - T^*)^{1+\gamma}$, $\tilde{d} = \tilde{d}_T$ is unknown, $\gamma > 0$ is known. Again, the biggest difference in the mean here depends on $T$, $T^*$ and $\gamma$, similarly as in the first model (1.1). Note that, instead of (4), Steinebach [14] assumed the following weak invariance principle for the process $\{Y(t) : 0 \leqslant t < \infty\}$, namely that, for every $T > 0$, there is a Wiener process $\{W_T(t) : 0 \leqslant t \leqslant T^*\}$ such that

$$\sup_{1 \leqslant t \leqslant T^*} |Y(T^*) - Y(T^* - t) - W_T(t)| / t^{1/(2+\delta)} = \mathrm{O}_{\mathrm{P}}(1) \quad (T \to \infty). \tag{9}$$

The reason is that small approximation rates were required near the change-point $T^*$, but only in a weak sense, whereas we need strong approximations for our permutation principles below. Here, too, the processes $\{Z(t)\}, \{Y(t)\}$, and $\{Y^*(t)\}$ could be replaced by a family of processes $\{Z_T(t)\}, \{Y_T(t)\}$, and $\{Y_T^*(t)\},\ T > 0$.

One is now interested in testing the null hypothesis of "no change in the drift", i.e.

$$H_0 : T^* = T$$

against the alternative of "a smooth (gradual) change in the drift", i.e.

$$H_1 : 0 < T^* < T, \quad \tilde{d} \neq 0.$$

Basic examples fulfilling the conditions above are again partial sums of i.i.d. random variables and renewal processes based on i.i.d. waiting times (cf. Steinebach [14] for more details). As in model 1.2, we assume that we have observed $\{S(t) : t \geqslant 0\}$ at discrete time points $t_i = iT/N$, and set $\Delta S_{i,T} = S(t_i) - S(t_{i-1})$. The following test statistic is used:

$$T_N^{(2)} = \sqrt{\frac{N}{T\widehat{b}_T^2}} \max_{1 \leqslant k < N} \frac{\left|\sum_{i=1}^N (i-k)_+^\gamma (\Delta S_{i,T} - \overline{\Delta S}_N)\right|}{\left(\sum_{i=1}^{N-k} i^{2\gamma} - \frac{1}{N}\left(\sum_{i=1}^{N-k} i^\gamma\right)^2\right)^{1/2}}, \tag{10}$$

where $\overline{\Delta S}_T = \frac{1}{N}\sum_{i=1}^N \Delta S_{i,T}$, and $\widehat{b}_T^2 = \frac{1}{T}\sum_{i=1}^N (\Delta S_{i,T} - \overline{\Delta S}_T)^2$.

Steinebach [14] assumed a slightly different weight, which is asymptotically equivalent to the one used above. However, it turns out, that the above weight gives much better results for the permutation statistic, which is due to the fact, that it is the maximum-likelihood statistic under Gaussian errors. The results obtained in Steinebach [14] remain valid.

**Remark 5.** The magnitude of $\tilde{d}$ is completely different from that of $d$ in the first model. However, $d := \tilde{d}(1+\gamma)T^{1+\gamma}/N$ is comparable to it, which can easily be seen via the mean value theorem.

Similar to Theorem 1, the following null asymptotic applies (cf. Steinebach [14]):

**Theorem 6.** *If* (9) *holds,* $N = N(T) \to \infty$ *and* $N = \mathrm{O}(T)$ *as* $T \to \infty$, *then, under* $H_0$, *for all* $x \in \mathbb{R}$:

$$P(\alpha_N T_N^{(2)} - \beta_N \leqslant x) \to \exp(-2e^{-x}),$$

*where* $\alpha_N = \sqrt{2 \log \log N}$ *and* $\beta_N = \beta_N(\gamma)$ *is as in Theorem* 1 (*with N replacing* $n$).

## 2. Rank and permutation statistics in case of a gradual change under i.i.d. errors

In order to derive distributional asymptotics for the permutation statistics, we shall make use of the following theorem for the corresponding rank statistics. In the case $\gamma = 1$, it was proven by Slabý [13].

**Theorem 7.** *Let* $\mathbf{R} = (R_1, \ldots, R_n)$ *be a random permutation of* $(1, \ldots, n)$, *and* $a_n(1), \ldots, a_n(n)$ *be scores satisfying*

$$\frac{1}{n} \sum_{i=1}^{n} (a_n(i) - \overline{a}_n)^2 \geqslant D_1, \tag{11}$$

*and*

$$\frac{1}{n} \sum_{i=1}^{n} |a_n(i) - \overline{a}_n|^{2+\delta} \leqslant D_2, \tag{12}$$

*where* $D_1$, $D_2$ *and* $\delta$ *are some positive constants, and* $\overline{a}_n = \frac{1}{n} \sum_{i=1}^{n} a_n(i)$. *Then, for fixed* $\gamma > 0$ *and all* $x \in \mathbb{R}$, *as* $n \to \infty$

$$P(\alpha_n T_n(\mathbf{R}) - \beta_n \leqslant x) \to \exp(-2e^{-x}),$$

*where*

$$T_n(\mathbf{R}) = \frac{1}{\sigma_n(\mathbf{a})} \max_{1 \leqslant k < n} \frac{\left|\sum_{i=1}^{n} (i-k)_+^{\gamma} (a_n(R_i) - \overline{a}_n)\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2\right)^{1/2}}.$$

*Here* $\sigma_n^2(\mathbf{a}) = \frac{1}{n} \sum_{i=1}^{n} (a_n(i) - \overline{a}_n)^2$, *the variance of* $a_n(R_1)$, $\alpha_n = \sqrt{2 \log \log n}$ *and* $\beta_n = \beta_n(\gamma)$ *is as in Theorem* 1.

In the proof of this theorem we apply the following weak embedding:

**Theorem 8.** *Let* $a_n(1), \ldots, a_n(n)$ *be scores satisfying* (11) *and* (12). *Then, on a rich enough probability space, there is a sequence of stochastic processes* $\{\widetilde{\Pi}_n(k) : 1 \leqslant k \leqslant n\}$ $(n = 1, 2, \ldots)$ *with*

$$\{\widetilde{\Pi}_n(k) : 1 \leqslant k \leqslant n\} \stackrel{\mathscr{D}}{=} \left\{ \frac{1}{\sqrt{\sigma_n^2(\mathbf{a})}} \sum_{i=1}^{k} (a_n(\pi_n(i)) - \overline{a}_n) : 1 \leqslant k \leqslant n \right\},$$

*where* $(\pi_n(1), \ldots, \pi_n(n))$ *is a random permutation of* $(1, 2, \ldots, n)$, $\sigma_n^2(\mathbf{a}) := \frac{1}{n} \sum_{i=1}^{n} (a_n(i) - \overline{a}_n)^2$, $\overline{a}_n = \frac{1}{n} \sum_{i=1}^{n} a_n(i)$, *and there is a fixed Brownian bridge* $\{B(t) : 0 \leqslant t \leqslant 1\}$ *such that, for* $0 \leqslant \nu < \min\left(\frac{\delta}{2(2+\delta)}, \frac{1}{4}\right)$,

$$\max_{1 \leqslant k < n} \left(\frac{k(n-k)}{n}\right)^{\nu} \frac{n}{\sqrt{k(n-k)}} \left| \frac{1}{\sqrt{n}} \widetilde{\Pi}_n(k) - B(k/n) \right| = O_P(1).$$

The proof goes along the lines of Theorem 1 of Einmahl and Mason [4], by replacing the Hájek-Rényi inequality (cf. [4, p. 110]) resp. Lemma 13 there with the following lemmas:

**Lemma 9.** *Let* $M(0) = 0$, $M(1), \ldots, M(m)$, $m \geqslant 1$, *be a mean* 0, *square-integrable martingale, and* $a(1) \geqslant \cdots \geqslant a(m) \geqslant 0$ *be constants. Then, for* $1 < s \leqslant 2$ *and* $\lambda > 0$,

$$P\left(\max_{1 \leqslant i \leqslant m} a_i |M(i)| > \lambda\right) \leqslant 2^{s-1} \frac{1}{\lambda^s} \sum_{i=1}^{m} a_i^s E|M(i) - M(i-1)|^s.$$

**Proof.** Confer Lemma 9 in Häusler and Mason [5], or Lemma 5.1.2 in Kirch [9] together with Einmahl [3]. □

**Lemma 10.** *Let* $a_n(1), \ldots, a_n(n)$ *be scores with* $\sum_{i=1}^{n} a_n(i) = 0$, *and* $(\pi_n(1), \ldots, \pi_n(n))$ *be a random permutation as in Theorem* 8. *Then, for* $1 \leqslant i \leqslant n$ *and* $1 \leqslant s \leqslant 2$,

$$E\left|\sum_{j=1}^{i} a_n(\pi_n(j))\right|^s \leqslant 2 \min(i, n-i) \frac{1}{n} \sum_{j=1}^{n} |a_n(j)|^s.$$

**Proof.** Confer Lemma 5.1.3. in Kirch [9] and Mason [12]. □

Now we have the tools to prove Theorem 7:

**Proof** (*Theorem 7*). First note that

$$\begin{aligned} n \sum_{i=1}^{n-k} i^{2\gamma} - \left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2 &= (n-k) \sum_{i=1}^{n-k} \left(i^{\gamma} - \frac{1}{n-k} \sum_{j=1}^{n-k} j^{\gamma}\right)^2 + k \sum_{i=1}^{n-k} i^{2\gamma} \\ &\geqslant k \int_0^{n-k} x^{2\gamma} \, \mathrm{d}x = k \frac{1}{2\gamma+1} (n-k)^{2\gamma+1}. \end{aligned} \tag{13}$$

Now, from Theorem 8 with $v = 0$, uniformly in $k \in [1, n/2]$:

$$\frac{1}{\sigma_n(\mathbf{a})} \sum_{i=1}^{k} (a_n(R_{n-i+1}) - \overline{a}_n) = \sqrt{n} B\left(\frac{k}{n}\right) + O_P\left(\sqrt{\frac{k(n-k)}{n}}\right)$$
$$= \sqrt{n} B\left(\frac{k}{n}\right) + O_P(\sqrt{k}).$$

Since $\left\{\sqrt{n} B\left(\frac{k}{n}\right) : k = 0, \ldots, n\right\} \stackrel{\mathcal{D}}{=} \left\{W(k) - \frac{k}{n} W(n) : k = 0, \ldots, n\right\}$, where $\{W(t) : t \geqslant 0\}$ is a standard Wiener process, we conclude from the law of the iterated logarithm

$$\frac{1}{\sigma_n(\mathbf{a})} \max_{n-\log n<k<n} \frac{\left|\sum_{i=1}^{n} (i-k)_+^{\gamma} (a_n(R_i) - \overline{a}_n)\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2\right)^{1/2}}$$
$$= \frac{1}{\sigma_n(\mathbf{a})} \max_{1<k<\log n} \frac{\left|\sum_{l=1}^{k} (l^{\gamma} - (l-1)^{\gamma}) \sum_{i=1}^{k-l+1} (a_n(R_{n-i+1}) - \overline{a}_n)\right|}{\left(\sum_{i=1}^{k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{k} i^{\gamma}\right)^2\right)^{1/2}}$$
$$= O_P\left(\max_{1<k<\log n} \frac{\left|\sum_{l=1}^{k} (l^{\gamma} - (l-1)^{\gamma}) \left(W(k-l+1) - \frac{k-l+1}{n} W(n)\right)\right|}{\left(\sum_{i=1}^{k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{k} i^{\gamma}\right)^2\right)^{1/2}}\right)$$
$$+ O_P\left(\max_{1<k<\log n} \frac{\left|\sum_{l=1}^{k} (l^{\gamma} - (l-1)^{\gamma}) \sqrt{k-l+1}\right|}{\left(\sum_{i=1}^{k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{k} i^{\gamma}\right)^2\right)^{1/2}}\right)$$
$$= o_P\left(\sqrt{\log\log n}\right).$$

Hence it suffices to investigate the maximum over $k \in [1, n - \log n]$. Let

$$\widehat{T}_n := \max_{1 \leqslant k \leqslant n-\log n} \frac{\left|\sum_{i=1}^{n} (i-k)_+^{\gamma} (X_i - \frac{1}{n} \sum_{l=1}^{n} X_l)\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2\right)^{1/2}}$$

resp.

$$\widetilde{T}_n := \max_{1 \leqslant k \leqslant n-\log n} \frac{\left|\sum_{i=1}^{n} (i-k)_+^{\gamma} (\widetilde{\Pi}_n(i) - \widetilde{\Pi}_n(i-1))\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^{\gamma}\right)^2\right)^{1/2}}$$

be the corresponding test statistics based on i.i.d. $N(0, 1)$ random variables $X_i$ resp. on the distributionally equivalent versions of $a_n(R_i)$. We choose $X_i$ such that $B\left(\frac{k}{n}\right) = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{k} X_i - \frac{k}{n} \sum_{i=1}^{n} X_i\right)$, with $\{B(t)\}$ denoting the Brownian bridge of Theorem 8.

By the same application of the law of the iterated logarithm as above,

$$\max_{n-\log\, n\leqslant k\leqslant n} \frac{\left|\sum_{i=1}^{n}(i-k)_+^{\gamma}(X_i-\frac{1}{n}\sum_{i=1}^{n}X_i)\right|}{\left(\sum_{i=1}^{n-k}i^{2\gamma}-\frac{1}{n}\left(\sum_{i=1}^{n-k}i^{\gamma}\right)^2\right)^{1/2}}=\mathrm{o}_P\left(\sqrt{\log\,\log\, n}\right).$$

Since $\alpha_n\widetilde{T}_n-\beta_n=(\alpha_n\widehat{T}_n-\beta_n)+\alpha_n(\widetilde{T}_n-\widehat{T}_n)$, and since Theorem 1 implies that $\alpha_n\widehat{T}_n-\beta_n$ has a limiting Gumbel distribution, it suffices to show that $\alpha_n(\widetilde{T}_n-\widehat{T}_n)=\mathrm{o}_P(1)$. We set $Y_{in}:=\widetilde{\Pi}_n(i)-\widetilde{\Pi}_n(i-1)-(X_i-\overline{X}_n)$, where $\overline{X}_n=\frac{1}{n}\sum_{i=1}^{n}X_i$, and $S_n(l):=\sum_{i=1}^{l}Y_{in}$.Then,

$$\begin{aligned}
&|\widetilde{T}_n-\widehat{T}_n|\\
&\quad\leqslant \max_{1\leqslant k\leqslant n-\log\, n}\sqrt{\frac{n}{n\sum_{i=1}^{n-k}i^{2\gamma}-\left(\sum_{i=1}^{n-k}i^{\gamma}\right)^2}}\left|\sum_{i=1}^{n}(i-k)_+^{\gamma}Y_{in}\right|\\
&\quad\leqslant \max_{1\leqslant k\leqslant n-\log\, n}\sqrt{\frac{n}{n\sum_{i=1}^{n-k}i^{2\gamma}-\left(\sum_{i=1}^{n-k}i^{\gamma}\right)^2}}\sum_{l=1}^{n-k}|S_n(l+k-1)|(l^{\gamma}-(l-1)^{\gamma})\\
&\quad\leqslant \max_{1\leqslant k<n}\left(\frac{k(n-k)}{n}\right)^{\nu}\frac{n}{\sqrt{k(n-k)}}\left|\frac{1}{\sqrt{n}}\widetilde{\Pi}_n(k)-B\left(\frac{k}{n}\right)\right|\\
&\qquad\times\max_{1\leqslant k\leqslant n-\log\, n} n^{\nu}\sum_{l=1}^{n-k}\frac{((l+k-1)(n-l-k+1))^{1/2-\nu}}{\sqrt{n\sum_{i=1}^{n-k}i^{2\gamma}-\left(\sum_{i=1}^{n-k}i^{\gamma}\right)^2}}(l^{\gamma}-(l-1)^{\gamma}),
\end{aligned}$$

where $0<\nu<\min\left(\frac{\delta}{2(2+\delta)},\frac{1}{4}\right)$ as in Theorem 8. This theorem also implies

$$\max_{1\leqslant k<n}\left(\frac{k(n-k)}{n}\right)^{\nu}\frac{n}{\sqrt{k(n-k)}}\left|\frac{1}{\sqrt{n}}\widetilde{\Pi}_n(k)-B\left(\frac{k}{n}\right)\right|=\mathrm{O}_P(1),$$

which means, that it suffices to show

$$\begin{aligned}
&\max_{1\leqslant k\leqslant n-\log\, n} n^{\nu}\sum_{l=1}^{n-k}\frac{((l+k-1)(n-l-k+1))^{1/2-\nu}}{\sqrt{n\sum_{i=1}^{n-k}i^{2\gamma}-\left(\sum_{i=1}^{n-k}i^{\gamma}\right)^2}}(l^{\gamma}-(l-1)^{\gamma})\\
&\quad=\mathrm{o}((\log\,\log\, n)^{-1/2}).
\end{aligned}$$

The latter rate can be obtained through a straightforward calculation, taking (13) into account together with the following estimate:

$$n\sum_{i=1}^{n-k}i^{2\gamma}-\left(\sum_{i=1}^{n-k}i^{\gamma}\right)^2\geqslant c_{\gamma}n(n-k)^{2\gamma+1}\quad\text{for all } n\geqslant n_{\gamma}, \tag{14}$$

where $c_{\gamma}>0$ and $n_{\gamma}$ depends only on $\gamma$. This completes the proof. For details we refer to Kirch [9], Corollary 5.2.3. □

We are now ready to study the following permutation statistic:

$$T_n^{(1)}(\mathbf{R}) = \frac{1}{\hat{\sigma}_n} \max_{1 \leqslant k < n} \frac{\left|\sum_{i=1}^n (i-k)_+^\gamma (X_{R_i} - \overline{X}_n)\right|}{\left(\sum_{i=1}^{n-k} i^{2\gamma} - \frac{1}{n}\left(\sum_{i=1}^{n-k} i^\gamma\right)^2\right)^{1/2}},$$

where $\mathbf{R} = (R_1, \ldots, R_n)$ is a random permutation of $(1, \ldots, n)$. We consider the conditional distribution of $T_n^{(1)}(\mathbf{R})$ given the original observations $X_1, \ldots, X_n$, i.e. the randomness is only generated by the random permutation $\mathbf{R} = (R_1, \ldots, R_n)$.

The following theorem proves that this statistic conditionally on the given observations has a.s. the same asymptotic behavior—both under the null hypothesis and under the alternative—as that of $T_n^{(1)}$ under the null hypothesis (cf. Theorem 1).

**Theorem 11.** *Let $X_1, \ldots, X_n$ be observations satisfying* (1) *and* (2). *Moreover, let $|d| = |d_n| \leqslant D$. Then, for all $x \in \mathbb{R}$, as $n \to \infty$,*

$$P(\alpha_n T_n^{(1)}(\mathbf{R}) - \beta_n \leqslant x \mid X_1, \ldots, X_n) \to \exp(-2e^{-x}) \quad a.s.,$$

*where $\alpha_n,\ \beta_n = \beta_n(\gamma)$ are as in Theorem* 1.

**Proof.** It is sufficient to verify the assumptions of Theorem 7 with $a_n(i) = X_i$, $i = 1, \ldots, n$. First we have

$$\overline{X}_n = \mu + \bar{e}_n + d_n n^{-\gamma-1} \sum_{l=1}^n (l - m_n)_+^\gamma.$$

Hence

$$\frac{1}{n}\sum_{i=1}^n (X_i - \overline{X}_n)^2 \geqslant \frac{1}{n}\sum_{i=1}^n (e_i - \bar{e}_n)^2 + 2d_n n^{-\gamma} \frac{1}{n}\sum_{i=1}^n (i-m_n)_+^\gamma e_i - 2d_n n^{-\gamma-1} \sum_{l=1}^{n-m_n} l^\gamma \frac{1}{n}\sum_{i=1}^n e_i.$$

It is enough to show that the second term converges to 0 a.s., because then, by the strong law of large numbers,

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X}_n)^2 \geqslant \operatorname{var}\, e_1 \quad \text{a.s.}$$

Now, by partial summation,

$$\sum_{i=1}^n (i-m_n)_+^\gamma e_i = S_n(n-m_n)_+^\gamma - \sum_{i=1}^{n-1} S_i((i+1-m_n)_+^\gamma - (i-m_n)_+^\gamma), \tag{15}$$

where $S_i := \sum_{j=1}^{i} e_j$, and, from the law of the iterated logarithm,

$$\begin{aligned}&\frac{1}{n^{\gamma+1}} \sum_{i=1}^{n-1} S_i((i+1-m_n)_+^\gamma - (i-m_n)_+^\gamma)\\&\quad = \mathrm{O}\left(\frac{1}{n^{\gamma+1}} \sum_{i=1}^{n-1} i^{3/4}((i+1-m_n)_+^\gamma - (i-m_n)_+^\gamma)\right) = \mathrm{o}(1) \quad \text{a.s.,}\end{aligned}$$

where the last estimate follows via the mean value theorem. Using (15) together with the strong law of large numbers, we get indeed, as $n \to \infty$,

$$\begin{aligned}&\frac{d_n}{n^{\gamma+1}} \sum_{i=1}^{n} (i-m_n)_+^\gamma e_i\\&\quad = \frac{d_n(n-m_n)_+^\gamma}{n^\gamma} \frac{S_n}{n} - \frac{d_n}{n^{\gamma+1}} \sum_{i=1}^{n-1} S_i((i+1-m_n)_+^\gamma - (i-m_n)_+^\gamma) \to 0 \quad \text{a.s.}\end{aligned}$$

On the other hand, for suitable constants $c$ and $C$, and $n \geqslant n_0$,

$$\begin{aligned}&\frac{1}{n} \sum_{i=1}^{n} |X_i - \overline{X}_n|^{2+\delta}\\&\quad = \frac{1}{n} \sum_{i=1}^{n} \left| e_i - \overline{e}_n + d_n n^{-\gamma} \left( (i-m_n)_+^\gamma - \frac{1}{n} \sum_{l=1}^{n} (l-m_n)_+^\gamma \right) \right|^{2+\delta}\\&\quad \leqslant c\, \frac{1}{n} \sum_{i=1}^{n} |e_i|^{2+\delta} + c|\overline{e}_n|^{2+\delta} + c\, d_n^{2+\delta} n^{-2\gamma-\delta\gamma-1} \sum_{i=1}^{n-m_n} i^{2\gamma+\gamma\delta}\\&\qquad + c\, d_n^{2+\delta} n^{-2\gamma-\delta\gamma-2-\delta} \left( \sum_{l=1}^{n-m_n} l^\gamma \right)^{2+\delta} \leqslant C \quad \text{a.s.}\end{aligned}$$

An application of Theorem 7 now completes the proof. □

## 3. Permutation statistics for changes of stochastic processes under strong invariance

Next we study models 1.2 and 1.3. For model 1.2, we first need to investigate the asymptotic behavior of the corresponding rank statistic:

**Theorem 12.** *Let* $(R_1, \ldots, R_n)$ *be a random permutation of* $(1, \ldots, n)$, *and* $a_n(1), \ldots, a_n(n)$ *be scores satisfying the following conditions*:

$$\sum_{i=1}^{n} a_n(i) = 0, \quad \frac{1}{n} \sum_{i=1}^{n} a_n^2(i) \to 1, \tag{16}$$

*and*

$$\frac{1}{n}\max_{1\leqslant i\leqslant n} a_n^2(i)\to 0. \tag{17}$$

*Then, as* $n\to\infty$,

$$\max_{1\leqslant k\leqslant n}\frac{1}{\sqrt{n}}\left|\sum_{i=1}^{k} a_n(R_i)\right| \xrightarrow{\mathscr{D}} \sup_{0\leqslant t\leqslant 1}|B(t)|,$$

*where* $\{B(t): 0\leqslant t\leqslant 1\}$ *denotes a Brownian bridge.*

**Proof.** It follows from Theorem 24.2 in Billingsley [2]. □

**Lemma 13.**

(1) *Let* $X_{1n},\ldots,X_{nn}$ *be independent r.v.'s with* $EX_{in}^4\leqslant D<\infty$ *for all* $i,n$. *Then*

$$\frac{1}{n}\sum_{i=1}^{n}(X_{in}-EX_{in})\to 0 \quad a.s. \quad (n\to\infty).$$

(2) *Let* $\{W_n(t): t\geqslant 0\}$, $n\in\mathbb{N}$, *be Wiener processes and f be a positive function of* $n$, *then*

$$W_n(f(n))=\mathrm{O}(\sqrt{f(n)\log n}) \quad a.s. \quad (n\to\infty).$$

**Proof.** (1) It follows immediately from Markov's inequality.
(2) Cf. Kirch [9], Theorem 10.0.2. □

In the sequel we assume that there is a 1-1-correspondence between $N$ and $T$, which is necessary to get a countable triangular array in $N$, and, in turn, allows us to use the preceding lemma.

Moreover, we assume $T^*=\theta T$, $0<\theta\leqslant 1$, and $N=\mathrm{o}(T^{1-2/(2+\delta)})$. Let $N^*=\lfloor \frac{NT^*}{T}\rfloor=\theta N(1+\mathrm{o}(1))$ and

$$\Delta Y_i=\begin{cases} b\left(Y\left(i\dfrac{T}{N}\right)-Y\left((i-1)\dfrac{T}{N}\right)\right), & i\leqslant N^*,\\ b\left(Y(T^*)-Y\left(\dfrac{N^*T}{N}\right)\right)+b^*Y^*\left(\dfrac{(N^*+1)T}{N}-T^*\right), & i=N^*+1,\\ b^*\left(Y^*\left(i\dfrac{T}{N}-T^*\right)-Y^*\left((i-1)\dfrac{T}{N}-T^*\right)\right), & i\geqslant N^*+2.\end{cases} \tag{18}$$

**Lemma 14.** (1) *It holds, as* $N\to\infty$,

$$\overline{\Delta Y}=\frac{1}{N}\sum_{i=1}^{N}\Delta Y_i=\mathrm{O}\left(\frac{\sqrt{T\log N}}{N}\right) \quad a.s.$$

(2) (a) *For* $s = 2, 3, 4$, *as* $N \to \infty$,

$$\frac{N^{(s-2)/2}}{T^{s/2}} \sum_{i=1}^{N} (\Delta Y_i)^s \to EW(1)^s(\theta b^s + (1-\theta)(b^*)^s) \quad a.s.,$$

*where* $W(1)$ *has a standard normal distribution.*
(b) *For* $v > 0$, *as* $N \to \infty$,

$$\frac{N^{(v-2)/2}}{T^{v/2}} \sum\nolimits_{i=1}^{N} |\Delta Y_i - \overline{\Delta Y}|^v = O(1) \qquad a.s.$$

(3) *For* $v > 0$, *as* $N \to \infty$,

$$\frac{N^{(v-2)/2}}{T^{v/2}} \max_{1 \leqslant i \leqslant N} |\Delta Y_i - \overline{\Delta Y}|^v = o(1) \quad a.s.$$

**Proof.** The proof makes use of (3)–(5) in combination with Lemma 13 (for details confer Kirch [9, Theorem 10.0.1]). □

We are now prepared to investigate the following permutation statistics:

$$M_T(\mathbf{R}) = \max_{1 \leqslant k \leqslant N} \left\{ \frac{1}{\sqrt{T}} \frac{1}{\widehat{b}_T} \left| \sum_{i=1}^{k} (\Delta Z_{R_i,T} - \overline{\Delta Z}_T) \right| \right\},$$

and

$$\widetilde{M}_T(\mathbf{R}) = \max_{1 \leqslant k \leqslant N} \left\{ \frac{1}{\sqrt{T}} \frac{1}{\widehat{c}_T} \left| \sum_{i=1}^{k} (\widetilde{\Delta Z}^2_{R_i,T} - \overline{\widetilde{\Delta Z}}{}^2_T) \right| \right\}.$$

Here again, $\mathbf{R} = (R_1, \ldots, R_n)$ denotes a random permutation of $(1, \ldots, n)$.

**Theorem 15.** *Let* $\{Z(t) : t \geqslant 0\}$ *be a process according to model* (3). *Let* $T^* = \theta T$, $0 < \theta \leqslant 1$, $N = o(T^{1-2/(2+\delta)})$, *and in* (2) *also* $a = a^*$. *Then, for all* $x \in \mathbb{R}$, *as* $T \to \infty$,

(1) $$P(M_T(\mathbf{R}) \leqslant x \mid Z(t), 0 \leqslant t \leqslant T) \to P\left( \sup_{0 \leqslant t \leqslant 1} |B(t)| \leqslant x \right) \quad a.s.$$

(2) $$P(\widetilde{M}_T(\mathbf{R}) \leqslant x \mid Z(t), 0 \leqslant t \leqslant T) \to P\left( \sup_{0 \leqslant t \leqslant 1} |B(t)| \leqslant x \right) \quad a.s.,$$

*where* $\{B(t) : 0 \leqslant t \leqslant 1\}$ *is a Brownian bridge.*

**Proof.** First note that, for the increments of $\{Z(t)\}$, we have

$$\Delta Z_{i,T} = \begin{cases} a\dfrac{T}{N} + \Delta Y_i, & i \leqslant N^*, \\ a\left(T^* - N^*\dfrac{T}{N}\right) + a^*\left((N^*+1)\dfrac{T}{N} - T^*\right) + \Delta Y_{N^*+1}, & i = N^*+1, \\ a^*\dfrac{T}{N} + \Delta Y_i^*, & i \geqslant N^*+2, \end{cases}$$

with $\Delta Y_i$ as in (18).

Now, for the proof of (1), consider the scores $a_N(i) = (1/\hat{b}_T)\sqrt{N/T}(\Delta Z_{i,T} - \overline{\Delta Z_{i,T}})$, $i = 1, \ldots, N$. Obviously, $\sum_{i=1}^N a_N(i)=0$ and $1/N\sum_{i=1}^N a_N^2(i)=1$, which means that it is sufficient to verify assumption (17) of Theorem 12.

In the sequel, $c$ and $C$ denote suitable constants which may be different in different places. We first consider the case $\theta < 1$ and $a \neq a^*$. Here, for sufficiently large $T$,

$$\begin{aligned} \widehat{b}_T^2 &= \frac{1}{T}\sum_{i=1}^N \Delta Z_{i,T}^2 - \frac{N}{T}\overline{\Delta_T Z}^2 \\ &= \frac{1}{T}\sum_{i=1}^N \Delta a_i^2 - \frac{N}{T}\overline{\Delta a}^2 + \frac{1}{T}\sum_{i=1}^N (\Delta Y_i)^2 - \frac{N}{T}(\overline{\Delta Y})^2 - 2\frac{1}{T}(aT^* + a^*(T-T^*))\overline{\Delta Y} \\ &\quad + \frac{2ab}{N}Y\left(N^*\frac{T}{N}\right) + \frac{2a^*b^*}{N}\left(Y^*(T-T^*) - Y^*\left((N^*+1)\frac{T}{N} - T^*\right)\right) \\ &\quad + \frac{2}{T}\left(a\left(T^* - N^*\frac{T}{N}\right) + a^*\left((N^*+1)\frac{T}{N} - T^*\right)\right)\Delta Y_{N^*+1} \\ &\geqslant c\frac{T}{N} \quad \text{a.s.,} \end{aligned} \tag{19}$$

where

$$\Delta a_i = \begin{cases} a\dfrac{T}{N}, & i \leqslant N^*, \\ a\left(T^* - N^*\dfrac{T}{N}\right) + a^*\left((N^*+1)\dfrac{T}{N} - T^*\right), & i = N^*+1, \\ a^*\dfrac{T}{N}, & i \geqslant N^*+2, \end{cases}$$

and $\overline{\Delta a} = \frac{1}{N}\sum_{i=1}^N \Delta a_i = \frac{1}{N}(aT^* + a^*(T-T^*))$. The last inequality in (19) follows from the fact that the first terms are the dominating ones. Indeed, since $\theta < 1$, $a \neq a^*$, for $T$ sufficiently large,

$$\begin{aligned}
&\frac{1}{T}\sum_{i=1}^{N}\Delta a_i^2-\frac{N}{T}\overline{\Delta a}^2\\
&\quad\geqslant a^2\frac{T}{N^2}N^*+a^*\frac{T}{N^2}(N-N^*-1)-a^2\frac{(T^*)^2}{TN}-(a^*)^2\frac{(T-T^*)^2}{TN}-2aa^*\frac{T^*(T-T^*)}{TN}\\
&\quad=(1+\mathrm{o}(1))\left(a^2\frac{T}{N}\theta(1-\theta)+(a^*)^2\frac{T}{N}\theta(1-\theta)-2aa^*\frac{T}{N}\theta(1-\theta)\right)-\frac{(a^*)^2T}{N^2}\\
&\quad=(1+\mathrm{o}(1))\left(\frac{T}{N}\theta(1-\theta)(a-a^*)^2\right)-(a^*)^2\frac{T}{N^2}\geqslant c\frac{T}{N}\quad\text{a.s.}
\end{aligned}\tag{20}$$

Next we prove that the other terms are of smaller order and hence are negligible. Lemma 13(2) gives

$$\begin{aligned}
&\frac{2ab}{N}Y\left(N^*\frac{T}{N}\right)+\frac{2a^*b^*}{N}\left(Y(T-T^*)-Y\left((N^*+1)\frac{T}{N}-T^*\right)\right)\\
&\quad=\frac{2ab}{N}W_T\left(N^*\frac{T}{N}\right)+\frac{2a^*b^*}{N}\left(W^*(T-T^*)-W^*\left((N^*+1)\frac{T}{N}-T^*\right)\right)+\mathrm{O}\left(\frac{T^{1/(2+\delta)}}{N}\right)\\
&\quad=\mathrm{O}\left(\frac{\sqrt{T\log N}}{N}\right)\quad\text{a.s.}
\end{aligned}\tag{21}$$

Since $T^*-N^*T/N\leqslant T/N$ and $(N^*+1)(T/N)-T^*\leqslant T/N$, we also get

$$\begin{aligned}
&\left|\frac{2}{T}\left(a\left(T^*-N^*\frac{T}{N}\right)+a^*\left((N^*+1)\frac{T}{N}-T^*\right)\right)\Delta Y_{N^*+1}\right|\\
&\quad\leqslant\frac{2}{N}(|a|+|a^*|)\left(|b|\left|W(T^*)-W\left(N^*\frac{T}{N}\right)\right|+|b^*|\left|W^*\left((N^*+1)\frac{T}{N}-T^*\right)\right|\right)\\
&\qquad+\mathrm{O}\left(\frac{T^{1/(2+\delta)}}{N}\right)=\mathrm{O}\left(\frac{\sqrt{T\log N}}{N}\right)\quad\text{a.s.}
\end{aligned}\tag{22}$$

Lemma 14 further implies

$$\frac{1}{T}\sum_{i=1}^{N}(\Delta Y_i)^2-\frac{N}{T}(\overline{\Delta Y})^2-2\frac{1}{T}(aT^*+a^*(T-T^*))\overline{\Delta Y}=\mathrm{O}\left(1+\frac{\log N}{N}+\frac{\sqrt{T\log N}}{N}\right)\quad\text{a.s.,}$$

which proves (19). Note that

$$\Delta a_i-\overline{\Delta a}=\begin{cases}(a-a^*)\dfrac{T-T^*}{N}, & i\leqslant N^*,\\[2mm] (a-a^*)\dfrac{\vartheta T-T^*}{N}, & i=N^*+1,\\[2mm] (a^*-a)\dfrac{T^*}{N}, & i\geqslant N^*+2,\end{cases}$$

for some $0\leqslant\vartheta\leqslant 1$, hence

$$\max_{1\leqslant i\leqslant N}(\Delta a_i-\overline{\Delta a})^2=\begin{cases}\left(\dfrac{T-T^*}{N}(a-a^*)\right)^2, & T^*\leqslant T/2,\\ \left(\dfrac{T^*}{N}(a-a^*)\right)^2, & T^*>T/2.\end{cases}$$

On combining (19), Lemma 14 (1) and Lemma 14(2(a)) we finally get (17), since

$$\begin{aligned}\frac{1}{N}\max_{1\leqslant i\leqslant N}a_N^2(i)&\leqslant 2\,\frac{1}{T\widehat{b}_T^2}\max_{1\leqslant i\leqslant N}(\Delta a_i-\overline{\Delta a})^2+2\frac{1}{T\widehat{b}_T^2}\max_{1\leqslant i\leqslant N}(\Delta Y_i-\overline{\Delta Y})^2\\&\leqslant\frac{2}{c}\,\frac{1}{N}\,(a-a^*)^2+\frac{2}{c}\,\frac{N}{T}\left(\frac{1}{T}\sum_{i=1}^{N}(\Delta Y_i)^2-\frac{N}{T}(\overline{\Delta Y})^2\right)\to 0\quad\text{a.s.}\end{aligned}\tag{23}$$

On the other hand, if $\theta=1$ or $a=a^*$, we obtain from Lemma 14,

$$\begin{aligned}\widehat{b}_T^2&=\frac{1}{T}\sum_{i=1}^{N}(\Delta Z_{i,T}-\overline{\Delta_T Z})^2=\frac{1}{T}\sum_{i=1}^{N}(\Delta Y_i)^2-\frac{N}{T}\,(\overline{\Delta Y})^2\\&\to\theta b^2+(1-\theta)(b^*)^2\geqslant c>0\quad\text{a.s.,}\end{aligned}\tag{24}$$

for $T$ sufficiently large. Using Lemma 14 (3), we arrive at (17), i.e.

$$\frac{1}{N}\max_{1\leqslant i\leqslant N}a_N^2(i)=\frac{1}{\widehat{b}_T^2 T}\max_{1\leqslant i\leqslant N}(\Delta Y_i-\overline{\Delta Y})^2\to 0\quad\text{a.s.,}\tag{25}$$

which completes the proof of (1).

For the proof of (2), consider $a_N(i)=(1/\sqrt{T}\widehat{c}_T)((\Delta Y_i-\overline{\Delta Y})^2-(1/N)\sum_{l=1}^{N}(\Delta Y_l-\overline{\Delta Y})^2)$. It suffices again to verify the assumptions of Theorem 12.

Since $a=a^*$, we get $\frac{1}{N}\sum_{i=1}^{N}a_N^2(i)=1$. Similarly as above, Lemma 14 gives

$$\frac{N}{T^2}\sum_{i=1}^{N}(\Delta Y_i-\overline{\Delta Y})^4\to 3(\theta b^4+(1-\theta)(b^*)^4)\quad\text{a.s.,}$$

and

$$(\widehat{b}_T)^2=\left(\frac{1}{T}\sum_{i=1}^{N}(\Delta Y_i)^2-\frac{N}{T}\overline{\Delta Y}^2\right)^2\to(\theta b^2+(1-\theta)(b^*)^2)^2\quad\text{a.s.}\tag{26}$$

From Jensen's inequality we conclude

$$\begin{aligned}\lim_{T\to\infty}\frac{N}{T}\widehat{c}_T^2&=\lim_{T\to\infty}\left(\frac{N}{T^2}\sum_{i=1}^{N}(\Delta Y_i-\overline{\Delta Y})^4-(\widehat{b}_T^2)^2\right)\\&=3(\theta b^4+(1-\theta)(b^*)^4)-(\theta b^2+(1-\theta)(b^*)^2)^2\\&\geqslant 2(\theta b^4+(1-\theta)(b^*)^4)>0\quad\text{a.s.}\end{aligned}$$

So, an application of Lemma 14 results in

$$\frac{1}{N} \max_{1\leqslant k\leqslant N} a_N^2(k) = \frac{1}{T\widehat{c}_T^2} \max_{1\leqslant k\leqslant N} \left( \left(\Delta Y_k - \overline{\Delta Y}\right)^2 - \frac{1}{N} \sum_{i=1}^{N} (\Delta Y_i - \overline{\Delta Y})^2 \right)^2$$
$$\leqslant C \left( \frac{N}{T^2} \max_{1\leqslant k\leqslant N} (\Delta Y_k - \overline{\Delta Y})^4 + \frac{1}{N} \left( \frac{1}{T} \sum_{i=1}^{N} (\Delta Y_i - \overline{\Delta Y})^2 \right)^2 \right) \to 0 \quad \text{a.s.},$$

which completes the proof of (2). □

Finally we turn to model 1.3 and investigate the permutation analogue of (10), i.e. the statistic

$$T_N^{(2)}(\mathbf{R}) = \sqrt{\frac{N}{T\ \widehat{b}_T^2}} \max_{1\leqslant k<N} \left\{ \frac{\left|\sum_{i=1}^{N} (i-k)_+^{\gamma} (\Delta S_{R_i,T} - \overline{\Delta S}_N)\right|}{\left(\sum_{i=1}^{N-k} i^{2\gamma} - \frac{1}{N}\left(\sum_{i=1}^{N-k} i^{\gamma}\right)^2\right)^{1/2}} \right\}.$$

The following asymptotic applies:

**Theorem 16.** *Let* $\{S(t) : t\geqslant 0\}$ *be a process according to model* (8). *Assume* $T^* = \theta T$, $0<\theta\leqslant 1$, *and* $N\sqrt{\log N} = \mathrm{o}(\min(T^{1-2/(2+\delta)}, T^{1/2+\gamma}))$. *Then, for all* $x \in \mathbb{R}$, *as* $T \to \infty$,

$$P(\alpha_N T_N^{(2)}(\mathbf{R}) - \beta_N \leqslant x \mid S(t), 0\leqslant t\leqslant T) \to \exp(-2e^{-x}) \qquad a.s.,$$

*where* $\alpha_N$, $\beta_N = \beta_N(\gamma)$ *are as in Theorem* 1 (*with N replacing n*).

**Proof.** First note that, for the increments of $\{S(t)\}$, we have

$$\Delta S_{i,T} = \begin{cases} \Delta Y_i, & i\leqslant N^*, \\ \Delta Y_{N^*+1} + \tilde{d}\left(\dfrac{(N^*+1)T}{N} - T^*\right)^{1+\gamma}, & i = N^*+1, \\ \Delta Y_i^* + \tilde{d}\left(\left(\dfrac{iT}{N} - T^*\right)^{1+\gamma} - \left(\dfrac{(i-1)T}{N} - T^*\right)^{1+\gamma}\right), & i\geqslant N^*+2. \end{cases}$$

In case of the null hypothesis, i.e. for $\theta = 1$, we can immediately verify the assumptions of Theorem 7 for $a_n(i) := \sqrt{(N/T)}\Delta S_{i,T}$ by using Lemma 14.

On the other hand, in case of $\theta<1$, we use $a_n(i) := (N/T^{1+\gamma})\Delta S_{i,T}$. First, via the mean value theorem,

$$\frac{N^{1+\delta}}{T^{(1+\gamma)(2+\delta)}} \left( \sum_{i=N^*+2}^{N} \left| \left(i\frac{T}{N} - T^*\right)^{1+\gamma} - \left((i-1)\frac{T}{N} - T^*\right)^{1+\gamma} \right|^{2+\delta} \right.$$
$$\left. + \left|(N^*+1)\frac{T}{N} - T^*\right|^{(1+\gamma)(2+\delta)} \right) = \mathrm{O}\left( \frac{N^{1+\delta}}{T^{(1+\gamma)(2+\delta)}} N \frac{T^{(1+\gamma)(2+\delta)}}{N^{2+\delta}} \right) = \mathrm{O}(1),$$

which, together with Lemma 14, gives

$$\frac{1}{N}\sum_{i=1}^{N}\left|\frac{N}{T^{1+\gamma}}\Delta S_{i,T}-\frac{N}{T^{1+\gamma}}\overline{\Delta S_n}\right|^{2+\delta}=\mathrm{O}(1)\quad \text{a.s.}$$

In order to verify the second assumption of Theorem 7, we first realize, by using partial summation, the mean value theorem and Lemmas 13 resp. 14, that

$$\begin{aligned}
&\frac{N}{T^{2+2\gamma}}\left(\sum_{i=N^*+2}^{N}\Delta Y_i\left(\left(\frac{iT}{N}-T^*\right)^{1+\gamma}-\left(\frac{(i-1)T}{N}-T^*\right)^{1+\gamma}\right)+\Delta Y_{N^*+1}\left(\frac{(N^*+1)T}{N}-T^*\right)^{1+\gamma}\right)\\
&\quad=\frac{N^2}{T^{2+2\gamma}}\overline{\Delta Y}\left((T-T^*)^{1+\gamma}-\left(\frac{N-1}{N}T-T^*\right)^{1+\gamma}\right)\\
&\qquad-\frac{N}{T^{2+2\gamma}}\sum_{k=N^*+2}^{N-1}\left(bY(T^*)+b^*Y^*\left(k\frac{T}{N}-T^*\right)\right)\left(\left((k+1)\frac{T}{N}-T^*\right)^{1+\gamma}\right.\\
&\qquad\qquad\left.-2\left(k\frac{T}{N}-T^*\right)^{1+\gamma}+\left((k-1)\frac{T}{N}-T^*\right)^{1+\gamma}\right)\\
&\qquad-\frac{N}{T^{2+2\gamma}}\left(bY(T^*)+b^*Y^*\left((N^*+1)\frac{T}{N}-T^*\right)\right)\\
&\qquad\qquad\times\left(\left((N^*+2)\frac{T}{N}-T^*\right)^{1+\gamma}-2\left((N^*+1)\frac{T}{N}-T^*\right)^{1+\gamma}\right)\\
&\quad=\mathrm{o}(1)+\mathrm{O}\left(\frac{1}{T^{1+\gamma}}\sum_{k=N^*+1}^{N}\left|bY(T^*)+b^*Y^*\left(k\frac{T}{N}-T^*\right)\right|\right)\\
&\quad=\mathrm{o}(1)+\mathrm{O}\left(\frac{N\sqrt{\log N}}{T^{1/2+\gamma}}\right)+\mathrm{O}\left(\frac{NT^{1/(2+\delta)}}{T^{1+\gamma}}\right)=\mathrm{o}(1)\quad \text{a.s.}
\end{aligned}\tag{27}$$

Next we have

$$\begin{aligned}
&\frac{N}{T^{2+2\gamma}}\left(\sum_{i=N^*+2}^{N}\left(\left(i\frac{T}{N}-T^*\right)^{1+\gamma}-\left((i-1)\frac{T}{N}-T^*\right)^{1+\gamma}\right)^2+\left((N^*+1)\frac{T}{N}-T^*\right)^{2(1+\gamma)}\right)\\
&\quad\geqslant\frac{1}{N}(1+\gamma)^2\sum_{i=N^*+1}^{N-1}\left(\frac{i}{N}-\theta\right)^{2\gamma}\geqslant\frac{1}{N}(1+\gamma)^2\int_{N^*}^{N-1}\left(\frac{x}{N}-\theta\right)^{2\gamma}\mathrm{d}x\\
&\quad=(1+\mathrm{o}(1))\frac{(1+\gamma)^2}{2\gamma+1}(1-\theta)^{2\gamma+1},
\end{aligned}$$

which shows that

$$\frac{\tilde{d}^2 N}{T^{2+2\gamma}}\left(\sum_{i=N^*+2}^{N}\left(\left(i\frac{T}{N}-T^*\right)^{1+\gamma}-\left((i-1)\frac{T}{N}-T^*\right)^{1+\gamma}\right)^2+\left((N^*+1)\frac{T}{N}-T^*\right)^{2(1+\gamma)}\right)$$
$$-\frac{\tilde{d}^2 N^2}{T^{2+2\gamma}}\left(\frac{1}{N}(T-T^*)^{1+\gamma}\right)^2 \geqslant (1+\mathrm{o}(1))\,\frac{(1-\theta)^{2\gamma+1}}{2\gamma+1}(\gamma^2+\theta(2\gamma+1)). \tag{28}$$

On combining (27), (28) and Lemma 14, we get indeed, for large $T$,

$$\frac{N}{T^{2+2\gamma}}\sum_{i=1}^{N}(\Delta S_{i,T}-\overline{\Delta S_n})^2 \geqslant c(\theta)$$

with some $c(\theta)>0$, which completes the proof. $\square$

## 4. Simulations

So far, we have only proven that the permutation principle is asymptotically applicable for processes satisfying models 1.1 to 1.3. Now we want to describe the results of some simulation studies to get an idea, how good the permutation method is in comparison to the original method. However, we abstain from giving the results in the i.i.d. case here due to limitation of space. The results are very similar to those of the gradual change, in particular for the case of partial sums.

### *4.1. Change in the mean of a stochastic process under strong invariance*

The following simulations are based on partial sums of normally distributed random variables (with variance 1) (cf. Horváth and Steinebach [6, Example 1.1],), and on a Poisson process (cf. Horváth and Steinebach [6, Example 1.2]). More specifically, we simulated the increments of the partial sums as i.i.d. r.v.' s, and the increments of the Poisson process were taken at times $1, 2, \ldots$ (instead of $i\frac{T}{N}$, $i=1,\ldots,N$, since this means only a scaling of the underlying r.v.' s). Other than that, we used the following parameters:

(1) $N=100, 200$
(2) $N^*=\frac{1}{4}N, \frac{1}{2}N, \frac{3}{4}N$
(3) $d := a^*-a=0, 1, 2, 3, 4$

Here $N^*$ is the change-point, and we are in the case of the null hypothesis for $d=0$.

Due to similarity of results and limitation of space we present only a small part of the simulation study in Figs. 1 and 2 and just show some graphs for the sake of visualization. For Tables containing simulated critical values for the null hypothesis as well as permutational quantiles, we refer the interested reader to Kirch and Steinebach [10]. In the latter preprint, we also discuss the i.i.d. case and give simulated $\alpha$-resp. $\beta$-errors for both methods, the asymptotic and the permutational one.

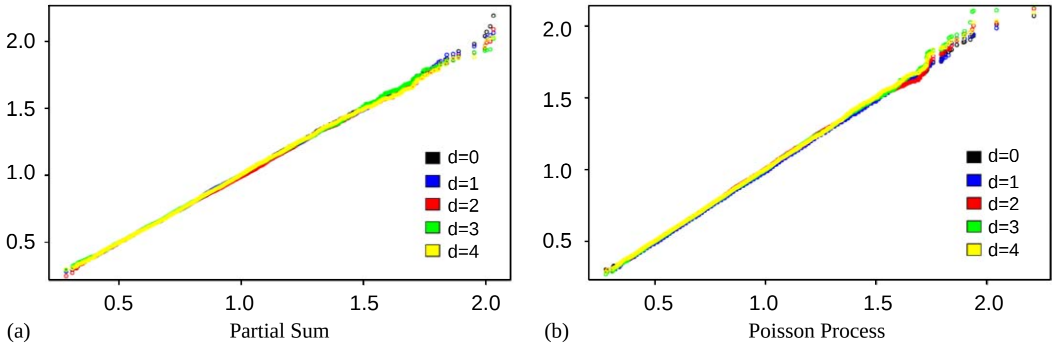


Fig. 1. QQ-Plots of $M_T$ (under $H_0$) against $M_T(\mathbf{R})$ for $N = 100$, $N^* = 75$.

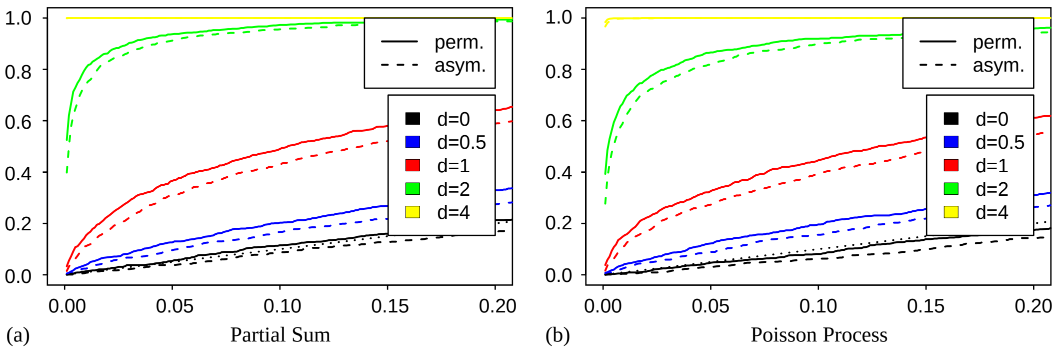


Fig. 2. Size-power-curves of $M_T(\mathbf{R})$ with respect to the asymptotic distribution and with respect to the permutation distribution for $N = 100$, $N^* = 75$.

Since we were interested in getting a better impression of how well the permutational distribution fits the real null distribution, we created quantile-quantile-plots of the one against the other. More precisely, we did the following:

(1) Exact distribution: Determine the empirical distribution function of $M_T$ (under $H_0$) based on 10 000 samples of length $N$.
(2) Permutation distribution: Determine the empirical distribution function of $M_N(\mathbf{R})$ (under particular realizations of $H_0$ or $H_1$) based on 10 000 permutations.
(3) Draw a QQ-plot of the null distribution from step 1 against the permutation distributions from step 2. Different colors represent different changes in the mean.

The results are to be found in Fig. 1.

We realize, that the permutation distribution fits the null distribution very well. Moreover, the result does not depend on the alternative.

Next we were interested in how well the test performs—and also how well it performs in comparison to the asymptotic one. For this reason we created size-power-curves of both methods under the null hypothesis and under alternatives.

We created these curves using the following algorithm:

(1) Simulate process $Z$ of length $N$ according to model (3).
(2) Calculate the empirical distribution function of the permutation statistic $M_T(\mathbf{R})$ based on 10 000 permutations.
(3) Calculate $M = M_T(Z)$, i.e. the value of the statistic for our sample $Z$.
(4) Calculate the $p$-value of $M$ with respect to the asymptotic distribution, i.e. $P\left(\sup_{0\leqslant t\leqslant 1}|B(t)| > M\right)$.
(5) Calculate the $p$-value of $M$ with respect to the permutation distribution from step (2), i.e. $P(M_T(\mathbf{R}) > M)$.
(6) Plot the empirical distribution function of the $p$-values from step (4) resp. (5) based on 1000 repetitions in the interval $(0, 0.2)$.

We did this for samples under the null hypothesis and various alternatives, for partial sums as well as Poisson processes.

What we get is a plot that shows the actual $\alpha$-errors resp. $1-(\beta$-errors) on the $y$-axis for the chosen quantiles on the $x$-axis, i.e. a plot that demonstrates very well the power of the test. So, the graph for the null hypothesis should be close to the diagonal (which is given by the dotted line), and the alternatives should be as steep as possible.

The results are presented in Fig. 2.

On comparing the asymptotic quantiles with the simulated null quantiles we realize, that they are too small. This is also confirmed by the size–power curves. Even though both methods apparently perform well, we do have a better fit under the permutation method. Under the null hypothesis ($d = a^* - a = 0$), the solid line (representing the permutation method) fits better to the dotted line (the one we wish to get). Moreover, under alternatives the lines representing the permutation method are also steeper, which means that the power of this test is better than the power of the asymptotic one.

Moreover, we were interested in the standard deviation of the critical values obtained by the permutation method. Under the null hypothesis (Poisson process, $N = 100$), we got a standard deviation of 0.01 for the 90%-quantile and of 0.019 for the 99%-quantile; for the partial sums the standard deviation was even smaller. The results are comparable for different parameters. As before, we used 1000 repetitions of step (1) to (4) of the first simulation above.

Computing time is not a problem here. For example, the calculation of the permutation quantiles for a series of length 100 takes approximately 3 s, and for length 200 approximately 5 s, using a Celeron with 466 MHz and 384 MB RAM and the software package R, Version 1.2.3.

### 4.2. Gradual change in the mean of a stochastic process under strong invariance

The following simulations are based on partial sums of normally distributed r.v. (with variance 1) (cf. Steinebach [14, Example 1.1]) and on a Poisson process (cf. [14, Example 1.2]). More precisely, we simulated the increments of the partial sums as i.i.d. r.v.'s, and the increments of the Poisson process were taken at times 1, 2, ... (instead of $i(T/N)$, $i = 1, \ldots, N$, as above). The following parameters were chosen:

(1) $N = 100, 200$
(2) $N^* = \frac{1}{4}N, \frac{1}{2}N, \frac{3}{4}N$
(3) $d = 0, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$

Here $N^*$ is the change-point, and the null hypothesis is given for $d=0$. The parameter $d$ has been rescaled as in Remark 5. More precisely, the increments of the change were chosen as $(d/(1+\gamma)N^{\gamma})((i-N^*)_+^{1+\gamma} - ((i-1)-N^*)_+^{1+\gamma})$. Note that the latter expression depends on $T$ only through $N$.

As in Section 4.1 we created QQ-Plots of the simulated null distribution vs. various permutation distributions in order to get an idea on how well the approximation fits. The results can be found in Fig. 3.

We realize that the matches (and thus the critical values) are quite good, but decline, if $\gamma < 1$, as the change becomes more obvious. On the other hand this leads to a greater power of the test, since the critical values are only too small if we are already under an alternative.

Moreover we have some kind of "step behavior" for the Poisson process. Apparently there are several permutations leading to the same maximal value (i.e. the value of the statistic). This, however, does not seem to influence the accuracy of the quantiles as will be shown by the size-power-curves. Remember that there are 10 000 points in the plot.

Note that here (in contrast to the i.i.d case) the consistency of the test is not guaranteed, since the estimator for $b$ is unbounded under the alternative (which violates condition (2.4) of Steinebach [14]).

This is why we were also interested in the power of the test. As in Section 4.1 we created size-power-curves of the asymptotic method as well as the permutation method. Note that for $\gamma = 0.25$ we do not know the asymptotic quantiles, since $H_{0.25}$ is not known.

The results can be found in Fig. 4.

First of all we realize, that the test gives good results for $\gamma = 0.25$, where we do not have the asymptotic test available. Also for $\gamma = 0.5$ the permutation test performs quite well, while the $\alpha$-errors of the asymptotic one are far too high. For $\gamma > 1$ both methods perform well, although the power under the permutation method is always greater than the power under the asymptotic method. The plot on $(0, 1)$ also shows, that the asymptotic curve (in contrast to the permutational one) is too high between 0.15 and 1. However, this is not a problem for the test, since one would hardly choose any critical value in that range.

We also notice that the power declines with increasing $\gamma$; for $\gamma=2$ it is almost impossible to distinguish between any alternatives. However this is not surprising, since for $\gamma = 2$ (and $N^* = \frac{3}{4}N$) we have an effective mean difference of approximately $d/16$, which is not very much.

When we used $\tilde{d}$, instead of $d$, and $T = N$ (which changes $\tilde{d}$ slightly), the critical values decreased significantly. Nevertheless, this did not seem to affect the permutation method at all—apparently the permutation quantiles were still smaller than the value of the test statistic for the unpermuted observations. With the asymptotic method, however, we only obtained good $\beta$-errors for smaller $\tilde{d}$'s, but observed a sudden jump in the $\beta$-errors (up to 100%) as soon as $\tilde{d}$ got larger. This jump, e.g., occurred at $\tilde{d} = 2$ for the 90%-quantile with $\gamma = 0.5$, $N = 100, 200$.

Again, we were also interested in the standard deviation of the critical values obtained by the permutation method. Under the null hypothesis (Poisson process, $N=100$, $\gamma=1$), we got a standard deviation of 0.28 for the 90%-quantile and of 0.96 for the 99%-quantile; for the partial sums the standard deviation was even smaller. As before, we used 1000 repetitions of step (1) to (4) from the first simulation.

For our simulations we used again the software package R, Version 1.2.3. On a Celeron with 466 MHz and 384 MB RAM the calculation of the permutation quantiles takes approximately 10 s in the case of 100 observations, and 30 s in the case of 200 (using 10 000 permutations).

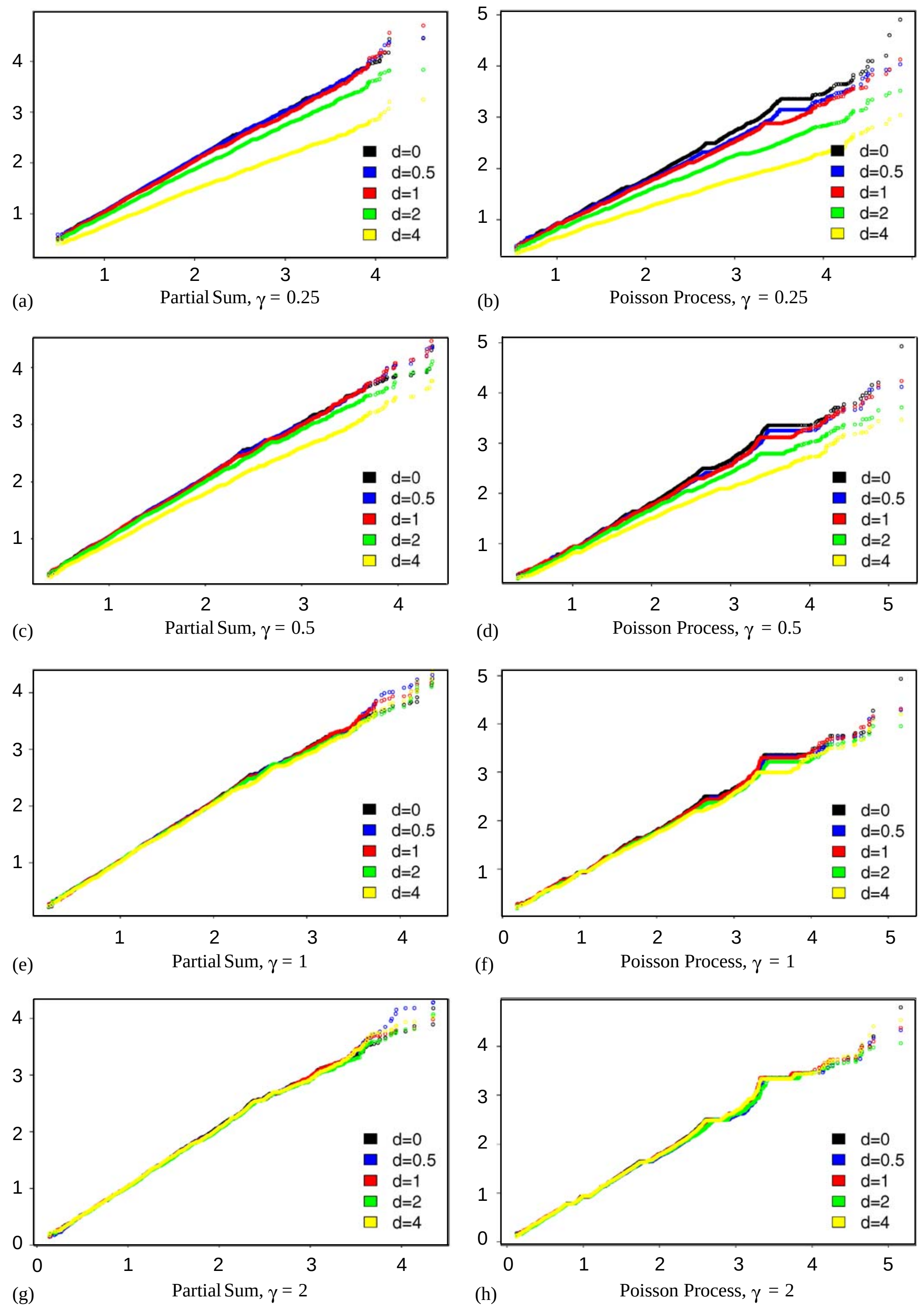


Fig. 3. QQ-Plots of $T_N^{(2)}$ (under $H_0$) against $T_n^{(2)}(\mathbf{R})$ for $N = 100$, $N^* = 75$. (a) Partial sum, $\gamma = 0.25$, (b) Poisson process, $\gamma = 0.25$, (c) partial sum, $\gamma = 0.5$, (d) Poisson process, $\gamma = 0.5$, (e) partial sum, $\gamma = 1$, (f) Poisson process, $\gamma = 1$, (g) partial sum, $\gamma = 2$ and (h) Poisson process, $\gamma = 2$.

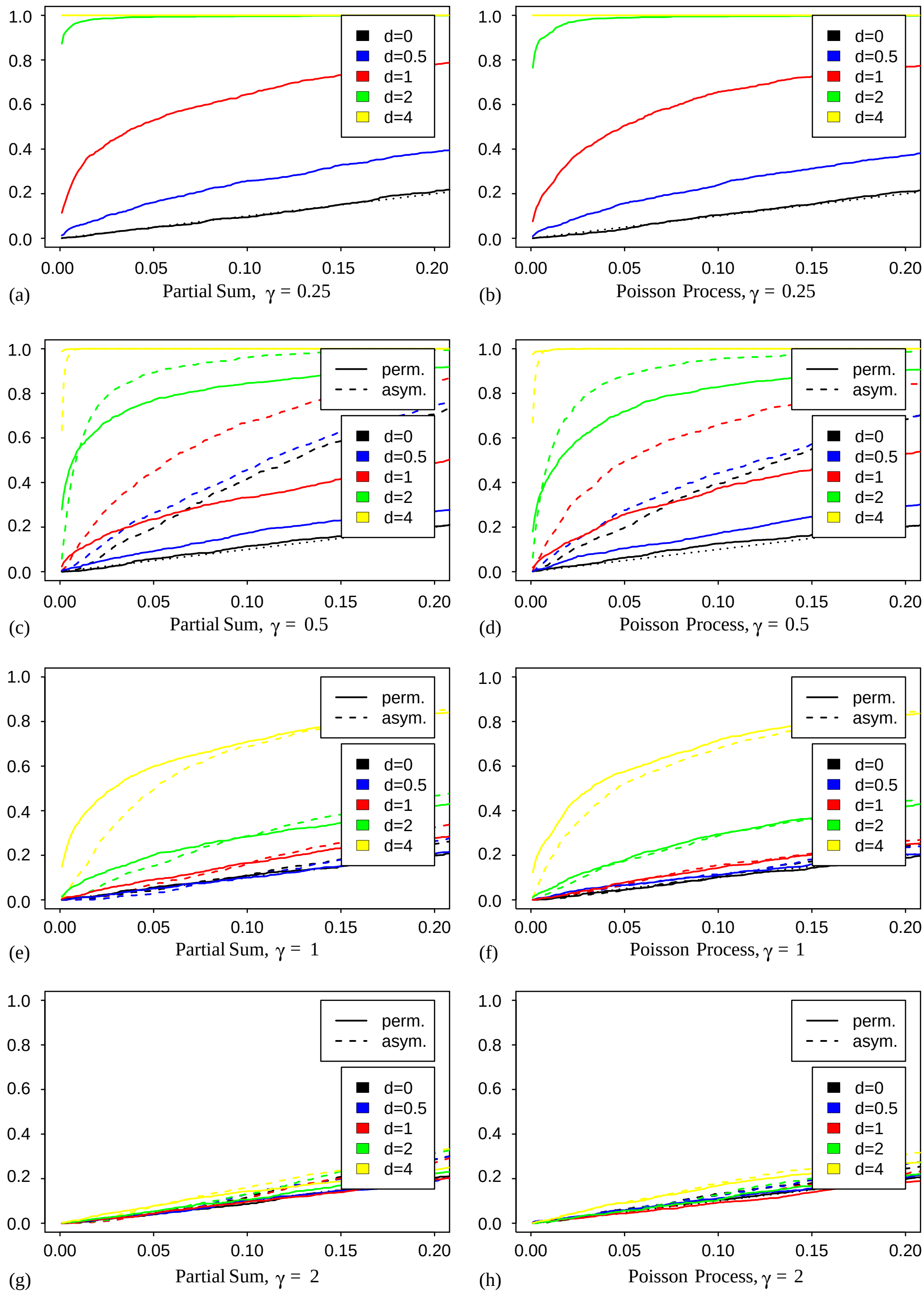


Fig. 4. Size–power curves of $T_N^{(2)}(\mathbf{R})$ with respect to the asymptotic distribution and with respect to the permutation distribution for $N = 100$, $N^* = 75$. (a) Partial sum, $\gamma = 0.25$ (b) Poisson process, $\gamma = 0.25$, (c) partial sum, $\gamma = 0.5$, (d) Poisson process, $\gamma = 0.5$, (e) partial sum, $\gamma = 1$, (f) Poisson process, $\gamma = 1$, (g) partial sum, $\gamma = 2$ and (h) Poisson process, $\gamma = 2$.

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