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A nonparametric test for the change of the density function in strong mixing processes

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Abstract

In this paper, we consider the problem of testing for a change of the marginal density of a strong mixing process. The test statistic is constructed based on the sequential kernel estimate. In order to derive the asymptotic distribution of the test statistic, we first show that a functional central limit theorem holds for the sequential density estimator under some regularity conditions. Based on the result, we show that the limiting distribution of the test statistic is a function of independent Brownian bridges.

MSC: Primary 62M07; secondary 62G09

Keywords: A change point problem; Sequential density estimate; Strong mixing processes; Functional central limit theorem

1. Introduction

Since Page (1955), the problem of testing for a parameter change in random samples has generated much interest in many areas of statistics. For a review of earlier work, readers are referred to Hinkley (1971), Brown et al. (1975), Zacks (1983), Csörgő and Horváth (1988), Krishaiah and Miao (1988), Brodsky and Darkhovsky (1993) and Csörgő and Horváth (1997). Since economic time series are frequently affected by monetary policy and critical social events, the change point problem in time series models has attracted considerable attention from many authors. See, for example,

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Wichern et al. (1976), Picard (1985), Krämer et al. (1988), Tang and MacNeil (1993), Inclán and Tiao (1994), Antoch et al. (1997), Lee and Park (2001), and the papers cited in those articles.

Bai (1994) considered the problem of testing for a distributional change of the errors in stationary ARMA models and proposed a test based on the residual sequential empirical process. See also Kanagawa et al. (1997). The concept behind Bai's approach is to compare the differences between the empirical process based on the first part of the residuals and that based on the rest of the residuals, and accept the existence of a distributional change when the differences are large. In his paper, the iid assumption imposed on the errors plays an important role since in that case the residual sequential empirical process converges weakly to a Kiefer process. However, his result does not hold for dependent observations since the relevant empirical process has a limiting process whose covariance structure depends upon their autocorrelations. As pointed out by Durbin (1973) and Lee and Wei (1999), this causes serious problems in calculating the critical value for a given significance level. It is well known that the difficulties arising in the empirical process from dependent observations are often circumvented by adopting the density estimation approach. For example, see Takahata and Yoshihara (1987) and Lee and Na (2000). Motivated by this viewpoint, we consider a procedure to detect a distributional change based on density estimates.

This paper is devoted to testing for a change of the marginal density of dependent observations. In particular, we focus on strong mixing processes since they cover a large class of stationary processes including the famous ARMA and GARCH models. The organization of this paper is as follows. In Section 2, we provide the main result of this paper: we introduce the sequential kernel estimate of the marginal density in the same spirit of Bai, and construct a basic process from which a test statistic is generated. Then a functional central limit theorem for the sequential kernel estimate is established under certain regularity conditions. Based on this result, the limiting distribution of the test statistic is derived. Finally, in Section 3 the proofs of the results presented in Section 2 are provided.

2. Main results

Let $\{X_t, t \geqslant 1\}$ be a stationary strong mixing process satisfying

$$\alpha(k) = \sup\{|P(A \cap B) - P(A)P(B)| : A \in \mathscr{M}_1^t, B \in \mathscr{M}_{t+k}^{\infty}\} \to 0, \tag{2.1}$$

where $\mathscr{M}_a^b = \sigma(X_a, \ldots, X_b)$ is the σ-field generated by $X_a, \ldots, X_b$ and $1 \leqslant a < b \leqslant \infty$. Suppose that one wishes to test a change of the marginal density of $\{X_t\}$. Towards this end, we set up the null and alternative hypotheses:

H_0: $X_1, \ldots, X_n$ have a common marginal density f vs.
H_1: For $0 < \theta < 1$,
(i) $X_1, \ldots, X_{[n\theta]}$ have a common marginal density f_1,
(ii) $X_{[n\theta]+1}, \ldots, X_n$ have a common marginal density f_2,

where f, f_1 and f_2 are all assumed unknown.

For $0 \leqslant s \leqslant 1$ and $x \in \mathbb{R}$, the sequential density estimate is defined by

$$f_{[ns]}(x) = \frac{1}{[ns]h} \sum_{j=1}^{[ns]} K\left(\frac{x - X_j}{h}\right),$$

where h is a bandwidth satisfying

$$h = h_n \to 0, \quad nh \to \infty \text{ as } n \to \infty,$$

and K is a kernel function.

When $s = 1$, $f_{[ns]}(x)$ becomes the usual kernel density estimate $f_n(x)$. Provided that H_1 holds, $f_{[ns]}$ can be viewed as an estimate of f_1 in (i). Similarly,

$$f^*_{n-[ns]}(x) = \frac{1}{(n-[ns])h} \sum_{j=[ns]+1}^{n} K\left(\frac{x - X_j}{h}\right)$$

indicates an estimate of f_2. Therefore, it is natural to consider $f_{[ns]}(x) - f^*_{n-[ns]}(x)$ as an estimate of $f_1(x) - f_2(x)$. Keeping this in mind, for $0 \leqslant s \leqslant 1$ and $x \in \mathbb{R}$, we define

$$d_n(s,x) = \left(\frac{nh}{f_n(x)\|K\|^2}\right)^{1/2} \frac{[ns]}{n} \left(\frac{n-[ns]}{n}\right) (f_{[ns]}(x) - f^*_{n-[ns]}(x)), \tag{2.2}$$

provided $f_n(x) \neq 0$. If $f_n(x) = 0$, (2.2) is defined to be zero. Here, $\|K\|^2$ denotes $\int_{\mathbb{R}} K^2(t)\,\mathrm{d}t$ which is assumed to be finite. A large value of $d_n(s,x)$ will indicate a situation in favor of H_1.

Provided that H_0 is true, we can define a partial sum process, which provides an insight for $d_n(s,x)$,

$$\begin{aligned} g_n(s,x) &= (f_n(x)\|K\|^2/nh)^{-1/2} \frac{[ns]}{n} (f_{[ns]}(x) - Ef_{[ns]}(x)) \\ &= (nhf_n(x)\|K\|^2)^{-1/2} \sum_{j=1}^{[ns]} \left(K\left(\frac{x-X_j}{h}\right) - EK\left(\frac{x-X_j}{h}\right)\right) \end{aligned}$$

if $f_n(x) > 0$, and $g_n(s,x) = 0$, otherwise. Then we can write

$$d_n(s,x) = g_n(s,x) - \frac{[ns]}{n} g_n(1,x). \tag{2.3}$$

Note that for fixed x, the partial sum process $\{g_n(s,x), 0 \leqslant s \leqslant 1\}$ can be viewed as a random element in the D space generated from a double array of random variables $\{K((x - X_j)/h),\ j = 1,\dots,n,\ n = 1,2,\dots\}$. It is well known that for independent $X_1,\dots,X_n$ and for fixed x, the process $\{g_n(s,x),\ 0 \leqslant s \leqslant 1\}$ converges weakly to a standard Brownian motion if and only if the sequence $\{g_n(1,x),\ n = 1,2,\dots\}$ converges to a standard normal random variable in distribution (cf. Billingsley, 1999, pp. 147–148). We will see later that an analogous result can be shown to hold in a strong mixing process.

Before we state our main results in this paper, we need to list some conditions imposed on the kernel function K and the density f.

(K1) The kernel function K is a symmetric density function such that

$$\|K\|_\infty = \sup_{t \in \mathbb{R}} K(t) < \infty, \quad \lim_{t \to \infty} tK(t) = 0, \quad \int_{\mathbb{R}} t^2 K(t)\,\mathrm{d}t < \infty.$$

(D1) The marginal density f is positive, twice continuously differentiable and

$$\sup_x f(x) < \infty, \quad \sup_x |f''(x)| < \infty.$$

(D2) For each $t_1 < t_2$, the joint density f_{t_1,t_2} of (X_{t_1}, X_{t_2}) exists and

$$\sup_{t_1<t_2} \sup_{x_1,x_2} f_{t_1,t_2}(x_1,x_2) < \infty,$$

and for each $t_1 < t_2 < t_3 < t_4$, the joint density f_{t_1,t_2,t_3,t_4} of $(X_{t_1}, X_{t_2}, X_{t_3}, X_{t_4})$ exists and

$$\sup_{t_1<\cdots<t_4} \sup_{x_1,\dots,x_4} f_{t_1,t_2,t_3,t_4}(x_1,x_2,x_3,x_4) < \infty.$$

Remark 1. Condition (K1) is assumed in many studies, and Condition (D1) is satisfied by a large class of densities including normal densities. For independent samples, Conditions (K1) and (D1) are sufficient to guarantee the consistency of the density estimator. However, for the strong mixing case, we need the additional Condition (D2) (cf. Bosq, 1996, Theorems 2.1 and 2.3).

The following result shows the weak convergence of $\{d_n(s,x),\ 0 \leqslant s \leqslant 1\}$ to a Brownian bridge.

Theorem 2.1. *Suppose that* (K1) *and* (D1)–(D2) *hold, the strong mixing coefficient* $\alpha(k)$ *defined in* (2.1) *satisfies*

$$\alpha(k) = O(k^{-\gamma}) \quad \text{for some } \gamma \geqslant 3$$

and the bandwidth $h = h_n$ *satisfies*

$$h_n = c_n n^{-1/5}, \tag{2.4}$$

where c_n *is any sequence with* $c_n \to \infty$ *and* $c_n n^{-\delta} \to 0$ *for all* $\delta > 0$. *Then under* H_0, *as* $n \to \infty$,

$$d_n(s,x) \xrightarrow{\mathrm{w}} W^0(s),$$

where $\{W^0(s),\ 0 \leqslant s \leqslant 1\}$ *is a Brownian bridge.*

Remark 2. Condition (2.4) is a usual condition in the density estimation context and $\log n$ can be considered as a possible candidate for c_n. The order except for c_n is optimal in a minimax sense (Bosq, 1996, p. 44).

If we define

$$T_n(x) = \max_{0 \leqslant s \leqslant 1} |d_n(s,x)|,$$

it is transparent that a large value of $T_n(x)$ indicates the distinctness of $f_1(x)$ and $f_2(x)$, which implies the existence of a change point. However, judging from only one point can lead to a false conclusion since a small value of $T_n(x)$ at only one value of x does not necessarily mean the coincidence of f_1 and f_2. Hence we should consider $T_n(x)$ at several values of x simultaneously. The following theorem concerns the weak convergence of $d_n(s,x)$ for finitely many values of x.

Theorem 2.2. *Let* $x_1,\dots,x_m$ *be distinct real numbers. Under the conditions in Theorem* 2.1, *as* $n\to\infty$,

$$(d_n(s,x_1),\dots,d_n(s,x_m)) \xrightarrow{\mathrm{w}} (W_1^0(s),\dots,W_m^0(s)), \tag{2.5}$$

where $W_1^0,\dots,W_m^0$ *are independent Brownian bridges.*

Remark 3. It seems impossible to guarantee the weak convergence of $\{d_n(s,x),\ 0\leqslant s\leqslant 1, x\in(-\infty,\infty)\}$ since random elements in the D space, all of whose projections to Euclidean space form independent random variables, may not exist.

Now we are ready to define the test statistic. For distinct $x_1,\dots,x_m$, we define

$$T_n = \max_{1\leqslant i\leqslant m} T_n(x_i).$$

Then we obtain the main result of this paper.

Theorem 2.3. *Assume that the conditions of Theorem* 2.1 *hold.*

(1) *Under* H_0, *as* $n\to\infty$,

$$T_n \xrightarrow{\mathrm{d}} \max_{1\leqslant i\leqslant m}\sup_{0\leqslant s\leqslant 1}|W_i^0(s)|,$$

where $W_1^0,\dots,W_m^0$ *are independent Brownian bridges. We reject* H_0 *if* T_n *is large.*

(2) *Suppose that both* f_1 *and* f_2 *satisfy Conditions* (D1) *and* (D2). *Then under* H_1, *as* $n\to\infty$,

$$T_n \xrightarrow{\mathrm{P}} \infty$$

if

$$f_1(x_i)\neq f_2(x_i) \quad \text{for some } x_i\in\{x_1,\dots,x_m\}.$$

The above theorem demonstrates that T_n constitutes a consistent test for testing H_0 vs. H_1. It is worthwhile to point out that one has to fix the points $x_1,\dots,x_m$; note that if the number of $A_n=\{x_{n1},\dots,x_{nm}\}$ goes to ∞ as $n\to\infty$, T_n defined over A_n diverges to ∞ with probability 1 under H_0, viz., $T_n=\max_{x\in A_n}T_n(x)\xrightarrow{\mathrm{P}}\infty$.

3. Proofs

Since Theorem 2.1 is a special case of Theorem 2.2, we only prove the latter.

Proof of Theorem 2.2. It suffices to show that

$$(g_n(s,x_1),\dots,g_n(s,x_m)) \xrightarrow{\mathrm{w}} (W_1(s),\dots,W_m(s)), \tag{3.1}$$

where $W_1, \ldots, W_m$ are independent standard Brownian motions, since (3.1) implies (2.5) in view of (2.3). In order to prove (3.1), we use the Cramer–Wold device to verify that

$$G_n(s) = \sum_{i=1}^{m} \lambda_i g_n(s, x_i) \xrightarrow{\mathrm{w}} \sum_{i=1}^{m} \lambda_i W_i(s) \tag{3.2}$$

for all $\lambda_1, \ldots, \lambda_m$.

Towards this end, we will introduce the processes approximating $\{G_n(s)\}$: $\{G_n^{(1)}(s)\}$, $\{G_n^{(2)}(s)\}$ and $\{G_n^{(3)}(s)\}$ in the following. The last one will give the weak convergence result in (3.2) as we will see later.

Replacing $f_n(x_i)$ by $f(x_i)$, we define

$$g_n'(s, x_i) = (nhf(x_i)\|K\|^2)^{-1/2} \sum_{j=1}^{[ns]} \left(K\left(\frac{x_i - X_j}{h}\right) - EK\left(\frac{x_i - X_j}{h}\right)\right), \quad i = 1, \ldots, m,$$

and

$$\begin{aligned} G_n^{(1)}(s) &= \sum_{i=1}^{m} \lambda_i g_n'(s, x_i) \\ &= (nh)^{-1/2} \sum_{j=1}^{[ns]} \sum_{i=1}^{m} \frac{\lambda_i}{(f(x_i)\|K\|^2)^{1/2}} \left(K\left(\frac{x_i - X_j}{h}\right) - EK\left(\frac{x_i - X_j}{h}\right)\right) \\ &\stackrel{\text{let}}{=} (nh)^{-1/2} \sum_{j=1}^{[ns]} K_{nj}. \end{aligned}$$

We split $\{K_{nj}\}$ into blocks of large size $p = p_n$ and small size $q = q_n$. For this task, we define $Y_{n1}, Y_{n1}', \ldots, Y_{nr}, Y_{nr}'$ as follows:

$$\begin{aligned} &Y_{n1} = K_{n1} + \cdots + K_{np}, \;\; Y_{n1}' = K_{n,p+1} + \cdots + K_{n,p+q}, \\ &\vdots \\ &Y_{nr} = K_{n,(r-1)(p+q)+1} + \cdots + K_{n,(r-1)(p+q)+p}, \\ &Y_{nr}' = K_{n,(r-1)(p+q)+p+1} + \cdots + K_{n,r(p+q)}, \end{aligned}$$

where $r = r_n = [n/(p+q)]$. Assume that $p = n^a$ and $q = n^b$, where the numbers a and b satisfy:

$$0 < b < a < \frac{2}{5}, \quad b > \frac{1}{\gamma}\left(\frac{13}{10} - a\right), \quad a\left(\gamma + \frac{3}{2}\right) > \frac{13}{10} + \frac{b}{2}. \tag{3.3}$$

Rewrite $G_n^{(1)}(s)$ as

$$G_n^{(1)}(s) = (nh)^{-1/2} \left(\sum_{k=1}^{u} Y_{nk} + \sum_{k=1}^{u} Y_{nk}' + K_{n,u(p+q)+1} + \cdots + K_{n,[ns]} \right),$$

where $u = u_n(s)$ is the largest integer such that

$$(p+q)u \leqslant [ns]. \tag{3.4}$$

Note that by (K1)

$$\|Y_{n1}\|_\infty = \text{esssup}\,|Y_{n1}| \leqslant 2p\sum_{i=1}^{m} \frac{|\lambda_i|\,\|K\|_\infty}{(f(x_i)\|K\|^2)^{1/2}} < \infty.$$

Owing to Bradley's coupling theorem for strong mixing random variables (cf. Bosq, 1996, p. 18), there exist iid random variables $Z_{n1},\ldots,Z_{nr}$, such that Z_{n1} has the same distribution as Y_{n1} and

$$P(|Y_{nk}-Z_{nk}| \geqslant \frac{\sqrt{nh}}{r\log n}) \leqslant 11\left(\frac{\|Y_{n1}\|_\infty}{\sqrt{nh}}\, r\log n\right)^{1/2}\alpha(q), \quad k=1,\ldots,r. \tag{3.5}$$

We define

$$G_n^{(2)}(s) = (nh)^{-1/2}\sum_{k=1}^{u} Z_{nk},$$

where u is the largest integer satisfying (3.4). We claim that

$$\sup_{0\leqslant s\leqslant 1} |G_n^{(1)}(s) - G_n^{(2)}(s)| = o_P(1). \tag{3.6}$$

We write

$$G_n^{(1)}(s) - G_n^{(2)}(s) = (nh)^{-1/2}\left(\sum_{k=1}^{u}(Y_{nk}-Z_{nk}) + \sum_{k=1}^{u} Y'_{nk} + K_{n,u(p+q)+1} + \cdots + K_{n,[ns]}\right).$$

Then,

$$\sup_{0\leqslant s\leqslant 1} |G_n^{(1)}(s) - G_n^{(2)}(s)| \leqslant (nh)^{-1/2}\left(\sum_{k=1}^{r}|Y_{nk}-Z_{nk}| + \max_{1\leqslant u\leqslant r}\left|\sum_{k=1}^{u} Y'_{nk}\right| + \sup_{0\leqslant s\leqslant 1}|K_{n,u(p+q)+1} + \cdots + K_{n,[ns]}|\right). \tag{3.7}$$

First, note that by the first condition of (3.3), (2.4) and (K1),

$$(nh)^{-1/2}\sup_{0\leqslant s\leqslant 1}|K_{n,u(p+q)+1} + \cdots + K_{n,[ns]}| = o_P(1), \tag{3.8}$$

since at most $p+q$ terms are involved in the above equation. Next, using (3.5) we obtain that for each $\varepsilon > 0$,

$$P((nh)^{-1/2}\sum_{k=1}^{r}|Y_{nk}-Z_{nk}| \geqslant \varepsilon) \leqslant rP(|Y_{n1}-Z_{n1}| \geqslant \varepsilon\sqrt{nh}/r)$$
$$= O(r^{5/4}p^{1/4}\alpha(q)h^{-1/4}(\log n)^{1/2}).$$

This together with the second condition of (3.3) yields

$$(nh)^{-1/2}\sum_{k=1}^{r}|Y_{nk}-Z_{nk}| = o_P(1). \tag{3.9}$$

Now it remains [for us to deal with the second term in (3.7). Let $Z'_{n1},\dots,Z'_{nr}$ be iid random variables such that Z'_{n1} has the same distribution as Y'_{n1} and

$$P\left(|Y'_{nk}-Z'_{nk}|\geqslant\frac{\sqrt{nh}}{r\log n}\right)\leqslant 11\left(\frac{\|Y'_{n1}\|_\infty}{\sqrt{nh}}r\log n\right)^{1/2}\alpha(p),\quad k=1,\dots,r, \tag{3.10}$$

where

$$\|Y'_{n1}\|_\infty=\operatorname{esssup}|Y'_{n1}|\leqslant 2q\sum_{i=1}^m\frac{|\lambda_i|\,\|K\|_\infty}{(f(x_i)\|K\|^2)^{1/2}}<\infty.$$

Note that

$$\max_{1\leqslant u\leqslant r}\left|\sum_{k=1}^u Y'_{nk}\right|\leqslant\max_{1\leqslant u\leqslant r}\left|\sum_{k=1}^u Z'_{nk}\right|+\sum_{k=1}^r|Y'_{nk}-Z'_{nk}|.$$

Using the fact that $\operatorname{var}(Z'_{n1})=O(qh)$ and Kolmogorov's maximal inequality, we can see that

$$(nh)^{-1/2}\max_{1\leqslant u\leqslant r}\left|\sum_{k=1}^u Z'_{nk}\right|=o_P(1). \tag{3.11}$$

Furthermore, using (3.10) and the third condition of (3.3) we have

$$(nh)^{-1/2}\sum_{k=1}^r|Y'_{nk}-Z'_{nk}|=o_P(1). \tag{3.12}$$

Then (3.6) follows from (3.8), (3.9), (3.11) and (3.12).

If we put

$$G_n^{(3)}(s)=(nh)^{-1/2}\sum_{k=1}^{[rs]}Z_{nk},$$

it follows from the definitions of r and $G_n^{(3)}(s)$ that

$$\sup_{0\leqslant s\leqslant 1}|G_n^{(2)}(s)-G_n^{(3)}(s)|=o_P(1). \tag{3.13}$$

Here, checking Lyapounov's condition (Bosq, 1996, Theorem 2.3), we have

$$G_n^{(3)}(1)\xrightarrow{\mathrm{d}}N(0,\lambda_1^2+\cdots+\lambda_m^2),$$

and therefore $G_n^{(3)}(s)$ converges weakly to $(\lambda_1^2+\cdots+\lambda_m^2)^{1/2}W_1(s)$ (cf. Billingsley, 1999, pp. 147–148). Since the two processes $\sum_{i=1}^m\lambda_iW_i(s)$ and $(\lambda_1^2+\cdots+\lambda_m^2)^{1/2}W_1(s)$ have the same distribution, we obtain

$$G_n^{(3)}(s)\xrightarrow{\mathrm{w}}\sum_{i=1}^m\lambda_iW_i(s),$$

which with (3.6) and (3.13) implies that

$$G_n^{(1)}(s)\xrightarrow{\mathrm{w}}\sum_{i=1}^m\lambda_iW_i(s).$$

By this result and the fact:

$$f_n(x_i) \xrightarrow{P} f(x_i) \quad \text{for } i = 1, \ldots, m,$$

which is due to Theorem 2.1 of Bosq (1996), we obtain (3.2). □

Proof of Theorem 2.3. (1) The result follows from Theorem 2.2 and the continuous mapping theorem.

(2) Suppose that there exists a change point θ with $0 < \theta < 1$. Without loss of generality, assume that $f_1(x_1) \neq f_2(x_1)$. Note that

$$d_n(\theta, x_1) = \left(\frac{nh}{f_n(x_1)\|K\|^2} \right)^{1/2} \frac{[n\theta]}{n} \frac{n - [n\theta]}{n} (f_{[n\theta]}(x_1) - f^*_{n-[n\theta]}(x_1))$$

and

$$f_n(x_1) = \frac{[n\theta]}{n} f_{[n\theta]}(x_1) + \frac{n - [n\theta]}{n} f^*_{n-[n\theta]}(x_1).$$

In view of Theorem 2.1 of Bosq (1996), we have

$$f_{[n\theta]}(x_1) \xrightarrow{P} f_1(x_1), \tag{3.14}$$

$$f^*_{n-[n\theta]}(x_1) \xrightarrow{P} f_2(x_1), \tag{3.15}$$

and thus

$$f_n(x_1) \xrightarrow{P} \theta f_1(x_1) + (1 - \theta) f_2(x_1). \tag{3.16}$$

From (3.14)–(3.16), we obtain

$$|d_n(\theta, x_1)| \xrightarrow{P} \infty.$$

Since $T_n \geqslant |d_n(\theta, x_1)|$, we establish the theorem. □

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References

Antoch, J., Hušková, M., Prášková, Z., 1997. Effect of dependence on statistics for determination of change. J. Statist. Plan. Inference 60, 291–310.

Bai, J., 1994. Weak convergence of the sequential empirical processes of residuals in ARMA models. Ann. Statist. 22, 2051–2061.

Billingsley, P., 1999. Convergence of probability measures, 2nd Edition. Wiley, New York.

Bosq, D., 1996. Nonparametric Statistics for Stochastic Processes: Estimation and Prediction. Springer, New York.

Brodsky, B.E., Darkhovsky, B.S., 1993. Nonparametric Methods in Change-Point Problems. Kluwer, Dordrecht.

Brown, R.L., Durbin, J., Evans, J.M., 1975. Techniques for testing the constancy of regression relationships over time. J. Roy. Statist. Soc. B 37, 149–163.

Csörgő, M., Horváth, L., 1988. Nonparametric methods for change point problems. In: Krishnaiah, P.R., Rao, C.R. (Eds.), Handbook of Statistics, Vol. 7. Elsevier, New York, pp. 403–425.

Csörgő, M., Horváth, L., 1997. Limit Theorems in Change-Point Analysis. Wiley, Chichester.

Durbin, J., 1973. Weak convergence of the sample distribution function when parameters are estimated. Ann. Statist. 1, 279–290.

Hinkley, D.V., 1971. Inference about the change-point from cumulative sum tests. Biometrika 58, 509–523.

Inclán, C., Tiao, G.C., 1994. Use of cumulative sums of squares for retrospective detection of changes of variances. J. Amer. Statist. Assoc. 89, 913–923.

Kanagawa, S., Takano, S., Yoshihara, K., 1997. Convergence of changepoint estimators for weakly dependent data. J. Nonparametr. Statist. 8, 379–392.

Krämer, W., Ploberger, W., Alt, R., 1988. Testing for structural change in dynamic models. Econometrica 56, 1355–1369.

Krishaiah, P.R., Miao, B.Q., 1988. Review about estimation of change points. In: Krishnaiah, P.R., Rao, C.R. (Eds.), Handbook of Statistics, Vol. 7. Elsevier, New York, pp. 375–402.

Lee, S., Na, S., 2000. A nonparametric goodness of fit test for strong mixing processes. Annales de l'I.S.U.P. 44, fasc 2–3, 3–20.

Lee, S., Park, S., 2001. The Cusum of squares test for scale changes in infinite order moving average processes. Scand. J. Statist. 28, 625–644.

Lee, S., Wei, C.Z., 1999. On residual empirical processes of stochastic regression models with applications to time series. Ann. Statist. 27, 237–261.

Page, E.S., 1955. A test for change in a parameter occurring at an unknown point. Biometricka 42, 523–527.

Picard, D., 1985. Testing and estimating change-points in time series. Adv. Appl. Probab. 17, 841–867.

Takahata, H., Yoshihara, K., 1987. Central limit theorems for integrated square error of nonparametric density estimators based on absolutely regular random sequences. Yokohama Math. J. 35, 95–111.

Tang, S.M., MacNeil, I.B., 1993. The effect of serial correlation on tests for parameter change at unknown time. Ann. Statist. 21, 552–575.

Wichern, D.W., Miller, R.B., Hsu, D.A., 1976. Changes of variance in first-order autoregressive time series models—with an application. Appl. Statist. 25, 248–256.

Zacks, S., 1983. Survey of classical and Bayesian approaches to the change-point problem: fixed sample and sequential procedures of testing and estimation. In: Rivzi, M.H., et al. (Ed.), Recent advances in statistics. Academic Press, New York, pp. 245–269.