

Detection of Structural Change in the Long-run Persistence in a Univariate Time Series*

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Abstract

In this paper, we investigate a test for structural change in the long-run persistence in a univariate time series. Our model has a unit root with no structural change under the null hypothesis, while under the alternative it changes from a unit-root process to a stationary one or vice versa. We propose a Lagrange multiplier-type test, a test with the quasi-differencing method, and ‘demeaned versions’ of these tests. We find that the demeaned versions of these tests have better finite-sample properties, although they are not necessarily superior in asymptotics to the other tests.

I. Introduction

Testing for a unit root has become common practice in time-series analysis, and the null hypothesis of a unit root is not rejected for many macroeconomic variables. In a practical analysis, we usually use as long a sample period as possible, but we sometimes divide the sample period into several sub-periods to analyse specific periods. For example, we may split the sample in 1990 to analyse the effect of the reunification of Germany, while we may be interested

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in the periods before and after the middle of 1997 when the Asian crises occurred. Applying a unit-root test for each sub-period, we sometimes encounter cases where the null of a unit root is rejected in one of the sub-periods, although it is not rejected in the other sub-period. Then, our interest becomes whether or not the long-run persistence in the time series has changed.

Recently, change in the long-run persistence has been investigated in several papers. Enders and Granger (1998) and Caner and Hansen (2001) considered the test for such change with a threshold autoregressive (AR) model, while Buseti and Taylor (2004), Kim (2000), Kim, Belaire-Franch and Amador (2002) and Leybourne and Taylor (2004) proposed the test for the null of stationarity with no structural change in persistence. On the contrary, Kurozumi (2002) and Leybourne *et al.* (2003) considered the null of a unit root and proposed the tests of a change in the long-run persistence. Chong (2001) investigated the limiting property of the least squares estimator in an AR model of order 1 with a single structural change in the AR parameter. Although there are a great deal of papers on structural change such as those of Andrews (1993), Andrews, Lee and Ploberger (1996), Andrews and Ploberger (1994), Sowell (1996) and references therein among others, most of them assume that the parameter of interest is an interior point of the parameter space. However, this is not the case in our situation and then those results cannot be applied directly to our problem.

Taking structural change into account in a model is important for statistical tests. For example, Perron (1989) showed that standard unit-root tests tend not to reject the unit-root hypothesis when a change in a constant and/or a linear trend exists, and he proposed tests for a unit-root using a model with a structural break in a deterministic term. While the purpose of his paper is to test the unit-root hypothesis in the whole sample period, it demonstrates the importance of considering structural change in a model. Perron (1989) was criticized by Banerjee, Lumsdaine and Stock (1992), Christiano (1992) and Zivot and Andrews (1992) because he assumed that the break point is known, while the latter studies insist that the break point should be unknown and decided depending on data. However, as explained in Perron (1994) there are situations where the time of the break is known and, therefore, it seems appropriate to consider the testing problem for both cases of a known and unknown break point, depending on the situation.

In this paper, we consider the model in which the process changes from non-stationarity with a unit root to stationarity or vice versa. Both cases of a known and unknown break point are investigated. We propose four tests for the null of no structural change in the long-run persistence: a Lagrange multiplier (LM)-type test, tests based on the quasi-differencing method used by Elliott, Rothenberg and Stock (1996) and Xiao and Phillips (1999), and

‘demeaned versions’ of these tests as used by Oya and Toda (1998) and Toda and Oya (1993). We derive the limiting distributions of these test statistics under local alternatives and compare the power functions. A Monte Carlo simulation is also conducted to study the finite-sample property. We found that the LM-type test is much more affected by the initial condition when data have no trend. As a whole, the demeaned versions of the tests perform better in a finite sample.

The plan of this paper is as follows. In section II, we investigate the LM-type test and the demeaned version test for a known break point. We derive the limiting distribution both under the null hypothesis and local alternatives, and the asymptotic local powers are compared. We also consider the tests with the quasi-differencing method. Section III treats the case where a break point is unknown, and the finite-sample properties are investigated in section IV. Section V gives empirical examples, and section VI concludes the paper.

II. Testing for stability in the long-run persistence with a known break point

LM-type tests

Let us consider the following model:

$$y_t = \mu_0 + \mu_1 t + x_t, \quad (1 - \alpha_t L)\psi(L)x_t = u_t \tag{1}$$

for $t = 1, \dots, T$, where $\{u_t\}$ is independently and identically distributed (i.i.d.) with mean 0 and variance σ^2 , L denotes the lag operator, $\psi(L)$ is the p th-order lag polynomial and all roots of $\psi(z) = 0$ lie outside the unit circle. Suppose that some shock occurred at time T_B^* and $T_B^*/T = \lambda^*$ is constant. Here, we consider the case where T_B^* is known, while the unknown case is treated in the next section.

The testing problem we are concerned with can be written as follows:

$$H_0 : \alpha_t = 1 \quad \forall t \text{ v.s.} \tag{2}$$

$$H_1^{10} : \begin{cases} \alpha_t = 1 & t \leq T_B^* \\ |\alpha_t| < 1 & t \geq T_B^* + 1 \end{cases} \quad \text{or} \quad H_1^{01} : \begin{cases} |\alpha_t| < 1 & t \leq T_B^* \\ \alpha_t = 1 & t \geq T_B^* + 1 \end{cases}$$

Note that $\{x_t\}$ is a unit-root process under H_0 . On the contrary, under H_1^{10} , it changes from a unit-root process to a stationary one, while the change is in the reverse direction under H_1^{01} . It is possible to consider the case of the mixture of H_1^{10} and H_1^{01} , under which $\{x_t\}$ shifts between two stationary processes possibly with different persistences.

Figure 1(a)–(d) display simulated realizations from the model (1) under the assumptions that $\{u_t\} \sim \text{n.i.d.}(0, 1)$, $\psi(L) = 1$, $\mu_0 = 5$, $\mu_1 = 0$ or 0.2 .

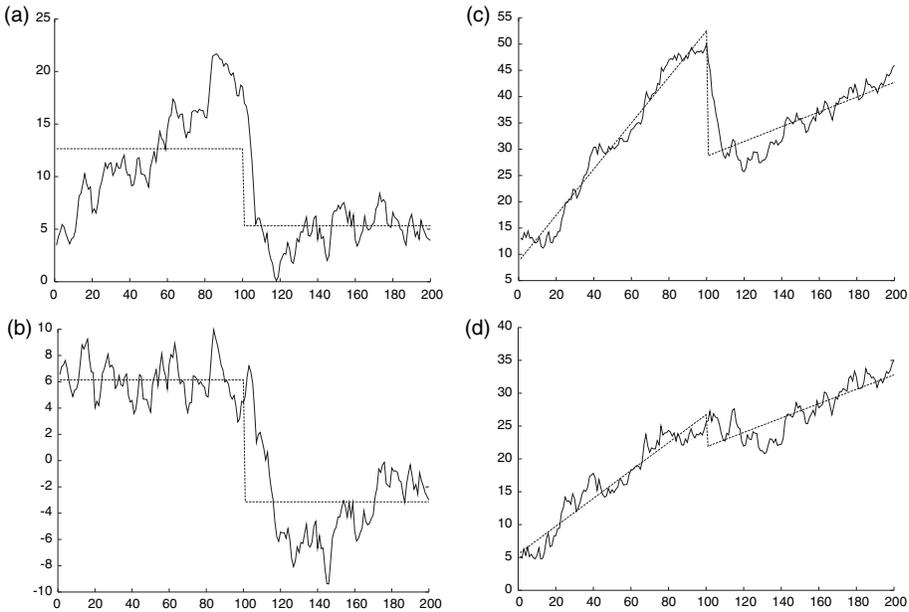


Figure 1. The simulated series; (a) non-trending case: $\alpha = 1 \rightarrow 0.85$, (b) non-trending case: $\alpha = 0.85 \rightarrow 1$, (c) trending case: $\alpha = 1 \rightarrow 0.85$, (d) trending case: $\alpha = 0.85 \rightarrow 1$

α_t changes from 1 to 0.85 or vice versa at $\lambda^* = 0.5$. Figure 1a, b corresponds to the cases without a linear trend ($\mu_1 = 0$) and Figure 1c, d corresponds to the cases with a linear trend ($\mu_1 = 0.2$). The dotted line in each figure is the estimated constant (trend). As we can see from the figures, variance of the process changes before/after the break point. In addition, the figures appear to show a structural break in a constant and/or a linear trend. These two phenomena sometimes appear in macroeconomic time series and hence, the model (1) may be seen as an alternative to the usual trend-break model.

As shown in the literature, the process $\{x_t\}$ in equation (1) can be expressed as

$$\Delta x_t = \rho_t x_{t-1} + \phi_{1t} \Delta x_{t-1} + \dots + \phi_{pt} \Delta x_{t-p} + u_t, \tag{3}$$

where

$$\rho_t = -(1 - \alpha_t)\psi(1)$$

and

$$\phi_{jt} = \alpha_t \psi_j - (1 - \alpha_t)(\psi_{j+1} + \dots + \psi_p), \quad 1 \leq j \leq p - 1, \quad \phi_{pt} = \alpha_t \psi_p. \tag{4}$$

Then, the testing problem (2) is equivalent to

$$H'_0 : \rho_t = 0 \quad \forall t \text{ v.s.} \tag{5}$$

$$H_1^{10'} : \begin{cases} \rho_t = \rho_1 = 0 & t \leq T_B^* \\ \rho_t = \rho_2 < 0 & t \geq T_B^* + 1 \end{cases} \quad \text{or} \quad H_1^{01'} : \begin{cases} \rho_t = \rho_1 < 0 & t \leq T_B^* \\ \rho_t = \rho_2 = 0 & t \geq T_B^* + 1 \end{cases}$$

Testing (5) by using the usual Wald test statistic is not convenient because we have to impose nonlinear restrictions on parameters to estimate equation (3). Instead, let us consider the LM test, which is easier to calculate because we only need the estimator under H_0' . Here notice that the log-likelihood function under $H_1^{10'}$ is different from that under $H_1^{01'}$. Then, if we construct the test statistic against $H_1^{10'}$, the test may not be able to detect $H_1^{01'}$ and vice versa. However, in practice, we are interested in whether or not a structural change occurred and we want to detect both $H_1^{10'}$ and $H_1^{01'}$. Then, we consider the model (3) with

$$\rho_t = \rho_1 \leq 0 \text{ for } t \leq T_B^* \quad \text{and} \quad \rho_t = \rho_2 \leq 0 \text{ for } t \geq T_B^* + 1 \quad (6)$$

and construct the test statistic to test H_0' . Note that the model (3) with (6) includes a stationary process possibly with a structural change. In this sense, our test is not designed to reject the specific alternative such as $H_1^{10'}$ and $H_1^{01'}$ but rather a wide class of alternatives that includes stationarity. However, as the model includes $H_1^{10'}$ and $H_1^{01'}$ as special cases, our test has considerable power against these alternatives and so we proceed to construct the test statistic based on the model (3) with (6).

Let us suppose that $\{u_t\}$ is normally distributed. As $x_t = y_t - \mu_0 - \mu_1 t$ from equation (1), we can substitute $y_t - \mu_0 - \mu_1 t$ for x_t in equation (3) and then the log-likelihood can be written as

$$\begin{aligned} \log \mathcal{L} = & \text{constant} - \frac{T}{2} \log \sigma^2 \\ & - \frac{1}{2\sigma^2} \sum_{t=1}^T \left\{ \Delta y_t - \mu_1 - \rho_t (y_{t-1} - y_0 - \mu_1(t-1)) - \sum_{j=1}^p \phi_{jt} (\Delta y_{t-j} - \mu_1) \right\}^2. \end{aligned} \quad (7)$$

Note that we replaced μ_0 by y_0 , as did Ahn (1993), Oya and Toda (1998) and Schmidt and Phillips (1992), because it is not identified under H_0 . As

$$\left. \frac{\partial \log \mathcal{L}}{\partial \mu_1} \right|_{H_0} = -\sigma^{-2} \sum_{t=1}^T u_t \left(-1 + \sum_{j=1}^p \phi_j \right),$$

where $\phi_j = \psi_j$ for all j under H_0 , we have the (approximate) maximum likelihood estimator (MLE) of μ_1 as

$$\tilde{\mu}_1 = T^{-1} \sum_{t=1}^T \Delta y_t$$

using the relation of

$$\sum_{t=1}^T \Delta y_t \simeq \sum_{t=1}^T \Delta y_{t-j}$$

(see Oya and Toda, 1998). Similarly, as

$$\left. \frac{\partial \log \mathcal{L}}{\partial \phi_{jt}} \right|_{H_0} = \sigma^{-2} \sum_{t=1}^T u_t (\Delta y_{t-j} - \mu_1),$$

the MLE of ϕ_j under H_0 is given by the following regression:

$$\Delta \tilde{x}_t = \tilde{\phi}_1 \Delta \tilde{x}_{t-1} + \dots + \tilde{\phi}_p \Delta \tilde{x}_{t-p} + \tilde{u}_t, \tag{8}$$

where $\tilde{x}_t = y_t - y_0 - \tilde{\mu}_1 t$. We also have

$$\tilde{\sigma}^2 = T^{-1} \sum_{t=1}^T \tilde{u}_t^2.$$

Using the above estimators we construct the LM-type test statistic. In the same way, as in Ahn (1993) and Toda and Oya (1993), the second derivative with the appropriate normalization is found to be asymptotically block diagonal between $\underline{\rho} = [\rho_1, \rho_2]'$ and the other parameters. Consequently, the LM test statistic becomes

$$\begin{aligned} LM_o^t(\lambda^*) &= \left(\frac{\partial \log \mathcal{L}}{\partial \underline{\rho}} \right)' \left(- \frac{\partial^2 \log \mathcal{L}}{\partial \underline{\rho} \partial \underline{\rho}'} \right)^{-1} \left(\frac{\partial \log \mathcal{L}}{\partial \underline{\rho}} \right) \\ &= \frac{(\partial \log \mathcal{L} / \partial \rho_1)^2}{-\partial^2 \log \mathcal{L} / \partial \rho_1^2} + \frac{(\partial \log \mathcal{L} / \partial \rho_2)^2}{-\partial^2 \log \mathcal{L} / \partial \rho_2^2}, \end{aligned} \tag{9}$$

which is evaluated under H_0 . The second equality holds because $\partial^2 \log \mathcal{L} / \partial \rho_1 \partial \rho_2 = 0$. As the first and second derivatives under H_0 are given by

$$\left. \frac{\partial \log \mathcal{L}}{\partial \rho_1} \right|_{H_0} = \frac{1}{\tilde{\sigma}^2} \sum_{t=1}^{T_B^*} \tilde{u}_t \tilde{x}_{t-1}, \quad \left. \frac{\partial \log \mathcal{L}}{\partial \rho_2} \right|_{H_0} = \frac{1}{\tilde{\sigma}^2} \sum_{t=T_B^*+1}^T \tilde{u}_t \tilde{x}_{t-1}, \tag{10}$$

$$\left. \frac{\partial^2 \log \mathcal{L}}{\partial \rho_1^2} \right|_{H_0} = \frac{-1}{\tilde{\sigma}^2} \sum_{t=1}^{T_B^*} \tilde{x}_{t-1}^2, \quad \left. \frac{\partial^2 \log \mathcal{L}}{\partial \rho_2^2} \right|_{H_0} = \frac{-1}{\tilde{\sigma}^2} \sum_{t=T_B^*+1}^T \tilde{x}_{t-1}^2, \tag{11}$$

we can express equation (9) as

$$LM_o^t(\lambda^*) = \frac{\left(\sum_{t=1}^{T_B^*} \tilde{u}_t \tilde{x}_{t-1} \right)^2}{\tilde{\sigma}^2 \sum_{t=1}^{T_B^*} \tilde{x}_{t-1}^2} + \frac{\left(\sum_{t=T_B^*+1}^T \tilde{u}_t \tilde{x}_{t-1} \right)^2}{\tilde{\sigma}^2 \sum_{t=T_B^*+1}^T \tilde{x}_{t-1}^2}. \tag{12}$$

Note that this test statistic is constructed for the two-sided alternative, which implies that it will reject the null hypothesis when the partial sums in parentheses of the right-hand side of equation (12) take large absolute values. However, our purpose is to detect a change in the long-run persistence and the alternative hypothesis is one-sided as given by (5). As the partial sums in parentheses of $LM_0^\tau(\lambda^*)$ tend to take negative values under $H_1^{10\ell}$ and $H_1^{01\ell}$, it is enough to reject the null hypothesis when they take large negative values. Then, we modify the test statistic (12) as

$$LM_1^\tau(\lambda^*) = \frac{\sum_{t=1}^{T_B^*} \tilde{u}_t \tilde{x}_{t-1}}{\tilde{\sigma} \sqrt{\sum_{t=1}^{T_B^*} \tilde{x}_{t-1}^2}} + \frac{\sum_{t=T_B^*+1}^T \tilde{u}_t \tilde{x}_{t-1}}{\tilde{\sigma} \sqrt{\sum_{t=T_B^*+1}^T \tilde{x}_{t-1}^2}}, \tag{13}$$

which rejects the null hypothesis when it takes small values. By making this modification, the power of the test is expected to be improved.

As in Oya and Toda (1998), we can show that the above test statistic is asymptotically equivalent to the sum of the t -statistics for ρ_1 and ρ_2 in the regression

$$\Delta \tilde{x}_t = \rho_1 D_{1t} \tilde{x}_{t-1} + \rho_2 D_{2t} \tilde{x}_{t-1} + \phi' \tilde{z}_{t-1} + e_t, \tag{14}$$

where $D_{1t} = 1$ for $t \leq T_B^*$ and zero otherwise, $D_{2t} = 1 - D_{1t}$, $\phi = [\phi_1, \dots, \phi_p]'$ and $\tilde{z}_{t-1} = [\Delta \tilde{x}_{t-1}, \dots, \Delta \tilde{x}_{t-p}]'$. Then, we define the test statistic for (5) as

$$LM^\tau(\lambda^*) = t_1^\tau(\lambda^*) + t_2^\tau(\lambda^*), \tag{15}$$

where t_1^τ and t_2^τ are t -statistics for ρ_1 and ρ_2 .

We also consider the ‘demeaned version’ of $LM^\tau(\lambda^*)$, i.e. the sum of the t -statistics for ρ_1 and ρ_2 in the regression

$$\Delta \tilde{x}_t = c_1 D_{1t} + \rho_1 D_{1t} \tilde{x}_{t-1} + c_2 D_{2t} + \rho_2 D_{2t} \tilde{x}_{t-1} + \phi' \tilde{z}_{t-1} + e_t. \tag{16}$$

We denote the demeaned version statistic as $LM_d^\tau(\lambda^*)$.

When y_t has no trend, i.e. if we know $\mu_1 = 0$ in the model (1), we define $\tilde{x}_t = y_t - y_0$ and construct the test statistics $LM^\mu(\lambda^*)$ and $LM_d^\mu(\lambda^*)$ exactly in the same way as $LM^\tau(\lambda^*)$ and $LM_d^\tau(\lambda^*)$.

Theorem 1 gives the limiting distributions of LM^τ , LM_d^τ , LM^μ and LM_d^μ under a sequence of local alternatives:

$$H_1^{10\ell} : \begin{cases} \alpha_t = \alpha_1 = 1 & t \leq T_B^* \\ \alpha_t = \alpha_2 = 1 - \frac{\theta_2^*}{T - T_B^*} & t \geq T_B^* + 1 \end{cases} \quad \text{or} \\ H_1^{01\ell} : \begin{cases} \alpha_t = \alpha_1 = 1 - \frac{\theta_1^*}{T_B^*} & t \leq T_B^* \\ \alpha_t = \alpha_2 = 1 & t \geq T_B^* + 1, \end{cases}$$

where θ_1^* and $\theta_2^* > 0$. We define the following functionals of a stochastic process $V(r)$ in generic form.

$$S(\lambda^*) = \frac{\frac{1}{2}(V^2(\lambda^*) - \lambda^*)}{\sqrt{\int_0^{\lambda^*} V^2(s)ds}} + \frac{\frac{1}{2}(V^2(1) - V^2(\lambda^*) - (1 - \lambda^*))}{\sqrt{\int_{\lambda^*}^1 V^2(s)ds}} \equiv S_1(\lambda^*) + S_2(\lambda^*), \tag{17}$$

$$S_d(\lambda^*) = \frac{\frac{\lambda^*}{2}(V^2(\lambda^*) - \lambda^*) - V(\lambda^*) \int_0^{\lambda^*} V(s)ds}{\sqrt{\lambda^{*2} \int_0^{\lambda^*} V^2(s)ds - \lambda^* \left(\int_0^{\lambda^*} V(s)ds\right)^2}} + \frac{\frac{1-\lambda^*}{2}(V^2(1) - V^2(\lambda^*) - (1 - \lambda^*)) - (V(1) - V(\lambda^*)) \int_{\lambda^*}^1 V(s)ds}{\sqrt{(1 - \lambda^*)^2 \int_{\lambda^*}^1 V^2(s)ds - (1 - \lambda^*) \left(\int_{\lambda^*}^1 V(s)ds\right)^2}} \equiv S_{d1}(\lambda^*) + S_{d2}(\lambda^*). \tag{18}$$

Theorem 1.

(i) Under $H_1^{10\ell}$, $LM^\tau(\lambda^*) \xrightarrow{d} S(\lambda^*)$ and $LM_d^\tau(\lambda^*) \xrightarrow{d} S_d(\lambda^*)$ with

$$V(r) = \begin{cases} W(r) - r\tilde{V}(1) & 0 \leq r \leq \lambda^* \\ \tilde{V}(r) - r\tilde{V}(1) & \lambda^* \leq r \leq 1 \end{cases}, \tag{19}$$

where $\tilde{V}(r) = e^{-\theta_2^*(r-\lambda^*)/(1-\lambda^*)} \{W(\lambda^*) - \tilde{W}(\theta_2^*/(1 - \lambda^*), \lambda^*)\} + \tilde{W}(\theta_2^*/(1 - \lambda^*), r)$ and $\tilde{W}(\theta, r)$ is the Ornstein–Uhlenbeck process defined by $d\tilde{W}(\theta, t) = -\theta\tilde{W}(\theta, t)dt + dW(t)$ for given θ .

(ii) Under $H_1^{10\ell}$, $LM^\mu(\lambda^*) \xrightarrow{d} S(\lambda^*)$ and $LM_d^\mu(\lambda^*) \xrightarrow{d} S_d(\lambda^*)$ with

$$V(r) = \begin{cases} W(r) & 0 \leq r \leq \lambda^* \\ \tilde{V}(r) & \lambda^* \leq r \leq 1 \end{cases}, \tag{20}$$

where $\tilde{V}(r)$ is the same as (i).

(iii) Under $H_1^{01\ell}$, $LM^\tau(\lambda^*) \xrightarrow{d} S(\lambda^*)$ and $LM_d^\tau(\lambda^*) \xrightarrow{d} S_d(\lambda^*)$ with

$$V(r) = \begin{cases} \tilde{W}(\theta_1^*/\lambda^*, r) - r\tilde{V}(1) & 0 \leq r \leq \lambda^* \\ \tilde{V}(r) - r\tilde{V}(1) & \lambda^* \leq r \leq 1 \end{cases} \tag{21}$$

where $\tilde{W}(\theta, r)$ is defined in (i) and $\tilde{V}(r) = \tilde{W}(\theta_1^*/\lambda^*, \lambda^*) + W(r) - W(\lambda^*)$.

(iv) Under $H_1^{01\ell}$, $LM^\mu(\lambda^*) \xrightarrow{d} S(\lambda^*)$ and $LM_d^\mu(\lambda^*) \xrightarrow{d} S_d(\lambda^*)$ with

$$V(r) = \begin{cases} \tilde{W}(\theta_1^*/\lambda^*, r) & 0 \leq r \leq \lambda^* \\ \tilde{V}(r) & \lambda^* \leq r \leq 1 \end{cases}, \tag{22}$$

where $\tilde{V}(r)$ is the same as (iii).

Remark 1. The limiting distributions under the null hypothesis are obtained by letting $\theta_1^* = 0$ for (iii) and (iv) and $\theta_2^* = 0$ for (i) and (ii). Under the null hypothesis, $V(r) = W(r) - rW(1)$ for the trending case and $V(r) = W(r)$ for the non-trending case.

Critical points of the above limiting distributions are tabulated in Table 1a. They are calculated by 10,000 iterations using the approximation

$$W(r) = \sum_{t=1}^{[1,000r]} \frac{\varepsilon_t}{\sqrt{1,000}}$$

TABLE 1
Critical values of the test statistics

λ^*	LM^μ			LM_d^μ		
	0.01	0.05	0.10	0.01	0.05	0.10
<i>(a) λ^* is known</i>						
0.1	-3.910	-3.045	-2.560	-5.724	-4.969	-4.552
0.2	-3.964	-3.030	-2.581	-5.796	-4.978	-4.539
0.3	-3.899	-3.056	-2.560	-5.786	-4.972	-4.526
0.4	-3.882	-3.063	-2.574	-5.707	-4.950	-4.541
0.5	-3.905	-3.063	-2.572	-5.659	-4.946	-4.522
0.6	-3.905	-3.064	-2.589	-5.664	-4.924	-4.536
0.7	-3.924	-3.065	-2.573	-5.728	-4.939	-4.536
0.8	-3.869	-3.044	-2.584	-5.733	-4.964	-4.529
0.9	-3.915	-3.018	-2.562	-5.725	-4.946	-4.562
	LM^ε			LM_d^ε		
0.1	-4.630	-3.801	-3.380	-5.926	-5.163	-4.761
0.2	-4.565	-3.843	-3.421	-5.896	-5.139	-4.744
0.3	-4.598	-3.836	-3.445	-5.844	-5.103	-4.730
0.4	-4.644	-3.893	-3.495	-5.831	-5.097	-4.700
0.5	-4.703	-3.983	-3.552	-5.773	-5.082	-4.711
0.6	-4.790	-4.005	-3.616	-5.847	-5.112	-4.729
0.7	-4.850	-4.097	-3.696	-5.923	-5.138	-4.745
0.8	-4.932	-4.173	-3.789	-5.847	-5.123	-4.753
0.9	-5.071	-4.342	-3.967	-5.951	-5.164	-4.755
		0.01		0.05		0.10
<i>(b) λ^* is unknown</i>						
inf- LM_d^μ		-7.097		-6.409		-6.057
avg- LM_d^μ		14.548		11.190		9.778
$\bar{\text{exp}}\text{-}LM_d^\mu$		22.014		17.208		15.116
inf- LM_d^ε		-7.122		-6.413		-6.038
avg- LM_d^ε		15.580		12.316		10.695
$\bar{\text{exp}}\text{-}LM_d^\varepsilon$		22.090		17.448		15.212
inf- LM_d^{QD}		-7.148		-6.442		-6.072
avg- LM_d^{QD}		14.528		11.433		9.982
$\bar{\text{exp}}\text{-}LM_d^{\text{QD}}$		21.937		17.339		15.266

where $\{\varepsilon_t\}$ is an independent standard normal variable and $[x]$ signifies the integer part of x . From Table 1a, we can see that critical values of LM^μ , LM_d^μ and LM_d^τ seem to be insensitive to the break fraction λ^* , while those of LM^τ tend to be smaller for larger values of λ^* .

Using the above result, we depict the local limiting powers as a function of θ_1^* and θ_2^* . Figure 2a shows the case, where the model does not have a linear trend when $\lambda^* = 0.5$. The centre of the horizontal axis corresponds to the null hypothesis, while the left-hand (right-hand) side from the centre shows the alternative hypothesis $H_1^{10\ell}$ ($H_1^{01\ell}$). We can see that the power of LM^μ dominates that of LM_d^μ for both directions of θ_1^* and θ_2^* , which may be theoretically expected, because LM_d^μ is constructed from the regression with an extra constant term. On the contrary, from Figure 2b, when data are trending, the power of the demeaned version test, LM_d^τ , dominates that of the LM-type test, LM^τ , under $H_1^{10\ell}$. To examine the reason of this strange relation, let us focus on the case where θ_2^* takes small values close to zero. The powers of LM^τ and LM_d^τ are 0.0388 and 0.0484 for $\theta_2^* = 1$, 0.0401 and 0.0582 for $\theta_2^* = 2$ and 0.0470 and 0.0740 for $\theta_2^* = 3$, respectively. These imply that both tests are biased, and LM^τ is more biased in a wider range of θ_2^* than LM_d^τ , which seems to partly explain why the demeaned version test dominates the LM-type.

The other interesting point is that the power function increases differently depending on the direction of the alternative: the test can detect the change from non-stationarity to stationarity more often than that in the reverse direction. One of the reasons is that, as is seen in Figure 1a–d, the unit-root process reverts to the mean value within a short period of time after it changes to stationarity, whereas the stationary process starts to deviate from the mean value relatively slowly when it becomes a unit-root process, so that the difference of the process before and after the break tends to be more evident under H_1^{10} . This tendency is observed even in finite samples as will be seen in later.

Tests using the quasi-differencing method

In the framework of a unit-root test, it is pointed out by Elliott *et al.* (1996) that the power of the test may increase if we estimate a non-stochastic term by generalized least squares (GLS), because the GLS estimator is efficient under the alternative of trend stationarity close to a unit-root process. For the test of a change in the long-run persistence, this point is also appropriate because under H_1^{10} (H_1^{01}), the process is trend stationary after (before) the break point. Leybourne *et al.* (2003) also used this quasi-differencing method for testing a change in the long-run persistence when the break point is unknown.

Let us define a quasi-differenced series as

$$\tilde{x}_t^{\text{QD}} = y_t - \tilde{\mu}_0^{\text{QD}} - \tilde{\mu}_1^{\text{QD}} t, \quad (23)$$

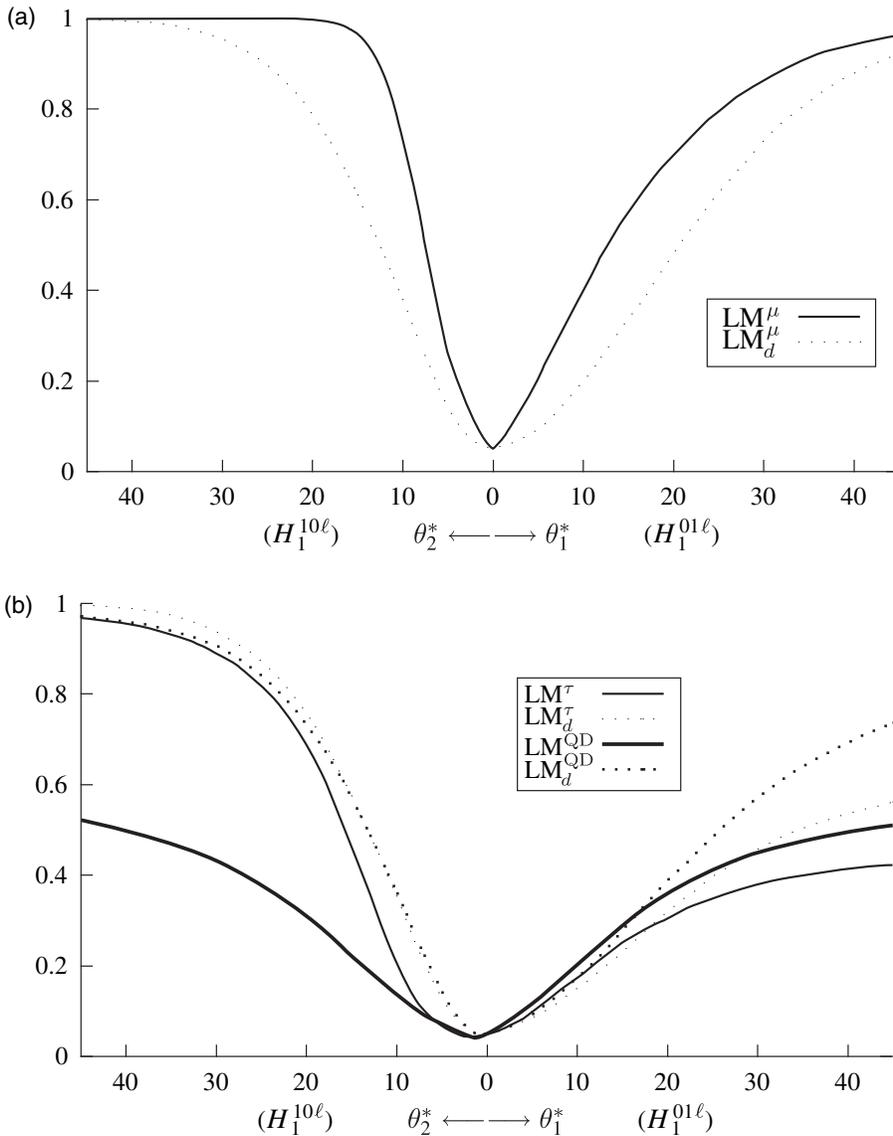


Figure 2. The asymptotic local powers; (a) non-trending case: $\lambda^* = 0.5$, (b) trending case: $\lambda^* = 0.5$

where $[\tilde{\mu}_0^{QD}, \tilde{\mu}_1^{QD}]'$ is obtained by regressing $y^{QD} = [y_1, \Delta_{\theta_1} y_2, \dots, \Delta_{\theta_1} y_{T_B^*}, \Delta_{\theta_2} y_{T_B^*+1}, \dots, \Delta_{\theta_2} y_T]'$ on $\xi^{QD} = [\xi_0^{QD}, \xi_1^{QD}]'$, where $\xi_0^{QD} = [1, \Delta_{\theta_1} 1, \dots, \Delta_{\theta_1} 1, \Delta_{\theta_2} 1, \dots, \Delta_{\theta_2} 1]'$ and $\xi_1^{QD} = [1, \Delta_{\theta_1} 2, \dots, \Delta_{\theta_1} T_B^*, \Delta_{\theta_2} (T_B^* + 1), \dots, \Delta_{\theta_2} T]'$ with $\Delta_{\theta_1} = 1 - (1 - \theta_1/T_B^*)L$ and $\Delta_{\theta_2} = 1 - (1 - \theta_2/(T - T_B^*))L$ being quasi-differencing operators for $t = 1, \dots, T_B^*$ and

$t = T_B^* + 1, \dots, T$, respectively. By using the quasi-differenced series \tilde{x}_t^{QD} , we construct the test statistics $\text{LM}^{\text{QD}}(\lambda^*)$ and $\text{LM}_d^{\text{QD}}(\lambda^*)$ in the same way as $\text{LM}^\tau(\lambda^*)$ and $\text{LM}_d^\tau(\lambda^*)$.

Theorem 2.

(i) Under $H_1^{0\ell}$, $\text{LM}^{\text{QD}}(\lambda^*) \xrightarrow{d} S(\lambda^*)$ and $\text{LM}_d^{\text{QD}}(\lambda^*) \xrightarrow{d} S_d(\lambda^*)$ with

$$V(r) = \begin{cases} W(r) - rJ_1 & 0 \leq r \leq \lambda^* \\ \tilde{V}(r) - rJ_1 & \lambda^* \leq r \leq 1 \end{cases}, \tag{24}$$

where $\tilde{V}(r)$ is the same as Theorem 1(i) and

$$\begin{aligned} J_1 = \frac{1}{\delta} & \left\{ \left(1 + \frac{\theta_2}{1 - \lambda^*} \right) W(1) + \left(\theta_1 - \frac{\lambda^* \theta_2}{1 - \lambda^*} \right) W(\lambda^*) \right. \\ & + \frac{\theta_1^2}{\lambda^{*2}} \int_0^{\lambda^*} sW(s)ds - \frac{\theta_2}{1 - \lambda^*} \int_{\lambda^*}^1 W(s)ds \\ & \left. + \frac{\theta_2 - \theta_2^*}{1 - \lambda^*} \int_{\lambda^*}^1 \left(1 + \frac{\theta_2}{1 - \lambda^*} s \right) \tilde{V}(s)ds \right\} \end{aligned} \tag{25}$$

with

$$\delta = \lambda^* \left(1 + \theta_1 + \frac{\theta_2^2}{3} \right) + (1 - \lambda^*) \left(1 + \theta_2 \frac{1 - \lambda^{*2}}{(1 - \lambda^*)^2} + \frac{\theta_2^2}{3} \frac{1 - \lambda^{*3}}{(1 - \lambda^*)^3} \right). \tag{26}$$

(ii) Under $H_1^{01\ell}$, $\text{LM}^{\text{QD}}(\lambda^*) \xrightarrow{d} S(\lambda^*)$ and $\text{LM}_d^{\text{QD}}(\lambda^*) \xrightarrow{d} S_d(\lambda^*)$ with

$$V(r) = \begin{cases} \tilde{W}(\theta_1^*/\lambda^*, r) - rJ_2 & 0 \leq r \leq \lambda^* \\ \tilde{V}(r) - rJ_2 & \lambda^* \leq r \leq 1 \end{cases} \tag{27}$$

where $\tilde{V}(r)$ is the same as Theorem 1(iii) and

$$\begin{aligned} J_2 = \frac{1}{\delta} & \left\{ \left(1 + \frac{\theta_2}{1 - \lambda^*} \right) W(1) + \left(\theta_1 - \frac{\lambda^* \theta_2}{1 - \lambda^*} \right) W(\lambda^*) \right. \\ & - \frac{\theta_1}{\lambda^*} \int_0^{\lambda^*} W(s)ds + \frac{\theta_1 - \theta_1^*}{\lambda^*} \int_0^{\lambda^*} \left(1 + \frac{\theta_1}{\lambda^*} s \right) \tilde{W}(\theta, s)ds \\ & \left. - \frac{\theta_2}{1 - \lambda^*} \int_{\lambda^*}^1 W(s)ds + \frac{\theta_2}{1 - \lambda^*} \int_{\lambda^*}^1 \left(1 + \frac{\theta_2}{1 - \lambda^*} s \right) \tilde{V}(s)ds \right\}. \end{aligned} \tag{28}$$

Remark 2. The null distribution is obtained by letting $\theta_2^* = 0$ for case (i) and $\theta_1^* = 0$ for case (ii).

Remark 3. We can consider the case where the model does not have a linear trend, but the test statistic constructed using the quasi-differencing method is shown to have the same limiting distribution as the LM-type test statistic (see

also Leybourne *et al.*, 2003). Then, we do not consider the quasi-differencing method for non-trending data.

Notice that the null distributions of $LM^{QD}(\lambda^*)$ and $LM_d^{QD}(\lambda^*)$ depend on θ_1 and θ_2 and then we have to decide these values to calculate percentage points of these distributions. If we choose $\theta_1 = \theta_1^*$, the power attains its maximum against the alternative of $H_1^{01\ell}$ for a fixed value of θ_1^* and then, by varying the values of $\theta_1 = \theta_1^*$, we obtain the power envelope against $H_1^{01\ell}$. Similarly, we can obtain the power envelope against $H_1^{10\ell}$ by choosing $\theta_2 = \theta_2^*$. Although we cannot construct a test that has the same power function as the envelope, this envelope has often been used to decide the local parameter. For example, King (1983) proposed to select the alternative point so that the power function of the test is tangent to the envelope at a power of 25%, 50% or 75%, while Tanaka (1996) considered the test the power function of which is tangent to the envelope at a power of 50%. Although these two papers consider models different from ours, their results show that the strategy of Tanaka (1996) works fairly well. Therefore, we choose the local parameter so that the power of the test can be tangent to the power envelope at a power of 50%. Following this strategy, we obtained θ_1 and θ_2 with which the power is tangent to both envelopes against $H_1^{01\ell}$ and $H_1^{10\ell}$ at a power of 50%. For $LM^{QD}(\lambda^*)$, $\{\theta_1, \theta_2, 5\%$ critical value $\}$ are given by $\{45, 50, -4.076\}$, $\{41, 40, -4.166\}$ and $\{35, 26, -4.344\}$ for $\lambda^* = 0.3, 0.5$ and 0.7 , and for $LM_d^{QD}(\lambda^*)$, they are $\{27.5, 18.5, -4.950\}$, $\{26, 13.25, -4.973\}$ and $\{24.75, 9.5, -5.042\}$.

The power functions of LM^{QD} and LM_d^{QD} are drawn in Figure 2b. As in the previous section, the demeaned version test is more powerful against $H_1^{10\ell}$ while it is less powerful against $H_1^{01\ell}$ when the alternative is close to the null. Note that LM^{QD} has very low power against $H_1^{10\ell}$. This is because our choice of the local parameter θ_2 is 41 at which the power attains 50%. If we use the other value of θ_2 , the shape of the power function will change and we may find the value of θ_2 which is more favourable than $\theta_2 = 41$ in view of the power. However, if we change the value of θ_2 , the power against $H_1^{01\ell}$ will also change and the test may not attain the power of 50% at $\theta_1^* = 40$ under $H_1^{01\ell}$. As we used the selection rule for θ_1 and θ_2 so that the test attains a power of 50% at $\theta_1 = \theta_1^*$ under $H_1^{01\ell}$ and $\theta_2 = \theta_2^*$ under $H_1^{10\ell}$, $\theta_1 = 41$ and $\theta_2 = 45$ are used in our analysis. Other selection rules may produce different results, but we do not pursue them.

III. Testing for stability in the long-run persistence with an unknown break point

In practice, it is often the case that we do not know the actual break point T_B^* and, for such a case, several testing procedures have been proposed in the

literature. One of the useful methods is to take the infimum of the test statistic in a closed interval:

$$\text{inf-LM}^\tau = \inf_{\lambda \in \Lambda} \text{LM}^\tau(\lambda), \quad (29)$$

where Λ is a closed set in $(0, 1)$. We also consider the test statistics of an average exponential form, as considered in Andrews *et al.* (1996) and Andrews and Ploberger (1994),

$$\begin{aligned} \text{avg-LM}^\tau &= \int_{\lambda \in \Lambda} (t_1^\tau(\lambda)^2 + t_2^\tau(\lambda)^2) d\lambda, \\ \overline{\text{exp-LM}}^\tau &= \log \int_{\lambda \in \Lambda} \exp(t_1^\tau(\lambda)^2 + t_2^\tau(\lambda)^2) d\lambda. \end{aligned}$$

Exactly in the same way, we consider the test statistics using the quasi-differencing method and the demeaned version statistics.

Theorem 3. Under H'_0 , $\text{inf-LM} \xrightarrow{d} \inf_{\lambda \in \Lambda} S(\lambda)$,

$$\begin{aligned} \text{avg-LM} &\xrightarrow{d} \int_{\lambda \in \Lambda} (S_1(\lambda)^2 + S_2(\lambda)^2) d\lambda \\ \text{and } \overline{\text{exp-LM}} &\xrightarrow{d} \log \int_{\lambda \in \Lambda} (S_1(\lambda)^2 + S_2(\lambda)^2) d\lambda, \end{aligned}$$

where $\text{LM} = \text{LM}^\tau$, LM_d^τ , LM^{QD} , LM_d^{QD} , LM^μ or LM_d^μ and $S(\lambda)$, $S_1(\lambda)$ and $S_2(\lambda)$ are corresponding distributions given by Theorems 1 and 2 with $\theta_1^* = \theta_2^* = 0$.

Critical values of the tests can be obtained once Λ is chosen, and many choices of Λ have been proposed in the literature for the sup- and inf-type tests, although they seem more or less arbitrary. Intuitively, the narrower interval of Λ will produce more powerful test, while such an interval excludes the possibility of a structural change that occurs at the end of the sample period. In this paper, we follow Leybourne *et al.* (2003) that chose $\Lambda = [0.2, 0.8]$. By choosing this interval, our tests become comparable with those of Leybourne *et al.* (2003). Critical values are tabulated in Table 1b only for the demeaned version statistics, because only the demeaned version tests are shown to be useful in practice from the Monte Carlo simulation. For the test using the quasi-differencing method, we used $\theta_1 = 26$ and $\theta_2 = 13.25$, which are the same values as in the case where the break point is known as $\lambda^* = 0.5$. Notice that we reject the null hypothesis when either $t_1^\tau(\lambda)$ or $t_2^\tau(\lambda)$ takes small values, which implies that the inf-type test rejects the null hypothesis when it takes small values. On the contrary, as the avg- and $\overline{\text{exp}}$ -type tests are increasing functions of $t_1^\tau(\lambda)^2$ and $t_2^\tau(\lambda)^2$, we reject the null hypothesis when they take large values.

IV. Finite-sample properties

In this section, we investigate the finite-sample properties of the test statistics in the previous sections. The following data generating process is considered:

$$y_t = c_t' \beta + x_t, \quad (1 - \alpha_t L)(1 - \psi L)x_t = u_t, \quad (30)$$

where $\{u_t\}$ is n.i.d.(0, 1) and $c_t = 1$ or $[1, t]'$. We set $\beta = 0$, α_1 and $\alpha_2 = 1, 0.95, 0.9, 0.8, 0.7$, $\psi = 0, \pm 0.5, 0.8$, $\lambda^* = 0.3, 0.5, 0.7$, and the sample size $T = 100, 200$ and 500 . The initial value of x_t is set equal to 0 and the first 100 observations are discarded. The level of significance is 0.05 and the number of replications is 1,000 in all experiments, performed by the GAUSS matrix programming language. Recall that when the true rejection probability is P , the standard error of the rejection frequency based on 1,000 replications of the experiment is given by $\{P(1 - P)/1,000\}^{1/2}$, so that, for example, the standard error is 0.007 for $P = 0.05$. It goes without saying that the accuracy of the experiment becomes better when the number of replications is increased.

Table 2 reports the size and power without a linear trend when the break point is known and $\psi = 0$. From the table, we can see that LM^μ has a reasonable empirical size close to 0.05 in all cases, while LM_d^μ tends to slightly over-reject the null hypothesis when $T = 100$. When $T = 200$ and 500 , both statistics have the empirical size close to the nominal one. As to the power, in almost all cases, LM_d^μ is more powerful than LM^μ . We can also see that the power of LM_d^μ against H_1^{01} increases as λ^* tends to 0, while it increases under H_1^{01} as the break occurs at the later point of the sample period.

The simulation results for the trending case are tabulated in Table 3. As in the case with no linear trend, the empirical size of the LM-type test is closer to the nominal one than the other statistics. The tests using the quasi-differencing method overly reject the null hypothesis, even when the sample size is 200, although LM_d^{QD} is not distorted as much as LM^{QD} . As to the power, the demeaned version tests are more powerful against both the alternatives. Considering the performance of the tests both under the null and the alternative, LM_d^c performs better than any other statistics.

In the case, where $\psi \neq 0$, the sizes of the tests are close to the entries on Tables 2 and 3, while the tests tend to be less powerful against H_1^{10} as ψ increases, although the difference is slight. The relation between the power against H_1^{01} and ψ is similar to the case of H_1^{10} , but the powers of the tests decrease considerably as ψ increases. For example, when $(\alpha_1, \alpha_2) = (0.8, 1)$, $\lambda^* = 0.5$ and $T = 200$, the power of LM_d^μ is 0.577, 0.493 and 0.325 for $\psi = 0, 0.5$ and 0.8 , while that of LM_d^c is 0.426, 0.308 and 0.175, respectively (details are available upon request). These results imply that it is difficult for our tests to find a change in persistence from stationarity to non-stationarity when the stable AR coefficient is close to 1.

TABLE 2
Size and power (non-trending case, λ^ is known)*

(α_1, α_2)	$\lambda^* = 0.3$		$\lambda^* = 0.5$		$\lambda^* = 0.7$	
	LM^μ	LM_d^μ	LM^μ	LM_d^μ	LM^μ	LM_d^μ
<i>T = 100</i>						
(1, 1)	0.054	0.089	0.055	0.105	0.055	0.076
(1, 0.95)	0.061	0.183	0.053	0.152	0.049	0.087
(1, 0.9)	0.076	0.409	0.067	0.367	0.076	0.245
(1, 0.8)	0.117	0.714	0.122	0.643	0.133	0.553
(1, 0.7)	0.133	0.869	0.154	0.816	0.183	0.732
(0.95, 1)	0.080	0.097	0.095	0.135	0.117	0.126
(0.9, 1)	0.119	0.104	0.165	0.166	0.201	0.195
(0.8, 1)	0.174	0.174	0.267	0.312	0.353	0.385
(0.7, 1)	0.265	0.290	0.377	0.478	0.459	0.611
<i>T = 200</i>						
(1, 1)	0.054	0.056	0.052	0.069	0.052	0.058
(1, 0.95)	0.091	0.309	0.080	0.260	0.092	0.183
(1, 0.9)	0.157	0.628	0.153	0.536	0.191	0.465
(1, 0.8)	0.214	0.916	0.218	0.846	0.255	0.745
(1, 0.7)	0.225	0.978	0.238	0.941	0.287	0.883
(0.95, 1)	0.103	0.090	0.140	0.136	0.202	0.180
(0.9, 1)	0.162	0.153	0.251	0.261	0.374	0.385
(0.8, 1)	0.311	0.328	0.486	0.577	0.594	0.758
(0.7, 1)	0.451	0.521	0.648	0.811	0.725	0.918
<i>T = 500</i>						
(1, 1)	0.053	0.056	0.044	0.064	0.045	0.070
(1, 0.95)	0.275	0.654	0.282	0.620	0.315	0.504
(1, 0.9)	0.327	0.944	0.375	0.901	0.402	0.816
(1, 0.8)	0.354	0.998	0.399	0.992	0.436	0.979
(1, 0.7)	0.359	1.000	0.406	1.000	0.461	0.994
(0.95, 1)	0.204	0.175	0.329	0.326	0.445	0.478
(0.9, 1)	0.365	0.398	0.551	0.702	0.684	0.851
(0.8, 1)	0.656	0.778	0.795	0.952	0.885	0.993
(0.7, 1)	0.793	0.937	0.893	0.995	0.938	1.000

As seen above, our finite-sample simulation shows that the power of LM^μ is very low, although it performs better than LM_d^μ in view of the asymptotic local power. This poor performance under the alternative in finite samples is partly because of the initial value condition. Table 4 summarizes the effect of the initial value on the power when $T = 1,000$. We see that LM^μ has reasonable power when $x_0 = 0$, whereas its power decreases dramatically when $x_0 = 10$, even if the sample size is 1,000. On the contrary, LM_d^μ seems to be robust to the initial value condition.¹ From these

¹We also checked the initial value effect for $T = 100, 200$ and 500, and this tendency also remains for these sample sizes.

TABLE 3
Size and power (trending case, λ^* is known)

(α_1, α_2)	$\lambda^* = 0.3$				$\lambda^* = 0.5$				$\lambda^* = 0.7$			
	LM^c	LM_d^c	LM^{QD}	LM_d^{QD}	LM^c	LM_d^c	LM^{QD}	LM_d^{QD}	LM^c	LM_d^c	LM^{QD}	LM_d^{QD}
<i>T = 100</i>												
(1, 1)	0.057	0.086	0.235	0.087	0.052	0.106	0.248	0.118	0.050	0.077	0.196	0.113
(1, 0.95)	0.072	0.133	0.200	0.103	0.058	0.130	0.204	0.129	0.050	0.063	0.129	0.086
(1, 0.9)	0.070	0.227	0.194	0.116	0.078	0.256	0.200	0.233	0.074	0.168	0.126	0.203
(1, 0.8)	0.135	0.384	0.251	0.185	0.213	0.506	0.265	0.435	0.346	0.466	0.269	0.512
(1, 0.7)	0.206	0.482	0.289	0.284	0.322	0.654	0.343	0.584	0.532	0.664	0.545	0.703
(0.95, 1)	0.085	0.101	0.283	0.128	0.077	0.136	0.286	0.162	0.081	0.109	0.243	0.150
(0.9, 1)	0.103	0.125	0.339	0.136	0.110	0.167	0.325	0.185	0.108	0.156	0.300	0.187
(0.8, 1)	0.158	0.190	0.416	0.185	0.166	0.248	0.424	0.260	0.177	0.257	0.408	0.314
(0.7, 1)	0.226	0.255	0.496	0.267	0.235	0.344	0.515	0.358	0.246	0.368	0.506	0.436
<i>T = 200</i>												
(1, 1)	0.065	0.061	0.236	0.062	0.060	0.071	0.211	0.084	0.054	0.062	0.150	0.090
(1, 0.95)	0.086	0.177	0.242	0.147	0.065	0.193	0.198	0.201	0.059	0.132	0.133	0.171
(1, 0.9)	0.159	0.354	0.299	0.252	0.183	0.422	0.282	0.441	0.263	0.383	0.180	0.426
(1, 0.8)	0.307	0.602	0.378	0.436	0.432	0.704	0.458	0.690	0.585	0.682	0.550	0.707
(1, 0.7)	0.361	0.690	0.419	0.511	0.537	0.811	0.568	0.775	0.697	0.815	0.740	0.854
(0.95, 1)	0.105	0.105	0.319	0.106	0.105	0.122	0.312	0.147	0.115	0.133	0.271	0.171
(0.9, 1)	0.159	0.152	0.398	0.148	0.166	0.211	0.417	0.224	0.186	0.252	0.411	0.299
(0.8, 1)	0.252	0.285	0.539	0.285	0.251	0.426	0.568	0.424	0.330	0.465	0.586	0.546
(0.7, 1)	0.344	0.441	0.625	0.398	0.342	0.536	0.653	0.561	0.380	0.560	0.649	0.669
<i>T = 500</i>												
(1, 1)	0.063	0.070	0.199	0.076	0.058	0.067	0.160	0.075	0.055	0.057	0.101	0.076
(1, 0.95)	0.268	0.485	0.398	0.449	0.302	0.518	0.305	0.551	0.331	0.434	0.157	0.489
(1, 0.9)	0.503	0.766	0.519	0.689	0.629	0.806	0.521	0.808	0.670	0.758	0.492	0.789
(1, 0.8)	0.595	0.852	0.581	0.753	0.755	0.927	0.704	0.910	0.850	0.941	0.783	0.940
(1, 0.7)	0.619	0.874	0.599	0.767	0.788	0.953	0.764	0.925	0.907	0.973	0.854	0.966
(0.95, 1)	0.181	0.170	0.383	0.176	0.192	0.237	0.411	0.249	0.234	0.289	0.399	0.348
(0.9, 1)	0.259	0.316	0.521	0.315	0.324	0.434	0.569	0.486	0.357	0.522	0.589	0.627
(0.8, 1)	0.386	0.560	0.672	0.521	0.408	0.604	0.655	0.686	0.420	0.624	0.662	0.796
(0.7, 1)	0.449	0.650	0.733	0.648	0.432	0.640	0.688	0.746	0.446	0.637	0.685	0.830

results, we recommend the use of the demeaned version test in practical analysis.

For the case, where the break point is unknown, we conducted the simulation for the same parameter settings as the known case with $\lambda^* = 0.5$, and tabulated the results in Tables 5 and 6. From our preliminary simulations, we found that the LM-type test and the test with the quasi-differencing method perform very poorly in finite samples and then, we report only the results of the demeaned version tests in the following.

Table 5 reports the simulation results when data have no trend. In the table, $LKSN^{\mu}$ denotes the ‘sequential statistic’ proposed by Leybourne *et al.* (2003).

TABLE 4
Effects of the initial value condition ($T = 1,000$, $\lambda^ = 0.5$ is known)*

(α_1, α_2)	$x_0 = 0$		$x_0 = 10$	
	LM^μ	LM_d^μ	LM^μ	LM_d^μ
(1, 1)	0.051	0.058	0.051	0.058
(1, 0.999)	0.060	0.064	0.068	0.073
(1, 0.995)	0.138	0.088	0.093	0.101
(1, 0.99)	0.280	0.175	0.099	0.201
(1, 0.95)	1.000	0.910	0.113	0.911
(1, 0.9)	1.000	1.000	0.140	0.999
(0.999, 1)	0.069	0.061	0.065	0.059
(0.995, 1)	0.126	0.079	0.067	0.062
(0.99, 1)	0.218	0.111	0.062	0.112
(0.95, 1)	0.764	0.662	0.032	0.747
(0.9, 1)	0.940	0.961	0.028	0.975

From the table, three tests proposed in this paper tend to slightly over-reject the null hypothesis when $T = 100$, while $LKSN^\mu$ has a large size distortion. As to the power, the $\overline{\text{exp}}$ -type test is more powerful than the inf -type and avg -type tests under H_1^{10} while the avg -type and $\overline{\text{exp}}$ -type tests have good performance under H_1^{01} . As a whole, $\overline{\text{exp}}\text{-}LM_d^\mu$ performs fairly well both under the null and the alternative. Therefore, we recommend using it when data have no trend.

The effect of ψ on the power of $\overline{\text{exp}}\text{-}LM_d^\mu$ is similar to the case where λ^* is known. The power tends to decrease as ψ becomes large, especially under H_1^{01} . The power is 0.480, 0.366 and 0.170 for $\psi = 0, 0.5$ and 0.8 , respectively, when $(\alpha_1, \alpha_2) = (0.8, 1)$ and $T = 200$.

The results when data are trending are reported in Table 6. Under the null hypothesis, the size of $\text{inf}\text{-}LM_d^{\text{QD}}$ is closest to the nominal size while $\text{avg}\text{-}LM_d^{\text{QD}}$ has a large size distortion, and the other tests have the similar performance under the null hypothesis. Under H_1^{10} , $\overline{\text{exp}}\text{-}LM_d^\tau$ outperforms the other tests, while the avg -type tests and $LKSN^\tau$ have low power when $T = 100$, although the power of the latter increases when $T = 200$ and 500 . On the contrary, under H_1^{01} , $\text{avg}\text{-}LM_d^\tau$ is most powerful when the alternative is close to the null hypothesis, while $LKSN^\tau$ becomes most powerful as the alternative tends to diverge from the null hypothesis. As a result, although none of the tests dominate others, $\overline{\text{exp}}\text{-}LM_d^\tau$, $\text{avg}\text{-}LM_d^\tau$ and $LKSN^\tau$ may be useful in practice for trending data.

The effect of ψ is similar to the non-trending case. The powers of these three statistics are 0.316, 0.406 and 0.511 for $\psi = 0, 0.225, 0.285$ and 0.409 for $\psi = 0.5$, and 0.232, 0.136 and 0.232 for $\psi = 0.8$, respectively, when $(\alpha_1, \alpha_2) = (0.8, 1)$ and $T = 200$.

TABLE 5
Size and power (non-trending case, $\lambda^ = 0.5$ is unknown)*

(α_1, α_2)	$inf-LM_d^\mu$	$avg-LM_d^\mu$	$\overline{exp-LM}_d^\mu$	$LKSN^\mu$
<i>T = 100</i>				
(1, 1)	0.080	0.089	0.098	0.179
(1, 0.95)	0.086	0.072	0.120	0.180
(1, 0.9)	0.190	0.093	0.268	0.222
(1, 0.8)	0.447	0.184	0.566	0.416
(1, 0.7)	0.630	0.337	0.750	0.635
(0.95, 1)	0.084	0.113	0.108	0.203
(0.9, 1)	0.099	0.146	0.125	0.279
(0.8, 1)	0.192	0.259	0.240	0.530
(0.7, 1)	0.299	0.388	0.383	0.758
<i>T = 200</i>				
(1, 1)	0.047	0.069	0.059	0.111
(1, 0.95)	0.114	0.089	0.170	0.142
(1, 0.9)	0.312	0.176	0.416	0.288
(1, 0.8)	0.645	0.481	0.791	0.750
(1, 0.7)	0.855	0.815	0.965	0.966
(0.95, 1)	0.100	0.127	0.113	0.205
(0.9, 1)	0.161	0.243	0.192	0.418
(0.8, 1)	0.361	0.543	0.480	0.852
(0.7, 1)	0.575	0.775	0.793	0.984
<i>T = 500</i>				
(1, 1)	0.046	0.071	0.053	0.079
(1, 0.95)	0.335	0.242	0.465	0.285
(1, 0.9)	0.721	0.629	0.862	0.765
(1, 0.8)	0.977	0.992	1.000	0.988
(1, 0.7)	1.000	1.000	1.000	0.998
(0.95, 1)	0.191	0.287	0.213	0.423
(0.9, 1)	0.447	0.656	0.604	0.833
(0.8, 1)	0.812	0.974	0.983	0.986
(0.7, 1)	0.955	0.999	1.000	1.000

V. Empirical examples

In this section, we present an empirical application of the tests proposed in the previous sections. We investigate the yen/dollar exchange rate (January 1974 to June 2001; denoted by 'Yen'), which is drawn in Figure 3. We calculated the test statistics for both models with/without a linear trend. Taking account of the finite-sample results investigated in the previous section, we calculated only $\overline{exp-LM}_d^\mu$ for the non-trending case and $avg-LM_d^\tau$, $\overline{exp-LM}_d^\tau$ and $LKSN^\tau$ for the trending case. The results are summarized in Table 7. The symbol ' μ ' in parentheses signifies that the model is estimated without a linear trend, while the symbol ' τ ' implies that a linear trend is included as a regressor.

TABLE 6
Size and power (trending case, $\lambda^ = 0.5$ is unknown)*

(α_1, α_2)	inf-LM_d^c	avg-LM_d^c	$\overline{\text{exp-LM}}_d^c$	inf-LM_d^{OD}	avg-LM_d^{OD}	$\overline{\text{exp-LM}}_d^{OD}$	LKSN ^c
<i>T = 100</i>							
(1, 1)	0.082	0.085	0.096	0.047	0.140	0.072	0.080
(1, 0.95)	0.084	0.071	0.114	0.048	0.110	0.090	0.074
(1, 0.9)	0.133	0.075	0.197	0.075	0.118	0.155	0.084
(1, 0.8)	0.275	0.110	0.421	0.159	0.148	0.330	0.165
(1, 0.7)	0.427	0.166	0.571	0.286	0.205	0.450	0.256
(0.95, 1)	0.095	0.133	0.119	0.060	0.187	0.093	0.085
(0.9, 1)	0.131	0.160	0.150	0.081	0.227	0.112	0.110
(0.8, 1)	0.172	0.234	0.202	0.106	0.316	0.161	0.220
(0.7, 1)	0.212	0.324	0.270	0.140	0.394	0.218	0.378
<i>T = 200</i>							
(1, 1)	0.072	0.074	0.077	0.049	0.118	0.060	0.072
(1, 0.95)	0.095	0.069	0.140	0.072	0.118	0.122	0.079
(1, 0.9)	0.217	0.120	0.336	0.170	0.173	0.308	0.131
(1, 0.8)	0.487	0.266	0.616	0.402	0.306	0.561	0.368
(1, 0.7)	0.665	0.423	0.778	0.581	0.440	0.717	0.732
(0.95, 1)	0.097	0.139	0.117	0.052	0.191	0.086	0.107
(0.9, 1)	0.137	0.214	0.155	0.085	0.276	0.120	0.175
(0.8, 1)	0.246	0.406	0.316	0.162	0.500	0.283	0.511
(0.7, 1)	0.359	0.510	0.469	0.283	0.615	0.453	0.802
<i>T = 500</i>							
(1, 1)	0.054	0.069	0.052	0.042	0.100	0.053	0.061
(1, 0.95)	0.274	0.173	0.398	0.241	0.215	0.377	0.135
(1, 0.9)	0.615	0.396	0.749	0.584	0.458	0.712	0.502
(1, 0.8)	0.864	0.775	0.930	0.849	0.765	0.902	0.964
(1, 0.7)	0.922	0.870	0.950	0.890	0.840	0.930	0.997
(0.95, 1)	0.135	0.232	0.154	0.104	0.308	0.155	0.217
(0.9, 1)	0.267	0.430	0.340	0.209	0.543	0.346	0.613
(0.8, 1)	0.496	0.626	0.600	0.448	0.750	0.654	0.974
(0.7, 1)	0.602	0.683	0.663	0.595	0.805	0.751	1.000

The lag length is decided by testing the significance of the coefficient estimate of y_{t-p-1} in the regression of y_t on $y_{t-1}, \dots, y_{t-p-1}$ and a constant or a constant and a linear trend. We assume that the maximum lag length ($p + 1$) is at most 18 and the test is continued until the coefficient estimate of y_{t-p-1} becomes significant at the 5% significance level. The lag length of 4 is selected for both models with/without a linear trend.

First, we consider the model without a linear trend. $\overline{\text{exp-LM}}_d$ rejects the null of a unit root in the whole period at the 10% significance level, which weakly implies a change in the long-run persistence. This is consistent with the results of the Buseti and Taylor (2004) and Leybourne and Taylor (2004) tests, which are designed to test the null of (trend) stationarity against



Figure 3. The yen-dollar exchange rate

a change in persistence. In Table 7, K_j for $j = 1, 2$ and 3 correspond to $\mathcal{H}_j^*(0)$ statistics proposed by Leybourne and Taylor (2004, p. 110), while BT_j for $j = 1, 2$ and 3 denote the Busetti and Taylor's tests given by equation (4.13) in their paper with $m = [12(T/100)^{1/4}] = 16$. Critical values of these statistics are given by Table 1 of Busetti and Taylor (2004). Four of these tests reject the null of stationarity and again, these results imply a change in persistence. We also conducted the augmented Dickey-Fuller (ADF)-GLS test proposed by Elliott *et al.* (1996) in the two sub-periods. The subscripts f and ℓ signify that the test is conducted in the former and the latter sub-periods, respectively. The whole sample is split at the date where the demeaned-type test, LM_d^μ , is minimized. In this case, LM_d^μ is minimized at September 1985, which is the same month as the Plaza Agreement, and the ADF-GLS test rejects the null of a unit root at the 10% and 1% significance levels before and after the break date, respectively. Judging from these results for the model without a linear trend, the exchange rate does not seem to have a unit-root in the whole period, and there is evidence of a change in persistence, although this tendency is not necessarily strong.

On the contrary, when we include a linear trend in the regressors, only K_3 rejects the null of no change in persistence at the 10% significance level and

TABLE 7
Summary of the test statistics

	<i>Yen</i> (μ)	<i>Yen</i> (τ)
avg-LM _d	—	5.46
exp-LM _d	16.62*	6.63
LKSN	—	-2.77
K ₁	24.15**	5.11
K ₂	2.56	1.09
K ₃	19.43***	2.42*
BT ₁	1.82*	0.34
BT ₂	0.64	0.08
BT ₃	0.72**	-0.37
Date	Sep.1985	Feb.1985
ADF-GLS _f	-1.78*	-1.54
ADF-GLS _l	-4.18***	-3.99***

Notes: The symbols *, ** and *** denote significance at the 10%, 5%, and 1% levels, respectively.

neither the null of a unit root nor the null of stationarity can be rejected by the other tests. One of the possible reasons for this discrepancy is that each test becomes less powerful when a linear trend is used. However, the ADF-GLS test rejects the unit-root hypothesis in the latter sub-period at the 1% significance level while the test does not reject it in the former sub-period, where the sample is split in February 1985, at which LM_d^c is minimized. Thus using the model with a linear trend, our conclusions are mixed. However, we can at least say that there seems to be no strong evidence supporting the unit-root hypothesis for the whole sample period.

VI. Concluding remarks

In this paper, we have investigated a test of a change in the long-run persistence in a univariate time series. We first proposed two types of tests, one is the LM-type test and the other is the 'demeaned version' of the LM-type test. We also considered the tests with the quasi-differencing method. From the Monte Carlo simulation, we found that the demeaned versions of the LM-type tests perform better than the tests with the quasi-differencing method. However, this result does not necessarily indicate that the quasi-differencing method is not useful for the test of a change in the long-run persistence. In our setting, the difficulty is that we have to decide *two* local parameters, θ_1 and θ_2 . We chose these parameters so that the power is tangent to the envelope at a power of 50% both under $H_1^{01\ell}$ and $H_1^{10\ell}$, but we can construct other tests using a different setting of θ_1 and θ_2 , which might have better finite-sample properties. One of the possibilities is that, as used in

Leybourne *et al.* (2003), we set $\theta = \theta_1 = \theta_2$ and investigate the performance of the test by changing one dimensional parameter θ . The choice of multiple local parameters need to be studied in the future.

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Appendix

We only outline the proofs here. Full details are available upon request.

Proof of Theorem 1.

(i) $\Delta\tilde{x}_t$ can be expressed as

$$\Delta\tilde{x}_t = \underline{\psi}'\tilde{z}_{t-1} - \psi(1)(\tilde{\mu}_1 - \mu_1) + u_t^*,$$

where $u_t^* = u_t - (1 - \alpha_t)\psi(L)x_{t-1}$ (31)

$\underline{\psi} = [\psi_1, \dots, \psi_p]'$ and $\tilde{z}_{t-1} = [\Delta\tilde{x}_{t-1}, \dots, \Delta\tilde{x}_{t-p}]'$. Using equation (31) it is shown that the product moment matrix of the regressors, $[D_{1t}\tilde{x}'_{t-1}, D_{2t}\tilde{x}'_{t-1}, \tilde{z}'_{t-1}]'$, with an appropriate weighting matrix is shown to be asymptotically block-diagonal, so that the t -statistics for ρ_1 and ρ_2 can be expressed as

$$t_i^*(\lambda^*) = \frac{T^{-1} \sum_{\mathcal{F}_i} \tilde{x}_{t-1} \{u_t^* - \psi(1)(\tilde{\mu}_1 - \mu_1)\}}{\sqrt{T^{-2} \tilde{\sigma}^2 \sum_{\mathcal{F}_i} \tilde{x}_{t-1}^2}} + o_p(1), \quad (32)$$

for $i = 1$ and 2 , where $\mathcal{F}_1 = \{t : 1 \leq t \leq T_B^*\}$ and $\mathcal{F}_2 = \{t : T_B^* + 1 \leq t \leq T\}$.

We first derive the limiting distribution of the denominator. Under $H_1^{10\ell}$ we have

$$T^{-1/2}x_{[Tr]} \Rightarrow \begin{cases} \sigma\psi^{-1}W(r) & 0 \leq r \leq \lambda^* \\ \sigma\psi^{-1}\tilde{V}(r) & \lambda^* \leq r \leq 1 \end{cases} \quad (33)$$

by the functional central limit theorem (FCLT), where $\psi = \psi(1)$ and ‘ \Rightarrow ’ signifies weak convergence. As $\tilde{\mu}_1 = \mu_1 + T^{-1}(x_T - x_0)$, this result induces that

$$T^{1/2}(\tilde{\mu}_1 - \mu_1) \Rightarrow \sigma\psi^{-1}\tilde{V}(1). \tag{34}$$

Noting that $\tilde{x}_t = x_t - (\tilde{\mu}_1 - \mu_1)t + \mu_0 - y_0$ and combining equations (33) and (34), we obtain $T^{-1/2}\tilde{x}_{[Tr]} \Rightarrow \sigma\psi^{-1}V(r)$. From this result and the continuous mapping theorem (CMT), we have

$$\begin{aligned} T^{-2} \sum_{t=1}^{T_B^*} \tilde{x}_{t-1}^2 &\Rightarrow \sigma^2\psi^{-2} \int_0^{\lambda^*} V^2(s)ds, \\ T^{-2} \sum_{t=T_B^*+1}^T \tilde{x}_{t-1}^2 &\Rightarrow \sigma^2\psi^{-2} \int_{\lambda^*}^1 V^2(s)ds. \end{aligned} \tag{35}$$

To show the convergence of the numerators of $t_1^{\tau}(\lambda^*)$ and $t_2^{\tau}(\lambda^*)$, we define

$$\tilde{S}_t = \sum_{j=1}^t (u_j^* - \bar{u}^*) = \sum_{j=1}^t u_j^* - \bar{u}^*t, \tag{36}$$

where

$$\bar{u}^* = T^{-1} \sum_{t=1}^T u_t^*.$$

Note that

$$T^{-1/2} \sum_{t=1}^{[Tr]} u_t^* \Rightarrow \sigma W(r) \quad \text{for } 0 \leq r \leq \lambda^*$$

while

$$T^{-1/2} \sum_{t=1}^{[Tr]} u_t^* \Rightarrow \sigma \tilde{V}(r) \quad \text{for } \lambda^* \leq r \leq 1.$$

Then, we can see that $T^{-1/2}\tilde{S}_{[Tr]} \Rightarrow \sigma V(r)$ for $0 \leq r \leq 1$. It is also shown that

$$T^{-1} \sum_{t=1}^{T_B^*} (u_t - \bar{u})^2 \rightarrow \sigma^2 \lambda^*.$$

Then,

$$\begin{aligned} &T^{-1} \sum_{t=1}^{T_B^*} \tilde{x}_{t-1} \{u_t^* - \psi(1)(\tilde{\mu}_1 - \mu_1)\} \\ &= (\psi T)^{-1} \sum_{t=1}^{T_B^*} \tilde{S}_{t-1} (u_t^* - \bar{u}^*) + o_p(1) \xrightarrow{d} \sigma^2 (2\psi)^{-1} (V^2(\lambda^*) - \lambda^*) \end{aligned} \tag{37}$$

by the FCLT and CMT. As $\tilde{\sigma}^2$ is shown to converge in probability to σ^2 under local alternatives, we have $t_1^\tau(\lambda^*) \xrightarrow{d} S_1(\lambda^*)$. Similarly, we can get $t_2^\tau(\lambda^*) \xrightarrow{d} S_2(\lambda^*)$. The theorem is established using these results.

- (ii) The limiting distribution of the demeaned version statistic and the result of (ii) can be derived in the same way as the proof of (i).
- (iii) Under $H_1^{01\ell}$ we have

$$T^{-1/2}x_{[Tr]} \Rightarrow \begin{cases} \sigma\psi^{-1}\tilde{W}(\theta_1^*/\lambda^*, r) & 0 \leq t \leq \lambda^* \\ \sigma\psi^{-1}\tilde{V}(r) \equiv \sigma\psi^{-1}(\tilde{W}(\theta_1^*/\lambda^*, \lambda^*) + W(r) - W(\lambda^*)) & \lambda^* \leq t \leq 1. \end{cases} \tag{38}$$

Then, in the same way as the proof of (i), we can establish the theorem. ■

Proof of Theorem 2. This theorem is proved in the same way as in Leybourne *et al.* (2003). First we derive the asymptotic behaviour of $\tilde{\mu}_0^{QD}$ and $\tilde{\mu}_1^{QD}$.

Lemma 1.

- (i) Under $H_1^{10\ell}$, $(\tilde{\mu}_0^{QD} - \mu_0) \xrightarrow{d} x_1$ and $T^{1/2}(\tilde{\mu}_1^{QD} - \mu_1) \xrightarrow{d} \sigma\psi^{-1}J_1/\delta$.
- (ii) Under $H_1^{01\ell}$, $(\tilde{\mu}_0^{QD} - \mu_0) \xrightarrow{d} x_1$ and $T^{1/2}(\tilde{\mu}_1^{QD} - \mu_1) \xrightarrow{d} \sigma\psi^{-1}J_2/\delta$.

Using equations (33), (38) and the above lemma, we have, under $H_1^{10\ell}$,

$$T^{-1/2}\tilde{x}_{[Tr]}^{QD} \Rightarrow \begin{cases} \sigma\psi^{-1}(W(r) - rJ_1) & 0 \leq t \leq \lambda^* \\ \sigma\psi^{-1}(\tilde{V}(r) - rJ_1) & \lambda^* \leq t \leq 1 \end{cases} \tag{39}$$

while under $H_1^{01\ell}$,

$$T^{-1/2}\tilde{x}_{[Tr]}^{QD} \Rightarrow \begin{cases} \sigma\psi^{-1}(\tilde{W}(\theta_1^*/\lambda^*, r) - rJ_2) & 0 \leq t \leq \lambda^* \\ \sigma\psi^{-1}(\tilde{V}(r) - rJ_2) & \lambda^* \leq t \leq 1. \end{cases} \tag{40}$$

Then, the statement of the theorem is proved exactly in the same way as the proof of the previous theorem. ■

Proof of Theorem 3. As shown in Theorems 1 and 2, $t_1^\tau(\lambda)$ and $t_2^\tau(\lambda)$ converge in distribution to $S_1(\lambda)$ and $S_2(\lambda)$ under H_0' . In the same way as Gregory and Hansen (1996), these convergences hold uniformly in λ . As inf-LM $^\tau$, avg-LM $^\tau$ and $\overline{\text{exp-LM}}^\tau$ are continuous functions of $S_1(\lambda)$ and $S_2(\lambda)$, the limiting distribution is obtained by the CMT. ■

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