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More powerful modifications of unit root tests allowing structural change

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It is well known that more powerful variants of Dickey–Fuller unit root tests are available. We apply two of these modifications, on the basis of simple maximum statistics and weighted symmetric estimation, to Perron tests allowing for structural change in trend of the additive outlier type. Local alternative asymptotic distributions of the modified test statistics are derived, and it is shown that their implementation can lead to appreciable finite sample and asymptotic gains in power over the standard tests. Also, these gains are largely comparable with those from GLS-based modifications to Perron tests, though some interesting differences do arise. This is the case for both exogenously and endogenously chosen break dates. For the latter choice, the new tests are applied to the Nelson–Plosser data.

Keywords: Structural change; Perron tests; MAX test; Weighted symmetric estimation; GLS estimation; Power comparison

1. Introduction

Widely used tests for a unit autoregressive root in the generating process of a time series, valid in the presence of a single structural break in trend at a known point in time, were developed by Perron [1–3] and Perron and Vogelsang [4]. These procedures are adaptations of the usual Dickey–Fuller tests through the incorporation of appropriate deterministic dummy variables. These may allow for a break in level (where the slope may or may not be known), a break in slope, or both.

It is now well known that more powerful elaborations of standard Dickey–Fuller tests are available. Several of these have been compared, both in terms of local asymptotic power and power in finite samples, by Leybourne *et al.* [5]. These authors consider two particular approaches, both of which are motivated by the time-reversibility of a strictly stationary linear process. Leybourne [6] proposed fitting the usual Dickey–Fuller regression both to the original series and to the time-reversed series, suggesting that a test be based simply on the maximum of the two Dickey–Fuller t-ratios (MAX). Pantula *et al.* [7] proposed weighted symmetric least squares estimation (WS) applied to the detrended series, as a computationally convenient

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approximation to maximization of the Gaussian likelihood. Leybourne *et al.* [5] found little in terms of power to choose between these approaches, and that for the cases they analysed little or no more power and, on occasion, less power was obtained from GLS-based detrending approaches such as Elliott *et al.* [8].

Given wide interest in unit root tests in the possible presence of a trend break, it is worth investigating to what extent the MAX and WS modification generate extra power relative to standard Perron Unit root tests, and to see how these compare with the power gains offered by alternative GLS modifications of Perron tests, recently introduced by Perron and Rodriguez [9].

Here, we consider three models – a break in mean in a non-trending series, a break in level only in an underlying linear trend and a break in both level and slope. In each case, there are two Perron tests – an additive outlier (AO) variant and an innovational outlier variant. Our interest is in the former, as models for the latter imply an evolving rather than an abrupt break in trend, so that considerations of time-reversibility are not directly applicable.

Section 2 of the paper analyses the case of a break in both level and slope. In particular, local alternative asymptotic distributions for both the MAX and WS modifications of the Perron test statistic are derived. Application to the special cases is then noted. Section 3 provides critical values for the test statistics of section 2, and reports the results of a simulation study that clearly demonstrates their small sample and asymptotic power advantages over the usual Perron tests. We also highlight some contrasts in behaviour between MAX/WS and GLS modifications that arise when different sample sizes are considered.

Tests analysed to this stage follow Perron [1] in employing a pre-specified break point. There are certain applications, for example, when institutional considerations suggest the possibility of a specific break location, where an analyst will feel comfortable in specifying that location exogenously – that is, without reference to the data – and these tests are frequently employed. However, as argued by Christiano [10], they are not strictly valid when reference, formal or informal, to the data plays a role in break date selection. Beginning with Zivot and Andrews [11] and Banerjee *et al.* [12], Perron tests have been extended to the case where break date selection is endogenized. Vogelsang and Perron [13] provide a thorough analysis of the available tests for various methods of break date choice. In Section 4 of this paper, we discuss the extension of the MAX/WS tests to the endogenous break date case. Section 5 provides simulation evidence of the retention of the power gains of MAX/WS relative to the standard tests and finds some further contrasts between MAX/WS and GLS modifications.

The totality of the power simulation evidence in sections 3 and 5 persuades that, while often there is little to choose among the MAX, WS and GLS modifications, a preferred strategy may be to consider the MAX tests, the most appropriate when small or moderately sized samples are under scrutiny, and the GLS-based tests when larger samples are available.

Finally, in section 6 of the paper, the new tests are applied, with endogenously chosen break dates, to the Nelson–Plosser data. Our conclusion remains that, in the aggregate, the evidence of stationarity, even allowing for breaking trends in these data, is far from substantial.

2. Unit root tests with a break in trend: exogenously selected break date

We begin with the case where the date of any break is chosen exogenously (and correctly) without reference to the data. This is the formulation originally analysed by Perron [1]. Suppose that the time series for T observations y_t is generated by

$$y_t = \alpha + \beta t + \gamma d_{1t}(\tau) + \delta d_{2t}(\tau) + u_t, \tag{1}$$

where

$$d_{1t}(\tau) \equiv 1[t > \tau T],$$
$$d_{2t}(\tau) \equiv (t - \tau T)1[t > \tau T],$$

allowing for a break at time $t = \tau T$, and

$$u_t = \rho u_{t-1} + v_t, \tag{2}$$
$$v_t = \sum_{i=1}^{\infty} \gamma_i \varepsilon_{t-i},$$

where $\rho = 1 + (c/T)$ with $c \in (-\infty, 0)$, and ε_t is IID$(0, \sigma_\varepsilon^2)$ with $E(|\varepsilon_t|^{4+h}) = \kappa < \infty$ for some $h > 0$. To ensure the stationarity of v_t, we also assume that $\sum_{i=1}^{\infty} i|\gamma_i| < \infty$. The variance of v_t is denoted as $\sigma^2 = \text{var}(v_t)$, which is given by $\sigma^2 = \sigma_\varepsilon^2 \gamma(1)^2$. With regard to the initial value of u_t, we follow Elliott [14] and assume that u_0 is drawn from a Gaussian distribution with mean zero and variance $\sigma^2(1-\rho^2)^{-1}$ and is independent of ε_t. Drawing u_0 from an unconditional distribution in this way, we feel, is more realistic than assuming u_0 is fixed, and is also more in keeping with the notion of time-reversibility.

The goal is to test the unit root null hypothesis, $\rho = 1$ in equation (2), in the presence of a possible break in trend. Equation (1) allows for simultaneous breaks in both level and slope. Later, we shall note the important special cases of this model.

Estimating equation (1) by OLS gives

$$y_t = \hat{\alpha} + \hat{\beta} t + \hat{\gamma} d_{1t}(\tau) + \hat{\delta} d_{2t}(\tau) + e_t. \tag{3}$$

Perron's test is then based on estimating the second stage model

$$e_t = \rho e_{t-1} + \phi d_{3t}(\tau) + \sum_{j=1}^{p} \{\phi_j d_{3,t-j}(\tau) + \pi_j \Delta e_{t-j}\} + w_t, \tag{4}$$

where $d_{3t}(\tau) = 1[t = \tau T + 1]$, a one-time dummy variable. Without including the p lagged terms in equation (4), the asymptotic distribution of the Perron's statistic will depend on some nuisance parameters such as σ and σ_ε^2. In order to ensure that the asymptotic distribution with general serial correlation structure on v_t is the same as the case with iid v_t, the number of lagged terms (p) should increase with the sample size, as demonstrated in Said and Dickey [15]. Specifically, it is assumed that $p \to \infty$ and $p^3/T \to 0$ as $T \to \infty$. Applying OLS, we have

$$e_t = \hat{\rho} e_{t-1} + \hat{\phi} d_{3t}(\tau) + \sum_{j=1}^{p} \{\hat{\phi}_j d_{3,t-j}(\tau) + \hat{\pi}_j \Delta e_{t-j}\} + \hat{w}_t \tag{5}$$

and the Perron unit root test is then the t-statistic for testing $\rho = 1$ based on equation (5):

$$\text{DF}_f(\tau) = \frac{\hat{\rho} - 1}{\widehat{\text{var}}(\hat{\rho})^{1/2}}, \tag{6}$$

where the subscript f denotes the standard test on the basis of *forward* residuals. It is entirely straightforward to show, by slightly modifying the results in Theorem 2 in Perron [1, 3], that

$$\text{DF}_f(\tau) \Longrightarrow R_2(\tau) \equiv \frac{H_{cf}(\tau)}{(g_{cf}(\tau) K_{cf}(\tau))^{1/2}}, \tag{7}$$

where $H_{cf}(\tau)$, $g_{cf}(\tau)$ and $K_{cf}(\tau)$ are defined in Theorem 2 in Perron [1, 3] with $\sigma^2 = \sigma_\varepsilon^2$, but in the definitions, the standard Brownian motion process $W(r)$, which is the limit of

$\sigma_\varepsilon^{-1} T^{-1/2} \sum_{t=1}^{rT} \varepsilon_t$, is now replaced by

$$J_c(r) = W_c(r) + (\mathrm{e}^{rc} - 1) Z_c. \tag{8}$$

Here, $W_c(r)$ is an Ornstein–Uhlenbeck process defined as

$$W_c(r) = c \int_0^r \mathrm{e}^{c(r-\lambda)} W(\lambda)\, \mathrm{d}\lambda + W(r)$$

and Z_c is a Gaussian random variable with mean zero and variance $(-2c)^{-1}$. Further details on the process $J_c(r)$ in equation (8) can be found in refs. [4, 5]. We now consider the possibility of extending to unit root testing in the possible presence of a trend break of the AO type two procedures that are known to add power to the standard Dickey–Fuller test.

2.1 *A test based on forward and reverse regressions*

Following Leybourne [6], we now consider the *reverse* series of y_t, which is $\tilde{y}_t \equiv y_{T+1-t}$. Then, it can be shown that the DGP for the reversed series $\tilde{y}_t$ is given by:

$$\tilde{y}_t = \alpha_0 + \beta_0 t + \gamma_0 d_{1t}(1-\tau) + \delta_0 d_{2t}(1-\tau) + \tilde{u}_t, \tag{9}$$

where $\alpha_0 = \alpha + \beta(T+1) + \gamma + \delta\{(1-\tau)T + 1\}$, $\beta_0 = -(\beta + \delta)$, $\gamma_0 = -(\gamma + \delta)$, $\delta_0 = \delta$ and $\tilde{u}_t = u_{T-t+1}$. The deterministic component of $\tilde{y}_t$ is the mirror image of the deterministic component of y_t and both have exactly the same break structure except that the break in the reverse series now occurs at time $(1-\tau)T$ instead of τT.

As for the forward series case in equations (3)–(5), the first step regression is

$$\tilde{y}_t = \tilde{\alpha} + \tilde{\beta} t + \tilde{\gamma} d_{1t}(1-\tau) + \tilde{\delta} d_{2t}(1-\tau) + z_t \tag{10}$$

and the second step regression is given by

$$z_t = \tilde{\rho} z_{t-1} + \tilde{\phi} d_{4t}(\tau) + \sum_{j=1}^{p} \{\tilde{\phi}_j d_{4,t-j}(\tau) + \tilde{\pi}_j \Delta z_{t-j}\} + \tilde{\eta}_t, \tag{11}$$

where $d_{4t}(\tau) \equiv 1[t = (1-\tau)T + 1]$. Note that, in the case $p = 0$, the one-time dummy is defined to exclude the data point where $e_{\tau T}$ 'depends on' $e_{\tau T+1}$, as follows since it is easily seen that $z_t \equiv e_{T+1-t}$, in this sense corresponding to the role of $d_{3t}(\tau)$ in the forward regression. We also note that once the first step regression for the forward series y_t is obtained as in equation (3), the first step regression in equation (10) does not need to be carried out again because of the fact that as just noted the residuals z_t in equation (10) can be obtained simply by reversing the residuals e_t in equation (3). Then a unit root test using the z_t is the t-statistic for testing $\rho = 1$ based on equation (11):

$$\mathrm{DF}_r(\tau) = \frac{\tilde{\rho} - 1}{\widetilde{\mathrm{var}}(\tilde{\rho})^{1/2}},$$

where the r subscript denotes the test is based on reverse residuals. The following theorem, proved in Appendix A, establishes the local alternative asymptotic distribution of $\mathrm{DF}_r(\tau)$.

THEOREM 1 *If y_t is generated by equations* (1) *and* (2) *and a test is based on the fitted model* (11), *then*

$$\mathrm{DF}_r(\tau) \Longrightarrow \frac{H_{cr}(\tau)}{(g_{cf}(\tau)K_{cf}(\tau))^{1/2}}$$

where

$$H_{cr}(\tau) \equiv \left(\frac{1-\tau}{\tau}\right)^3 \left[\frac{1}{12}\tau^3 G_{10} + G_{13}(G_{12} - G_{11}) - G_{14}\left\{G_{12} - G_{11} + \left(\frac{\tau}{1-\tau}\right)^3 G_{12}\right\}\right]$$

and

$$\begin{aligned}
G_{10} &\equiv \frac{1}{2}\{J_c(1)^2 - 1\} - (1-\tau)^{-1}\{J_c(1) - J_c(\tau)\}\{(1-\tau)J_c(1) - H_1 + H_\tau\} \\
&\quad - \tau^{-1}J_c(\tau)\{\tau J_c(1) - H_\tau\} \\
G_{11} &\equiv \frac{1}{2}\tau H_1 + \frac{1}{2}(H_1 - H_\tau) - M_1 \\
G_{12} &\equiv \frac{1}{2}(1+\tau)(H_1 - H_\tau) - (M_1 - M_\tau) \\
G_{13} &\equiv \frac{1-\tau}{2}J_c(1) + \frac{1}{2}J_c(\tau) - H_1 \\
G_{14} &\equiv \frac{1}{2}(1-\tau)\{J_c(1) - J_c(\tau)\} - (H_1 - H_\tau)
\end{aligned}$$

with

$$H_\kappa \equiv \int_0^\kappa J_c(r)\,\mathrm{d}r, \quad M_\kappa \equiv \int_0^\kappa r J_c(r)\,\mathrm{d}r,$$

and $J_c(r)$ is the process defined in equation (8).

In fact, as may be obvious, the limiting marginal distributions of $\mathrm{DF}_f(\tau)$ and $\mathrm{DF}_r(\tau)$ are the same, a conclusion that follows from Perron's result that $\mathrm{DF}_f(\tau)$ and $\mathrm{DF}_f(1-\tau)$ have the same limiting distribution (note also that the denominators of the two limiting functionals are identical). However, our interest is in the joint behaviour of these statistics as the Leybourne test statistic is based on the maximum of $\mathrm{DF}_f(\tau)$ and $\mathrm{DF}_r(\tau)$. Its local alternative asymptotic distribution is given by the following theorem.

THEOREM 2 *If y_t is generated by equations* (1) *and* (2), *then*

$$\mathrm{MAX}(\tau) \equiv \max\{\mathrm{DF}_f(\tau), \mathrm{DF}_r(\tau)\} \Longrightarrow \frac{\max\{H_{cf}(\tau), H_{cr}(\tau)\}}{(g_{cf}(\tau)K_{cf}(\tau))^{1/2}}.$$

The proof of Theorem 2 is a straightforward application of the continuous mapping theorem and hence is not presented. The null asymptotic distribution of $\mathrm{MAX}(\tau)$ (*i.e.*, when $c = 0$) is obtained simply by redefining $J_c(r) = W(r)$ in all relevant expressions.

2.2 *A test based on the weighted symmetric estimator*

Next we explore an extension of the Perron test using the weighted symmetric estimator suggested by Park and Fuller [16] and Pantula *et al.* [7]. The test statistic is constructed by

a two-step procedure. The first step regression is given by equation (3). For the second-step regression, in the non-augmented, $p = 0$, case, we define the following variables:

$$
\begin{aligned}
w_t &\equiv \left(\frac{t}{T}\right)^{1/2} 1[t \in S_1] + \left(2 - \left(\frac{t}{T}\right) - \frac{1}{T}\right)^{1/2} 1[t \in S_2],\\
z_t^d &\equiv w_t e_{t+1} 1[t \in S_1] + w_t e_{t-T+1} 1[t \in S_2],\\
z_t^i &\equiv w_t e_t 1[t \in S_1] + w_t e_{t-T+2} 1[t \in S_2],\\
d_{5t}(\tau) &\equiv 1[t = \tau T],\\
d_{6t}(\tau) &\equiv 1[t = (1+\tau)T - 1],
\end{aligned}
$$

where $S_1 \equiv \{1, 2, \ldots, T-1\}$ and $S_2 \equiv \{T, T+1, \ldots, 2(T-1)\}$. The second step regression is given by:

$$z_t^d = \hat{\rho} z_t^i + \hat{\phi}_1 d_{5t}(\tau) + \hat{\phi}_2 d_{6t}(\tau) + \hat{\eta}_t. \tag{12}$$

More generally, when augmentation is required to 'soak up' additional serial correlation, equation (12) is supplemented by lagged and leading changes in e_t, as set out in table 7 of Pantula *et al.* [7], and also lags and leads of the one-time dummies, corresponding to those introduced in the previous section. In the non-augmented case, the WS-Perron test statistic is defined as

$$\mathrm{WS}(\tau) \equiv \left[\frac{T-3}{2T-5}\right]^{1/2} \frac{\hat{\rho} - 1}{\widehat{\mathrm{var}}(\hat{\rho})^{1/2}},$$

where $\hat{\rho}$ is obtained from equation (12). The following theorem, proved in Appendix A, provides the local alternative distribution of $\mathrm{WS}(\tau)$. It can be shown that the same result holds in the more general augmented case.

THEOREM 3 *If y_t is generated by equations* (1) *and* (2) *with $\gamma(L) = 1$, and a test is based on equation* (12), *then*

$$\mathrm{WS}(\tau) \Longrightarrow \frac{H_{cf}(\tau) - K_{cf}(\tau) + F_1(\tau) + F_2(\tau)}{(g_{cf}(\tau) K_{cf}(\tau))^{1/2}},$$

where

$$
\begin{aligned}
F_1(\tau) \equiv g_{cf}(\tau) \Big\{ & H_1^2 + \frac{1}{4}B^2 + G^2(1-\tau)^2 + \frac{1}{4}D^2(1-\tau)^4 - H_1 B - 2(1-\tau)H_1 G\\
& -(1-\tau)^2 H_1 D + (1-\tau)BG + \frac{1}{2}(1-\tau)^2 BD + (1-\tau)^3 GD \Big\}\\
F_2(\tau) \equiv (1-\tau) & g_{cf}(\tau)\{G^2(2\tau - 1) - 2G(J_c(\tau) - H_1) + BG(2\tau - 1) - GD(1-\tau)^2\}.
\end{aligned}
$$

with

$$
\begin{aligned}
B &\equiv 6\tau^{-3}(2M_\tau - H_\tau \tau),\\
G &\equiv 2\tau^{-2}(1 - 2\tau + \tau^2)^{-1}\{H_1 \tau^2(\tau + 2) - 3M_1 \tau^2\\
&\quad - H_\tau \tau(4\tau - 1) + 3M_\tau(2\tau - 1)\},\\
D &\equiv 6\tau^{-3}(\tau^3 - 3\tau^2 + 3\tau - 1)^{-1}\{H_1 \tau^3(\tau + 1) - H_\tau \tau(4\tau^2 - 3\tau + 1)\\
&\quad - 2M_1 \tau^3 + 2M_\tau(3\tau^2 - 3\tau + 1)\}.
\end{aligned}
$$

Here, the definitions of H_k and M_k are given in Theorem 1.

Again, the limiting null distribution of $\mathrm{WS}(\tau)$ arise by redefining $J_c(r) = W(r)$ everywhere. We now describe how the MAX and WS test procedures apply to two special cases of the model (1). The proofs of the limiting local distributions of the test statistics in the models below proceed along lines very similar to those of Theorems 1–3 and are omitted.

2.3 *Break in level without linear trend*

Suppose y_t is generated by

$$y_t = \alpha + \gamma d_{1t}(\tau) + v_t.$$

Then we fit by OLS the regression

$$y_t = \hat{\alpha} + \hat{\gamma} d_{1t}(\tau) + e_t$$

and construct the $\mathrm{MAX}(\tau)$ test on the basis of secondary regressions (5) and (11), noting that here and throughout the z_t are simply the reverse of the e_t series, that is, $z_t = e_{T+1-t}$. The $\mathrm{WS}(\tau)$ test is based on the secondary regression (12), or its augmented variant when it is required to account for further serial correlation.

2.4 *Break in level with fixed slope*

Suppose y_t is generated by

$$y_t = \alpha + \beta t + \gamma d_{1t}(\tau) + v_t.$$

Then we fit by OLS the regression

$$y_t = \hat{\alpha} + \hat{\beta} t + \hat{\gamma} d_{1t}(\tau) + e_t$$

and construct the $\mathrm{MAX}(\tau)$ test on the basis of the secondary regressions (5) and (11). The $\mathrm{WS}(\tau)$ test is based on the secondary regression (12), or its augmented variant when it is required to account for further serial correlation.

3. Critical values of the test statistics, and power simulations: exogenous break case

To distinguish the different $\mathrm{MAX}(\tau)$ and $\mathrm{WS}(\tau)$ tests, in the sequel we will adopt the convention that a subscript 1 refers to a test with a break in level but no trend (section 2.3); subscript 2 to tests with a break in level and fixed slope (section 2.4); and subscript 3 to tests with a break in both level and slope (those initially introduced in section 2). Finite sample null critical values for the tests $\mathrm{MAX}(\tau)_i \equiv \max\{\mathrm{DF}_f(\tau)_i, \mathrm{DF}_r(\tau)_i\}$ and $\mathrm{WS}(\tau)_i$, $i = 1, \ldots, 3$ were based on simulating the DGP(1) with $c = 0$. We set $\alpha = \beta = \gamma = \delta = 0$ (allowable due to invariance of the tests) and $u_0 = 0$, $v_t = \varepsilon_t$ in equation (2).† Here, ε_t was generated as Gaussian white noise with unit variance. Asymptotic critical values, denoted $T = \infty$, were obtained by direct simulation of the appropriate limiting functionals with $c = 0$ using Gaussian discrete approximations with sample size 1000.‡ The results, based on performing 10,000 replications, are given in tables 1 and 2.

We now compare the 0.05-level powers of the tests $\mathrm{MAX}(\tau)_i$ and $\mathrm{WS}(\tau)_i$ in the exogenous break case with those of the corresponding standard Perron test [this latter test is simply given

†Under the null $c = 0$, the statistics are invariant to u_0, hence we can fix $u_0 = 0$ without loss of generality.
‡Note that $J_c(r) = W(r)$ when $c = 0$.

Table 1. Critical values of $\mathrm{MAX}(\tau)_i$ and $\mathrm{WS}(\tau)_i$ tests: exogenous break date.

	0.20/0.80			0.30/0.70			0.40/0.60			0.50		
τ	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
						$T = 50$						
MAX_1	−3.46	−2.83	−2.48	−3.54	−2.86	−2.52	−3.53	−2.89	−2.56	−3.54	−2.89	−2.57
MAX_2	−4.25	−3.56	−3.22	−4.23	−3.54	−3.20	−4.22	−3.50	−3.15	−4.17	−3.49	−3.14
MAX_3	−4.55	−3.87	−3.50	−4.68	−4.03	−3.69	−4.77	−4.12	−3.77	−4.80	−4.14	−3.79
WS_1	−3.65	−3.01	−2.66	−3.72	−3.07	−2.74	−3.69	−3.09	−2.77	−3.73	−3.11	−2.78
WS_2	−4.44	−3.75	−3.41	−4.40	−3.74	−3.40	−4.36	−3.68	−3.35	−4.31	−3.67	−3.33
WS_3	−4.68	−3.99	−3.63	−4.84	−4.17	−3.84	−4.97	−4.27	−3.94	−4.95	−4.30	−3.96
						$T = 100$						
MAX_1	−3.40	−2.74	−2.42	−3.48	−2.80	−2.48	−3.42	−2.83	−2.51	−3.41	−2.83	−2.50
MAX_2	−4.07	−3.45	−3.12	−4.04	−3.44	−3.11	−4.02	−3.39	−3.05	−4.03	−3.37	−3.04
MAX_3	−4.35	−3.75	−3.40	−4.48	−3.88	−3.56	−4.54	−3.96	−3.64	−4.58	−3.99	−3.66
WS_1	−3.57	−2.92	−2.59	−3.63	−2.98	−2.67	−3.60	−3.01	−2.71	−3.58	−3.01	−2.70
WS_2	−4.23	−3.63	−3.30	−4.19	−3.62	−3.31	−4.18	−3.56	−3.24	−4.15	−3.53	−3.21
WS_3	−4.50	−3.86	−3.52	−4.62	−4.01	−3.70	−4.64	−4.09	−3.78	−4.71	−4.13	−3.80
						$T = 200$						
MAX_1	−3.35	−2.76	−2.42	−3.38	−2.79	−2.49	−3.38	−2.83	−2.51	−3.44	−2.83	−2.52
MAX_2	−4.05	−3.37	−3.07	−3.96	−3.34	−3.04	−3.93	−3.33	−3.02	−3.93	−3.32	−3.00
MAX_3	−4.33	−3.67	−3.35	−4.44	−3.80	−3.49	−4.50	−3.86	−3.56	−4.49	−3.89	−3.61
WS_1	−3.51	−2.92	−2.60	−3.55	−2.97	−2.68	−3.54	−3.00	−2.70	−3.57	−3.00	−2.71
WS_2	−4.18	−3.55	−3.25	−4.12	−3.52	−3.23	−4.08	−3.48	−3.19	−4.06	−3.47	−3.16
WS_3	−4.46	−3.79	−3.47	−4.54	−3.93	−3.63	−4.61	−3.99	−3.71	−4.62	−4.02	−3.75
						$T = \infty$						
MAX_1	−3.29	−2.71	−2.40	−3.41	−2.75	−2.45	−3.40	−2.76	−2.48	−3.41	−2.80	−2.49
MAX_2	−3.94	−3.34	−3.03	−3.90	−3.31	−3.00	−3.87	−3.27	−2.98	−3.87	−3.25	−2.96
MAX_3	−4.23	−3.63	−3.32	−4.29	−3.75	−3.44	−4.34	−3.80	−3.50	−4.34	−3.81	−3.52
WS_1	−3.46	−2.88	−2.57	−3.54	−2.94	−2.64	−3.58	−2.95	−2.66	−3.53	−2.96	−2.67
WS_2	−4.11	−3.51	−3.21	−4.04	−3.48	−3.18	−4.01	−3.43	−3.14	−3.99	−3.40	−3.11
WS_3	−4.36	−3.75	−3.44	−4.41	−3.87	−3.58	−4.45	−3.92	−3.64	−4.47	−3.94	−3.66

by $\mathrm{DF}_f(\tau)_i$]. We also compare power with tests extending to our problem the GLS approach of Elliott *et al.* [8], proposed in Perron and Rodriguez [9]. These test statistics, which we denote $\mathrm{GLS}(\tau)_i$, differ in construction in two respects from the standard Perron $\mathrm{DF}_f(\tau)_i$ statistics. First, the initial stage regressions such as equation (3) are estimated by GLS as if the stochastic error u_t of equation (1) was pure AR(1) with parameter $\rho = 1 + (\bar{c}/T)$. Following the recommendation of Perron and Rodriguez, we set $\bar{c} = -22.5$. Then the parameters of the first stage regression are estimated by an OLS fit of $(y_1, y_2 - \rho y_1, y_3 - \rho y_2, \ldots, y_T - \rho y_{T-1})$ on the correspondingly transformed independent variables. The second-stage regression is then based on the residuals e_t defined as in equation (3) except that now the parameter estimates are

Table 2. Critical values of $\mathrm{MAX}(\hat{\tau})_i$ and $\mathrm{WS}(\hat{\tau})_i$ tests: endogenous break date.

	$T = 50$			$T = 100$			$T = 200$			$T = \infty$		
	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
MAX_1	−4.15	−3.49	−3.16	−4.06	−3.45	−3.14	−3.98	−3.40	−3.09	−3.91	−3.33	−3.05
MAX_2	−4.82	−4.17	−3.85	−4.72	−4.13	−3.78	−4.62	−3.98	−3.69	−4.49	−3.89	−3.62
MAX_3	−5.14	−4.46	−4.13	−4.97	−4.35	−4.02	−4.87	−4.28	−3.95	−4.78	−4.18	−3.88
WS_1	−4.25	−3.63	−3.30	−4.19	−3.57	−3.27	−4.10	−3.50	−3.18	−4.00	−3.44	−3.15
WS_2	−4.96	−4.32	−3.98	−4.80	−4.24	−3.90	−4.68	−4.11	−3.82	−4.53	−4.01	−3.76
WS_3	−5.24	−4.57	−4.24	−5.08	−4.43	−4.11	−4.96	−4.34	−4.02	−4.87	−4.28	−3.99

Table 3. Power of $DF_{f1}(\tau)$, $MAX_1(\tau)$, $WS_1(\tau)$, $GLS_1(\tau)$ at nominal 0.05-level: exogenous break date.

	τ	c				
		−5	−10	−15	−20	−25
			$T = 50$			
DF_{f1}	0.3	0.11	0.25	0.48	0.75	0.92
MAX_1	0.3	0.17	0.41	0.73	0.92	0.99
WS_1	0.3	0.15	0.38	0.70	0.91	0.98
GLS_1	0.3	0.11	0.28	0.56	0.82	0.95
DF_{f1}	0.5	0.10	0.22	0.45	0.73	0.90
MAX_1	0.5	0.15	0.38	0.69	0.90	0.98
WS_1	0.5	0.15	0.38	0.67	0.90	0.98
GLS_1	0.5	0.11	0.27	0.54	0.80	0.95
			$T = 100$			
DF_{f1}	0.3	0.11	0.25	0.49	0.74	0.91
MAX_1	0.3	0.16	0.40	0.70	0.91	0.98
WS_1	0.3	0.15	0.38	0.67	0.89	0.98
GLS_1	0.3	0.15	0.36	0.65	0.88	0.98
DF_{f1}	0.5	0.10	0.22	0.43	0.68	0.87
MAX_1	0.5	0.14	0.37	0.67	0.88	0.98
WS_1	0.5	0.15	0.37	0.66	0.88	0.98
GLS_1	0.5	0.13	0.33	0.62	0.85	0.97
			$T = \infty$			
DF_{f1}	0.3	0.12	0.26	0.49	0.72	0.89
MAX_1	0.3	0.18	0.43	0.73	0.92	0.99
WS_1	0.3	0.16	0.40	0.69	0.90	0.98
GLS_1	0.3	0.19	0.47	0.69	0.82	0.90
ENV	0.3	0.20	0.52	0.83	0.97	0.99
DF_{f1}	0.5	0.12	0.26	0.47	0.70	0.88
MAX_1	0.5	0.17	0.41	0.69	0.90	0.98
WS_1	0.5	0.16	0.38	0.65	0.87	0.97
GLS_1	0.5	0.19	0.46	0.68	0.82	0.90
ENV	0.5	0.20	0.52	0.83	0.97	0.99

GLS rather than OLS. Further, one-time dummies are not required in second-stage regression such as equation (5), which as before are fitted by OLS. At this point, in constructing all tests we impose $p = 0$.

The DGP for the power simulations is as described earlier, except that now we set $c < 0$ for the alternative and draw u_0 from a Gaussian distribution with mean zero and variance $(1 - \rho^2)^{-1}$ (independent of ε_t). Hence, y_t is a truly stationary AR(1) process with standard Gaussian white noise innovation.† Tables 3 and 4 gives the results for samples of $T = 50, 100$ at the nominal 0.05-level. Finite sample critical values are used, and a virtue of setting $p = 0$ in the fitted models implies that these are exact. For presentational brevity, we report only results for the cases of a break in level without linear trend ($i = 1$) and a break in both level and slope ($i = 3$). For these two cases, test powers are symmetric in $(\tau, 1 - \tau)$, so that we only report results for break fractions $\tau \leq 0.5$. Also given in Tables 3, and 4 under $T = \infty$, are the local asymptotic test powers, again obtained by the appropriate simulation of limiting functionals but with $c < 0$. We additionally provide, under ENV, the asymptotic Gaussian power envelope. This is obtained from constructing the point optimal test of the alternative $\bar{c} = c$; in this case, the Gaussian likelihood ratio statistic to test the null $\bar{c} = 0$ against the alternative $\bar{c} = c$. The statistic is calculated as the difference between the sums of squared

†We retain $\alpha = \beta = \gamma = \delta = 0$ as the tests remain invariant to these parameters under the alternative $c < 0$.

Table 4. Power of $DF_{f3}(\tau)$, $MAX_3(\tau)$, $WS_3(\tau)$, $GLS_3(\tau)$ at nominal 0.05-level: Exogenous break date.

	τ	c				
		−10	−15	−20	−25	−30
			$T = 50$			
DF_{f3}	0.3	0.12	0.24	0.43	0.66	0.85
MAX_3	0.3	0.15	0.31	0.54	0.78	0.93
WS_3	0.3	0.16	0.32	0.55	0.79	0.93
GLS_3	0.3	0.16	0.32	0.55	0.79	0.93
DF_{f3}	0.5	0.12	0.24	0.41	0.64	0.84
MAX_3	0.5	0.15	0.28	0.50	0.74	0.90
WS_3	0.5	0.15	0.28	0.50	0.74	0.90
GLS_3	0.5	0.15	0.29	0.51	0.74	0.90
			$T = 100$			
DF_{f3}	0.3	0.13	0.24	0.42	0.63	0.81
MAX_3	0.3	0.17	0.32	0.55	0.76	0.92
WS_3	0.3	0.17	0.32	0.55	0.77	0.91
GLS_3	0.3	0.18	0.35	0.59	0.81	0.93
DF_{f3}	0.5	0.11	0.21	0.36	0.56	0.75
MAX_3	0.5	0.14	0.28	0.48	0.70	0.88
WS_3	0.5	0.14	0.28	0.47	0.69	0.87
GLS_3	0.5	0.15	0.31	0.53	0.74	0.90
			$T = \infty$			
DF_{f3}	0.3	0.13	0.24	0.40	0.59	0.77
MAX_3	0.3	0.16	0.31	0.51	0.72	0.87
WS_3	0.3	0.16	0.31	0.51	0.71	0.87
GLS_3	0.3	0.17	0.33	0.54	0.75	0.89
ENV	0.3	0.17	0.34	0.55	0.76	0.91
DF_{f3}	0.5	0.11	0.21	0.35	0.52	0.71
MAX_3	0.5	0.14	0.27	0.46	0.67	0.84
WS_3	0.5	0.14	0.27	0.46	0.66	0.83
GLS_3	0.5	0.14	0.28	0.46	0.66	0.82
ENV	0.5	0.15	0.31	0.52	0.74	0.88

GLS residuals under the alternative and null. We derived the local alternative distribution of this point optimal test. The expression is a complicated (and not particularly informative) function of $J_c(r)$ and since the test is infeasible in practice (as c is unknown) here we simply report the powers obtained from simulating limiting functionals.

Turning now to analyse the finite sample results of table 3, the case of a trendless series with a possible change in level, we find that the MAX and WS tests are the two most powerful, with MAX slightly the more so. The GLS test is somewhat less powerful for $T = 50$, but makes up much of the deficit for $T = 100$. The DF_f test is always the least powerful, often by a considerable margin. Regarding the asymptotic results, MAX is slightly more powerful than WS. GLS performs better than MAX for small values of c, but this position is reversed for the larger values of c. All three tests substantially dominate DF_f. None of the tests, however, reach particularly close to the power envelope for mid-range values of c.† In table 4, which permits a possible break in both level and slope, we see that there is little to choose among MAX, WS and GLS when $T = 50$, though GLS performs slightly the better for $T = 100$. In terms of asymptotic power, MAX, WS and GLS perform very similarly in general, and also

†That this is also true for the GLS test is not surprising since it is constructed to be near optimal under the assumption that assuming u_0 is fixed at the value 0, which is a condition that we do not impose here.

achieve power levels that are now fairly near to the envelope. All three tests comprehensively outperform DF_f in both the finite sample and asymptotic contexts.

4. Endogenously selected break date

In practice, it is more often plausible to assume that the true break fraction τ is unknown and that an analyst would be unwilling to specify such a date through purely exogenous considerations. There are various proposals in the literature to endogenize the choice of break date, given for example by Zivot and Andrews [11], Banerjee *et al.* [12] and Vogelsang and Perron [13] among others. One of the popular choices is to obtain a break estimate by maximizing some statistics that test the significance of estimates of break parameters such as γ or δ in equation (1). For any value $\bar{\tau} \in \Gamma = [\bar{\tau}_1, \bar{\tau}_2]$, where $0 < \bar{\tau}_1 < \bar{\tau}_2 < 1$, we obtain for the case corresponding to equation (3) of possible breaks in both level and slope the following regression

$$y_t = \hat{\alpha} + \hat{\beta} t + \hat{\gamma} d_{1t}(\bar{\tau}) + \hat{\delta}_{2t}(\bar{\tau}) + e_t. \tag{13}$$

Then, the t-statistics $t_{\hat{\delta}}(\bar{\tau})$ for testing $\delta = 0$, on which break date selection is based, is a function of $\bar{\tau}$. The break estimate used in our subsequent theoretical analysis is obtained by maximizing the absolute value of the t-statistic over all possible values of $\bar{\tau}$ in Γ:

$$\hat{\tau} = \underset{\bar{\tau} \in \Gamma}{\operatorname{argmax}} |t_{\hat{\delta}}(\bar{\tau})|.$$

Once we obtain this break estimate, then the two regressions in equations (3) and (5) are carried out using $\hat{\tau}$ in place of τ. When there is no break ($\delta = 0$) in the DGP, Vogelsang and Parron [13] proved that

$$\mathrm{DF}_f(\hat{\tau}) \Longrightarrow R_2(\tau^*)$$

where $\tau^* = \arg\max_{\bar{\tau} \in \Gamma} |Q^A_{2,AO}(\bar{\tau})|$. Here $Q^A_{2,AO}(\bar{\tau})|$ is a limit functional of a Brownian motion and its expression is given on p. 1083 in ref. [13]. Here, it is again noted that the standard Brownian motion process $W(\cdot)$ is replaced by $J_c(\cdot)$. We now examine MAX and WS variants of the statistic $\mathrm{DF}_f(\hat{\tau})$. The following theorem shows the local alternative distribution of our statistic when the break estimator $\hat{\tau}$ is used in place of τ.

THEOREM 4 *If y_t is generated by equations* (1) *and* (2) *with $\delta = 0$, and a test is based on the fitted model* (11), *then*

$$\mathrm{DF}_r(\hat{\tau}) \Longrightarrow \frac{H_{cr}(\tau^*)}{(g_{cf}(\tau^*)K_{cf}(\tau^*))^{1/2}},$$

$$\mathrm{MAX}(\hat{\tau}) \equiv \max\{\mathrm{DF}_f(\hat{\tau}), \mathrm{DF}_r(\hat{\tau})\} \Longrightarrow \frac{\max\{H_{cf}(\tau^*), H_{cr}(\tau^*)\}}{(g_{cf}(\tau^*)K_{cf}(\tau^*))^{1/2}}.$$

The key step of the proof is to show that $T^{-1/2} t_{\hat{\delta}}(\bar{\tau}) \Rightarrow Q^A_{2,AO}(\bar{\tau})$ and recognize that maximizing $|T^{-1/2} t_{\hat{\delta}}(\bar{\tau})|$ and maximizing $|t_{\hat{\delta}}(\bar{\tau})|$ will give the same solution $\hat{\tau}$. For a detailed derivation of the result, see Vogelsang and Perron [13]. This result together with Theorems 1 and 2 is sufficient to prove Theorem 4.

Regarding the WS variant of the test statistic, we again estimate the break fraction τ from the first stage regression (13), and in particular from the absolute values of t-ratios associated with $\hat{\delta}$ for all possible break fraction, noting again from the definition of Γ that some trimming is required. Having found the estimate $\hat{\tau}$, the procedure is precisely as specified in section 2,

with $\hat{\tau}$, in place of τ. The limiting null distribution of the WS test statistic is exactly that given in Theorem 3, but with τ^* in place of τ. In the case of a break only in level, with or without a fixed slope, the break estimator will be based on t-ratios associated with $\hat{\gamma}$, so that in an obvious notation

$$\hat{\tau} = \underset{\bar{\tau}\in\Gamma}{\operatorname{argmax}}|t_{\hat{\gamma}}(\bar{\tau})|.$$

The remainder of the analysis in this case parallels the above-mentioned.

5. Critical values of the test statistics, and power simulations: endogenous break case

Table 2 gives finite sample and asymptotic critical values for all three variants of the test where the break date is chosen endogenously. In all cases, selection is based on the absolute value of the t-ratio associated with the estimated coefficient $\hat{\delta}$ on the slope break parameter, unless the slope is assumed to remain unchanged, when the significance of $\hat{\gamma}$ of equation (13) with no estimated slope break is substituted. The DGP is the same as that for Table 1, with search for the break fraction, both here and in all subsequent experiments, restricted to the range $0.15 \leq \bar{\tau} \leq 0.85$.

In tables 5 and 6, we examine the relative power performance of all the unit root tests. The simulation DGP is the same as that underlying in tables 3 and 4, and the power envelope is constructed in a manner analogous to that in tables 3 and 4, with the required modification that the sums of squared GLS residuals under the alternative and null are minimized across $\bar{\tau}$. From table 5, the case of a trendless series with a possible change in level, we see that the MAX and WS tests easily dominate the GLS test for $T = 50$, and all three tests behave very similarly for $T = 100$. As might be expected, the DF_f test is least powerful, but only marginally less than GLS when $T = 50$. Asymptotically, the GLS test is by some way the most powerful, and close to the power envelope for larger values of c. In table 6, where we allow a possible break in both level and slope, we see that there is generally little to choose among MAX, WS and GLS in either the finite sample or asymptotic contexts, with all tests fairly close to

Table 5. Power of $\mathrm{DF}_{f1}(\hat{\tau})$, $\mathrm{MAX}_1(\hat{\tau})$, $\mathrm{WS}_1(\hat{\tau})$, $\mathrm{GLS}_1(\hat{\tau})$ at nominal 0.05-level: endogenous break date.

	c				
	−5	−10	−15	−20	−25
		$T = 50$			
DF_{f1}	0.11	0.22	0.42	0.67	0.86
MAX_1	0.14	0.30	0.55	0.80	0.93
WS_1	0.13	0.28	0.53	0.78	0.92
GLS_1	0.10	0.23	0.44	0.70	0.88
		$T = 100$			
DF_{f1}	0.12	0.24	0.43	0.66	0.84
MAX_1	0.14	0.30	0.53	0.76	0.92
WS_1	0.14	0.29	0.52	0.73	0.90
GLS_1	0.14	0.30	0.53	0.73	0.90
		$T = \infty$			
DF_{f1}	0.12	0.27	0.46	0.67	0.84
MAX_1	0.16	0.33	0.56	0.78	0.91
WS_1	0.16	0.33	0.55	0.77	0.90
GLS_1	0.19	0.48	0.73	0.87	0.93
ENV	0.20	0.50	0.79	0.95	0.99

Table 6. Power of $DF_{f3}(\hat{\tau})$, $MAX_3(\hat{\tau})$, $WS_3(\hat{\tau})$, $GLS_3(\hat{\tau})$ at nominal 0.05-level: endogenous break date.

	c				
	−5	−10	−15	−20	−25
		$T = 50$			
DF_{f3}	0.12	0.22	0.39	0.61	0.82
MAX_3	0.13	0.25	0.45	0.68	0.87
WS_3	0.13	0.24	0.45	0.67	0.87
GLS_3	0.12	0.23	0.42	0.64	0.84
		$T = 100$			
DF_{f3}	0.12	0.22	0.37	0.56	0.76
MAX_3	0.13	0.24	0.44	0.65	0.83
WS_3	0.13	0.25	0.45	0.66	0.84
GLS_3	0.13	0.24	0.43	0.65	0.82
		$T = \infty$			
DF_{f3}	0.13	0.23	0.39	0.56	0.72
MAX_3	0.14	0.25	0.41	0.61	0.80
WS_3	0.14	0.24	0.40	0.60	0.78
GLS_3	0.13	0.25	0.42	0.65	0.82
ENV	0.14	0.28	0.45	0.66	0.84

the envelope. All three tests outperform DF_f, through the margin is not always particularly significant. Throughout tables 5 and 6, MAX and WS behave very similarly.

6. Finite sample size and power simulations with ARMA errors: endogenous break case

To gauge the relative performance of the tests in situations of perhaps more practical relevance, tables 7 and 8 report, respectively, for the break in level with no linear trend and the change in both level and slope cases, simulated finite sample sizes and powers from models with more general noise processes, assumed not to be known but approximable by an autoregression. The DGP is

$$y_t = u_t; \quad u_t = \rho u_{t-1} + v_t; \quad v_t - \phi v_{t-1} = \varepsilon_t - \theta_{\varepsilon_{t-1}}, \tag{14}$$

where again the ε_t are independent standard normal and $\rho = 1 + (c/T)$. Specifically, we consider noise process v_t that are either pure AR(1), $\theta = 0$, or pure MA(1), $\phi = 0$ in equation (14). In practice, this DGP will be unknown and we consider fitting to the first stage residual models with up to four lagged changes, *i.e.* p takes a maximum value of 4 in regressions such as (5). The order actually selected is determined by general-to-specific testing at the 0.10-level. This same approach is applied separately for all four tests, except that in computing the MAX statistic, we select the number of lags from the forward model, exactly as for DF_f, and impose that selected order on the model fitted to the reverse series. (In fact, with tedious but straight-forward algebra it can be shown that applying the same lag order selection process to the reversed series yields the same selected lag order.) We report size unadjusted powers (*i.e.*, rejection frequencies) here since size adjusted powers have little practical relevance as they are based on infeasible tests.

Examining table 7, the break in level model, first note that all four tests have approximately correct sizes, even for samples as short as $T = 50$ except in the case of a first-order moving average process, with $\phi = 0$ and θ positive in equation (14). It is, of course, well known

Table 7. Size/power of $DF_{f1}(\hat{\tau})$, $MAX_1(\hat{\tau})$, $WS_1(\hat{\tau})$, $GLS_1(\hat{\tau})$ at nominal 0.05-level: endogenous break date, $AR(1)$ or $MA(1)$ noise.

	$T = 50$				$T = 100$			
c	0	−15	−20	−25	0	−15	−20	−25
				$\phi = 0.5, \theta = 0$				
DF_{f1}	0.05	0.24	0.30	0.37	0.05	0.31	0.42	0.53
MAX_1	0.05	0.29	0.38	0.44	0.05	0.37	0.50	0.61
WS_1	0.05	0.26	0.34	0.39	0.05	0.34	0.45	0.56
GLS_1	0.04	0.25	0.33	0.40	0.04	0.37	0.52	0.63
				$\phi = -0.5, \theta = 0$				
DF_{f1}	0.07	0.37	0.53	0.69	0.06	0.40	0.58	0.74
MAX_1	0.07	0.44	0.62	0.77	0.06	0.48	0.67	0.82
WS_1	0.07	0.41	0.55	0.67	0.06	0.44	0.61	0.74
GLS_1	0.07	0.31	0.41	0.49	0.06	0.33	0.43	0.50
				$\phi = 0, \theta = 0.5$				
DF_{f1}	0.21	0.70	0.83	0.90	0.22	0.73	0.83	0.89
MAX_1	0.24	0.77	0.87	0.92	0.25	0.78	0.86	0.91
WS_1	0.20	0.64	0.73	0.80	0.17	0.63	0.75	0.83
GLS_1	0.22	0.62	0.70	0.72	0.21	0.56	0.58	0.59
				$\phi = 0, \theta = -0.5$				
DF_{f1}	0.07	0.38	0.49	0.58	0.06	0.38	0.50	0.61
MAX_1	0.08	0.46	0.57	0.66	0.07	0.44	0.57	0.67
WS_1	0.07	0.34	0.43	0.50	0.05	0.35	0.47	0.56
GLS_1	0.07	0.41	0.52	0.61	0.07	0.46	0.58	0.66

Table 8. Size/power of $DF_{f3}(\hat{\tau})$, $MAX_3(\hat{\tau})$, $WS_3(\hat{\tau})$, $GLS_3(\hat{\tau})$ at nominal 0.05-level: Endogenous break date, $AR(1)$ or $MA(1)$ noise.

	$T = 50$				$T = 100$			
c	0	−20	−25	−30	0	−20	−25	−30
				$\phi = 0.5, \theta = 0$				
DF_{f3}	0.06	0.15	0.19	0.23	0.06	0.23	0.30	0.39
MAX_3	0.07	0.21	0.25	0.30	0.06	0.26	0.36	0.45
WS_3	0.07	0.17	0.22	0.25	0.06	0.25	0.33	0.41
GLS_3	0.07	0.22	0.26	0.30	0.06	0.29	0.39	0.48
				$\phi = -0.5, \theta = 0$				
DF_{f3}	0.06	0.31	0.44	0.58	0.07	0.37	0.52	0.67
MAX_3	0.07	0.39	0.53	0.68	0.07	0.44	0.60	0.75
WS_3	0.07	0.33	0.44	0.55	0.07	0.40	0.55	0.68
GLS_3	0.07	0.29	0.37	0.45	0.06	0.32	0.43	0.54
				$\phi = 0, \theta = 0.5$				
DF_{f3}	0.20	0.63	0.75	0.82	0.30	0.77	0.86	0.91
MAX_3	0.24	0.72	0.82	0.88	0.32	0.82	0.89	0.93
WS_3	0.20	0.57	0.66	0.72	0.24	0.68	0.76	0.83
GLS_3	0.22	0.58	0.67	0.72	0.22	0.64	0.71	0.75
				$\phi = 0, \theta = -0.5$				
DF_{f3}	0.10	0.31	0.38	0.44	0.09	0.34	0.45	0.53
MAX_3	0.12	0.37	0.45	0.53	0.10	0.39	0.50	0.59
WS_3	0.09	0.27	0.33	0.39	0.08	0.30	0.39	0.47
GLS_3	0.12	0.38	0.46	0.54	0.10	0.40	0.51	0.59

that conventional Dickey–Fuller type tests suffer from this over-size problem for such DGPs. Otherwise, there are two pertinent features of the results of table 7. The MAX test exhibits superior power, being generally somewhat more powerful than WS and providing consistent worthwhile gains compared with the usual DF_f. Further, in certain circumstances, notably where the v_t of equation (14) exhibit negative first-order autocorrelation, GLS can have very low power, lower even than the standard DF_f test, and well below that of the MAX test. This is most clearly seen for v_t generated by a first-order autoregression with $\phi = -0.5$ and $\theta = 0$. Given the structure of the GLS test, this finding is perhaps not surprising. The main feature of this test is that, in the first stage regression, the deterministic component of the DGP is not estimated by OLS, but by quasi-GLS, as if the deviation from trend followed a first order autoregression with large positive parameter. When this is in fact the case, as for example for the DGPs analysed in tables 3–6, one would expect the GLS tests to be based on relatively precise estimates of the deterministic component and therefore to exhibit good power. As we now see, when that deviation from trend follows a very different process from that implicit in the GLS estimation, tests of very low power can result. The results of table 8, the change in both level and slope model, follow a qualitatively very similar pattern to those of table 7, with the differences in performance between the tests remaining just as emphatic.

Given the finite sample and asymptotic simulation evidence reported in this section and previous ones, our recommendation would be that for small moderate sample use, the MAX test should be implemented, mainly on the basis of the power results of tables 7 and 8. When larger samples are available, our local alternative asymptotic power results in tables 5 and 6 generally point in favour of the GLS test. Size reliability would not seem to be an issue that would point in favour of any particular test.

7. Application to the Nelson–Plosser data

Nelson and Plosser [17] carried out standard Dickey–Fuller tests on annual observations of the logarithms of 14 historical US macroeconomic time series, incorporating a linear trend in the test equations. The unit root null hypothesis was rejected at conventional significance levels for only one series, the unemployment rate. Subsequently, the remaining 13 series were frequently re-analysed from a number of perspectives. Their analysis formed a focus for the work of Perron [1], who found considerably more evidence of (broken) trend-stationarity with and exogenously selected trend break in 1929. Zivot and Andrews [11], among others, have repeated this analysis with break date selection endogenized. In this section, we seek evidence of broken trend-stationarity in these 13 series, again with endogenously chosen break date, through the more powerful AO type break model tests discussed in the previous two sections. We follow Perron [1] and others in the specification of break type – a break in both level and slope for real wages and the S&P500, and a break in level with fixed slope for the remaining 11 series. Otherwise, we follow the method of analysis on which the results of tables 7 and 8 are base, described in the previous Section. Break date selection is based on t-ratios associated with estimates of parameters on dummy variables in the first step regressions, and the number of lagged changes incorporated in the second stage regression, on which unit root tests are based, is determined by general-to-specific testing. Finite sample critical values, such as those to tables 1 and 2, were employed, and where decisions at the usual significance levels were not completely clear for particular sample sizes T, we obtained by simulation critical values specific to those sample sizes.

The results of our analysis are shown in table 9, which gives test statistic values for the four procedures and indicated rejections of the unit root null. The overall picture is that for 9

Table 9. Application to Nelson–Plosser data: endogenous break date.

	Break in level with fixed slope ($i = 2$)		
	Real GNP ($T = 62$)	Nominal GNP ($T = 62$)	Real GNP per cap ($T = 62$)
DF_{f2}	−3.91	−4.02	−3.56
MAX_2	−3.59	−3.85*	−3.56
WS_2	−3.74	−4.02*	−3.55
GLS_2	−2.93	−2.43	−3.27
	Ind. Prodn. ($T = 111$)	Employment ($T = 81$)	GNP deflator ($T = 82$)
DF_{f2}	−4.52**	−3.15	−3.40
MAX_2	−4.45**	−3.15	−2.20
WS_2	−4.66**	−3.32	−2.31
GLS_2	−4.46**	−3.09	−2.69
	Consumer prices ($T = 111$)	Wages ($T = 71$)	Money stock ($T = 82$)
DF_{f2}	−2.23	−3.25	−3.11
MAX_2	−2.23	−3.22	−3.11
WS_2	−2.69	−3.19	−3.24
GLS_2	−2.04	−2.62	−3.01
	Velocity ($T = 102$)	Bond yield ($T = 71$)	
DF_{f2}	−3.10	−0.01	
MAX_2	−2.85	−0.01	
WS_2	−2.94	−0.68	
GLS_2	−1.27	−0.69	
	Break in both level and slope ($i = 3$)		
	Real wages ($T = 71$)	S&P 500 ($T = 100$)	
DF_{f3}	−3.86	−4.91**	
MAX_3	−3.86	−4.91**	
WS_3	−4.35*	−4.93**	
GLS_3	−3.78	−3.73	

Note: Unit root null hypothesis rejected at: *10% level, **5% level, ***1%level.

of these 13 series, there are no rejections even at the 0.10 level. The GLS test finds only one rejection at that level or lower, the usual DF_f test finds one further such rejection, the MAX test finds a third, and the WS test a fourth. For samples of the current size, $T \leq 111$, this pattern of results does not contradict those from tables 7 and 8; it is entirely possible that GLS may have the lowest power of all the procedures. Certainly, its large sample power advantages are unlikely to play a prominent role here. Also, that neither MAX nor WS drastically overturns the evidence provided by the usual DF_f procedure is also perhaps unsurprising. Given that while our analysis suggests that these tests are, to a worthwhile extent, more powerful than DF_f tests, the magnitude of those power differences does not suggest dramatic reversals are liable to arise in any one practical situation.

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APPENDIX A

A.1 Proof of Theorem 1

The theorem can be proved directly from first principles. However, a more straightforward proof follows from exploitation of the result for the forward statistic given in equation (7). Obviously, adjustment must be made for the fact that in reverse time the break fraction $(1-\tau)$. We also note that the statistic is invariant with respect to α, β, γ and δ in equation (1) so that we set these parameters to zero without loss of generality.

First, we make the following observations: (i) the limiting distribution of the forward statistic $\mathrm{DF}_f(\tau)$ in equation (6) is a function of $J_c(r)$ only and (ii) $J_c(r)$ is obtained as the limit of the following forward series:

$$\sigma^{-1}T^{-1/2}(y_{rT}-y_1) \Longrightarrow J_c(r).$$

Hence, if $\tilde{J}_c(r)$ is defined as the limit of the corresponding reverse series (its existence will be shown below) as follows

$$\sigma^{-1}T^{-1/2}(\tilde{y}_{rT}-\tilde{y}_1) \Longrightarrow \tilde{J}_c(r),$$

then it is immediately obvious that the limiting functional for the reverse statistic $\mathrm{DF}_r(\tau)$ can be obtained by replacing $J_c(r)$ in the limiting functional for the forward statistic $\mathrm{DF}_f(\tau)$ with $\tilde{J}_c(r)$, but using the break fraction $(1-\tau)$ in place of τ. This observation is sufficient to immediately give the limiting null distribution of $\mathrm{DF}_r(\tau)$ in terms of $\tilde{J}_c(\cdot)$. Below, we demonstrate the existence of $\tilde{J}_c(r)$. It can be shown that

$$\begin{aligned}\sigma^{-1}T^{-1/2}(\tilde{y}_{sT}-\tilde{y}_1) &= -\sigma^{-1}\rho^2T^{-1/2}\sum_{t=(1-s)T+2}^{T}\rho^{(1-s)T-t}v_t+\sigma^{-1}(\rho^{1-sT}-1)T^{-1/2}y_T\\ &\equiv C_{1,T}+C_{2,T}.\end{aligned} \tag{A1}$$

Focusing on the first term, we have

$$C_{1,T} = -\sigma^{-1}\rho^2 \left\{ \rho^{-sT} \sum_{t=1}^{T} \rho^{T-t} v_t - \sum_{t=1}^{(1-s)T} \rho^{(1-s)T-t} v_t \right\} + o_p(1)$$
$$\Longrightarrow -\{\mathrm{e}^{-sc} W_c(1) - W_c(1-s)\}, \tag{A2}$$

where the result in the last line is obtained directly from Phillips [18]. Next, for the second term, we have

$$C_{2,T} = \sigma^{-1}(\rho^{-sT+1} - 1)T^{-1/2} y_T$$
$$\Longrightarrow (\mathrm{e}^{-sc} - 1)(W_c(1) + \mathrm{e}^c Z_c), \tag{A3}$$

where the last line is again obtained from Phillips [18]. Collecting the limits in equations (A2) and (A3) and using the definition of $J_c(r) = W_c(r) + (\mathrm{e}^{rc} - 1)Z_c$, we obtain

$$\sigma^{-1}T^{-1/2}(\tilde{y}_{sT} - \tilde{y}_1) \Longrightarrow -\{J_c(1) - J_c(1-s)\}.$$

Hence, the new process $\tilde{J}_c(s)$ exists and is given by $\tilde{J}_c(s) = -\{J_c(1) - J_c(1-s)\}$. The limiting functional in the theorem for the reverse statistic expressed in terms of $J_c(\cdot)$ is then simply obtained by substituting $-\{J_c(1) - J_c(1-s)\}$ for $\tilde{J}_c(s)$. We illustrate the procedure discussed so far in the following.

The limiting functional for the forward Perron statistic is given by

$$\mathrm{DF}_f(\tau) \Longrightarrow \frac{H_{cf}(\tau)}{(g_{cf}(\tau)K_{cf}(\tau))^{1/2}}$$

where

$$g_{cf}(\tau) \equiv \frac{1}{12}(1-\tau)^3,$$
$$H_{cf}(\tau) \equiv g_{cf}(\tau)D_{10} + D_{13}(D_{12} - D_{11}) - D_{14}\left\{D_{12} - D_{11} + \left(\frac{1-\tau}{\tau}\right)^3 D_{12}\right\}$$

and

$$K_{cf}(\tau) \equiv g_{cf}(\tau)D_9 + D_{11}(D_{12} - D_{11}) - D_{12}\left\{D_{12} - D_{11} + \left(\frac{1-\tau}{\tau}\right)^3 D_{12}\right\}$$

with $\sigma^2 = \sigma_\varepsilon^2$. All the functionals D_i are defined in Perron [1, 3], but with $W(\cdot)$ replaced by $J_c(\cdot)$. For example, the term D_{14} is given by

$$D_{14} \equiv \frac{1}{2}\tau J_c(\tau) - \int_0^\tau J_c(r)\,\mathrm{d}r.$$

First, we replace $J_c(\cdot)$ with $\tilde{J}_c(\cdot)$, but using the break fraction $1-\tau$ in place of τ, which results in the corresponding term for the reverse time statistic which we denote as

$$G_{14} \equiv \frac{1}{2}(1-\tau)\tilde{J}_c(1-\tau) - \int_0^{1-\tau} \tilde{J}_c(r)\,\mathrm{d}r.$$

By the transformation of replacing $\tilde{J}_c(s)$, with $-\{J_c(1) - J_c(1-s)\}$, we have

$$G_{14} = \frac{1}{2}(1-\tau)\{J_c(1) + J_c(\tau)\} - \int_\tau^1 J_c(r)\,\mathrm{d}r,$$

which is given in the theorem. The same procedure is carried out for all the functions in $g_{cf}(\tau)$, $H_{cf}(\tau)$ and $K_{cf}(\tau)$. It turns out that the denominator in the limiting functionals for the reverse time statistic with $H_{cr}(\tau)$ defined as in the theorem is equal to the denominator in the limiting functional for the forward Perron statistic given in equation (7). Collecting all the terms delivers the result in the theorem.

A.2 Proof of Theorem 3

As done in the proof for forward and reverse Perron statistics, we can, without loss of generality, set the coefficients α, β, γ and δ in equation (1) to zero. We may then write.

$$e_t = \breve{y}_t - \hat{\beta}\breve{t} - \hat{\gamma}\breve{d}_{1t}(\tau) - \hat{\delta}\breve{d}_{2t}(\tau),$$

where the overstrike denotes a deviation from the mean. We also make use of the following results regarding the OLS estimators from the first stage estimation in equation (3):

$$T^{1/2}\hat{\beta} \Longrightarrow B, \quad T^{-1/2}\hat{\gamma} \Longrightarrow G, \quad T^{1/2}\hat{\delta} \Longrightarrow D, \tag{A4}$$

where B, G and D are as defined in the statement of the theorem.

The OLS estimator $\hat{\rho}$ from the regression in equation (12) is given by:

$$\hat{\rho} = \frac{\sum_2^T e_t e_{t-1} - \tau e_{\tau T} e_{1+\tau T} - (1-\tau) e_{\tau T} e_{1+\tau T}}{\sum_3^T e_{t-1}^2 + T^{-1}\sum_3^T e_{t-1}^2 + T^{-1}e_1^2 + T^{-1}e_T^2 - \tau e_{\tau T}^2 - (1-\tau)e_{1+\tau T}^2}.$$

Hence, we have the following result:

$$\begin{aligned}
\frac{\hat{\rho}-1}{\widehat{\mathrm{var}}(\hat{\rho})^{1/2}} &= \hat{\sigma}^{-1}\left[T^{-2}\sum_3^T e_{t-1}^2 - T^{-2}e_{1+\tau T}^2\right]^{-1/2}\left[T^{-1}\left\{\sum_2^T e_{t-1}\Delta e_t - e_{\tau T}\Delta e_{1+\tau T}\right\}\right. \\
&\quad \left. - T^{-2}\left\{\sum_2^T e_{t-1}^2 - e_{\tau T}^2\right\} + T^{-1}e_2 e_1 + (1-\tau)T^{-1}(e_{1+\tau T}^2 - e_{\tau T}^2)\right] + o_p(1) \\
&= \hat{\sigma}^{-1}[T^{-3}g_T(\tau)]^{-1/2}\left[T^{-3}g_T(\tau)T^{-2}\left\{\sum_3^T e_{t-1}^2 - e_{1+\tau T}^2\right\}\right]^{-1/2} \\
&\quad \times \left[T^{-3}g_T(\tau)T^{-1}\left\{\sum_2^T e_{t-1}\Delta e_t - e_{\tau T}\Delta e_{1+\tau T}\right\}\right. \\
&\quad - T^{-3}g_T(\tau)T^{-2}\left\{\sum_2^T e_{t-1}^2 - e_{\tau T}^2\right\} \\
&\quad \left. + T^{-3}g_T(\tau)T^{-1}e_2 e_1 + (1-\tau)T^{-3}g_T(\tau)T^{-1}(e_{1+\tau T}^2 - e_{\tau T}^2)\right],
\end{aligned}$$

where $\hat{\sigma}^2 \equiv T^{-1}\sum_2^T \hat{\eta}_t^2$. Now it is straightforward using the results in equation (A4) to show that

$$T^{-3}g_T(\tau)T^{-2}\left\{\sum_3^T e_{t-1}^2 - e_{1+\tau T}^2\right\} \Longrightarrow K_{cf}(\tau),$$

$$T^{-3}g_T(\tau)T^{-1}\left\{\sum_2^T e_{t-1}\Delta e_t - e_{\tau T}\Delta e_{1+\tau T}\right\} \Longrightarrow H_{cf}(\tau),$$

$$T^{-3}g_T(\tau)T^{-1}e_2e_1 \Longrightarrow F_1(\tau),$$

$$(1-\tau)T^{-3}g_T(\tau)T^{-1}(e_{1+\tau T}^2 - e_{\tau T}^2) \Longrightarrow F_2(\tau),$$

and

$$\hat{\sigma}^2 \xrightarrow{p} \frac{1}{2}\sigma^2.$$

Hence, we have the required result:

$$\left[\frac{T-3}{2T-5}\right]^{1/2}\frac{\hat{\rho}-1}{\widehat{\mathrm{var}}(\hat{\rho})^{1/2}} \Longrightarrow \frac{H_{cf}(\tau) - K_{cf}(\tau) + F_1(\tau) + F_2(\tau)}{(g_{cf}(\tau)K_{cf}(\tau))^{1/2}}.$$

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