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A RANK TEST FOR CONSTANCY OF A LOCATION OR SCALE PARAMETER

F. LOMBARD

Department of Statistics, Rand Afrikaans University
PO Box 524, 2006 Auckland Park, South Africa

Key Words: changepoint model, periodogram, asymptotic distribution.

ABSTRACT

A nonlinear rank statistic is proposed to test whether a location or scale parameter of a continuous distribution has remained constant over time. The test is based on frequency domain characteristics of the data and uses the periodogram of appropriate rank scores. The test has good power against alternatives involving multiple abrupt up and down changes

1. INTRODUCTION

Consider a continuous distribution depending on a location or scale parameter θ. Let $X_1, \ldots, X_n$ be independent observations from the distribution with corresponding parameter values $\theta_1, \ldots, \theta_n$. Our interest is in testing the hypothesis of constancy, $H_0 : \theta_1 = \cdots = \theta_n$, against an alternative under which all the θ s are not the same. A particular case of this is the multiple change alternative

$$\theta_j = \begin{cases} \zeta_1 & \text{for } 1 \le j \le t_1 \\ \zeta_2 & \text{for } t_1 + 1 \le j \le t_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \zeta_{L+1} & \text{for } t_L + 1 \le j \le n \end{cases} \tag{1.1}$$

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which involves L unknown changepoints $1 < t_1 < \cdots < t_L < n$. A further problem is then to estimate these changepoints.

A general procedure for accomplishing this in the case where θ is a location parameter was given by Vostrikova (1981). An essential condition is that the changepoints are well separated in the sense that $\lim_{n\to\infty} t_k/n = \tau_k$ exists for all $k = 1, \ldots L$ and that $\tau_k - \tau_{k-1} > \alpha$ $(0 < \alpha < 1/2)$ for all $k = 1, \ldots, L+1$ $(\tau_0 \equiv 0, \tau_{L+1} \equiv 1)$. One begins by testing H_0 using the cumulative sum statistic

$$V_n = n^{1/2} \max_{1\le t\le n-1} \left| \sum_{j=1}^{t} (X_j - \bar{X}_n)/\{t(n-t)\}^{1/2} \right|, \tag{1.2}$$

rejecting H_0 for large values of V_n. If H_0 is not rejected one concludes that no changepoints are present. On the other hand. if H_0 is rejected an estimate of one of the changepoints is the t which maximizes $\sum_{j=1}^{t}(X_j - \bar{X}_n)/\{nt(n-t)\}^{1/2}$. Call the estimate $\hat{t}_{r_1}$. The data are now split into two sections, $X_1, \ldots, X_{\hat{t}_{r_1}}$ and $X_{\hat{t}_{r_1}+1}, \ldots, X_n$ and the preceding test, estimation and splitting procedure is applied to each of the two sections. One continues in this way until H_0 is not rejected on the data of any section. Vostrikova (1981) states that when $n \to \infty$ this procedure leads to consistent estimation of the number of changepoints as well as of their relative locations $\tau_1, \ldots, \tau_L$. Inclán and Tiao (1994, section 3) developed a similar procedure for the case where θ is the variance of a normal distribution with zero mean. Their test of H_0 is based on the centered cumulative sums of squares

$$D_t = \sum_{j=1}^{t} X_j^2 / \sum_{j=1}^{n} X_j^2 - t/n. \tag{1.3}$$

In the same situation, but allowing a constant unknown mean Chen and Gupta (1997, section 5) proposed to apply the Vostrikova (1981) algorithm with a test statistic derived from the Schwartz information criterion.

The success of these segmentation methods in finite sample size situations depends to a large extent on the abilities of the various test statistics to detect multiple changes in the value of θ. We will show that the tests mentioned above can have very poor power for some common changepoint configurations, for instance when an increase in θ is followed by a decrease of similar magnitude. This deficiency can be overcome by switching our viewpoint from the time

domain to the frequency domain. Define the periodogram of a sequence of numbers $a_1, \ldots, a_n$ by

$$I(\lambda) = n^{-1} \left| \sum_{t=1}^{n} a_t \exp(-i\lambda t) \right|^2, \ 0 < \lambda \le 0.5 \tag{1.4}$$

and denote the Fourier (or natural) frequencies by

$$\lambda_k = 2\pi k/n, \ k = 1, 2, \ldots, [n/2] \tag{1.5}$$

where, for all x, $[x]$ denotes the integer part of x. A distinguishing feature of many alternatives to constancy is that the periodogram of the θ_j sequence in (1.1) has, on the average, larger values at small k than at large k. In other words, a sequence of parameter values which is distinctly non-monotone when viewed in the time domain can exhibit some monotonicity feature in the frequency domain. Figure 1, for instance, shows a plot of the periodogram of a θ_j sequence of length $n = 100$ in a multiple changepoint model with jump sizes 0.5 and -0.5 at the 30 th and 71 st observations respectively.

These considerations suggest that a test of the constancy hypothesis be based on the correlation between the periodogram values $I(\lambda_k)$ of the data and a decreasing weight sequence $\{w_k\}$. In section 2 below we propose such a test statistic based on the weights $w_k = 1/k$ and show that it has good power against a wide range of alternatives to constancy. Replacing the data by appropriate functions of their ranks avoids the need to make specific assumptions regarding the type of distribution involved and also allows one to treat location and scale parameters in a unified manner. The test proposed in Section 2 will therefore be distribution free. The test statistic has a simple form of limiting distribution which allows easy computation of large-sample p-values. In Section 2 we also give some results from a Monte Carlo comparison between the power of the various tests. Section 3 gives details of the derivation of the limiting null distribution of the new test statistic.

2. THE TEST

We replace the raw data $X_1, \ldots, X_n$ by appropriate functions of their ranks

$$R_{n,t} = \text{ number of } X_j \le X_t, \ t = 1, \ldots, n.$$

That is, we replace X_t by the rank score

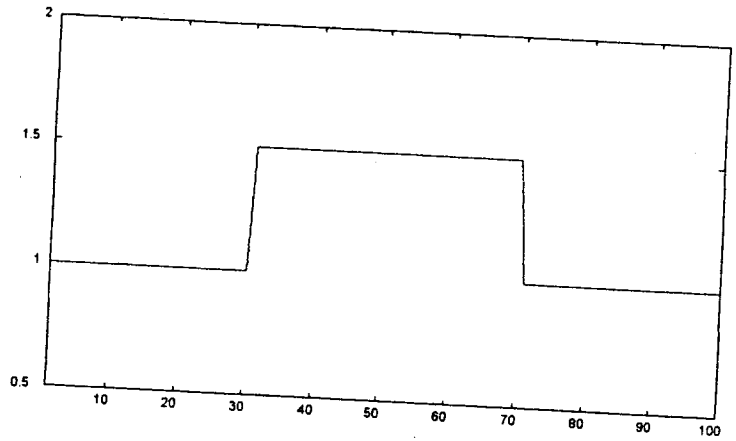

Figure 1a. Time sequence plot of parameter values. Parameter value on vertical, sequence number on horizontal.

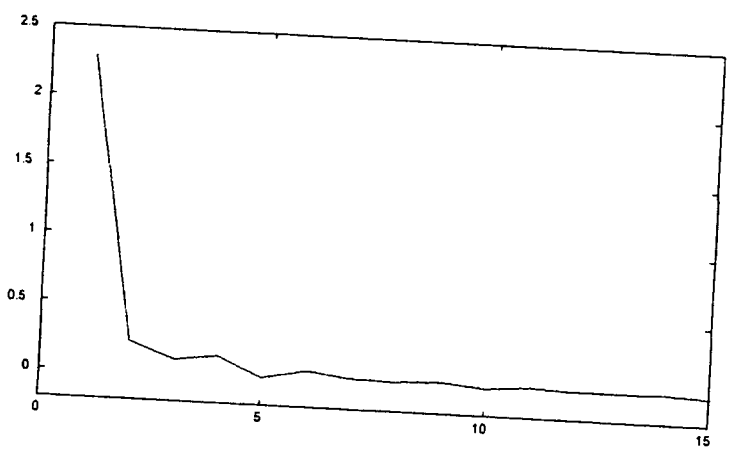

Figure 1b.

$$a_t = \phi(\frac{R_{n,t}}{n+1})$$

where ϕ is a function on the interval $(0,1)$. If θ is a location parameter typical choices for ϕ would be

$$\phi(u) = u \text{ (Wilcoxon scores) or } \phi(u) = \Phi^{-1}(u) \text{ (normal scores)}$$

where Φ^{-1} denotes the inverse of the standard normal distribution function, while if θ is a scale parameter one would typically use

$$\phi(u) = (u - 1/2)^2 \text{ or } \phi(u) = \left\{\Phi^{-1}(u)\right\}^2.$$

Let m denote the integer part of $(n-1)/2$. To test H_0 we propose to use the statistic

$$T_n = \sum_{k=1}^{m} \{I(\lambda_k)/\sigma_n^2 - 1\} w_k \tag{2.1}$$

where $I(\cdot)$, defined in (1.4), is the periodogram of the sequence of rank scores $a_1, \ldots, a_n$ and

$$\sigma_n^2 = var\{\sum_{t=1}^{n} \exp(-2\pi i \lambda_k t)\phi(\frac{R_{n,t}}{n+1})\} = (n-1)^{-1} \sum_{j=1}^{n} \phi^2(\frac{j}{n+1}).$$

Notice that $\sigma_n^2 \to \sigma^2$ as $n \to \infty$ where

$$\sigma^2 = \int_0^1 \phi^2 < \infty.$$

The null hypothesis is rejected when T_n is sufficiently large. The rationale for such a test was given in Section 1.

2.1. Asymptotic null distribution

It is shown in section 3 that the periodogram ordinates $I(\lambda_k)$, $k = 1, \ldots, m$ are "asymptotically independent and exponentially distributed" with mean σ^2. This leads one to expect that the distribution of T_n will tend to that of the random variable

$$T = \sum_{k=1}^{\infty} (e_k - 1) w_k \tag{2.2}$$

where the e_k are i.i.d. exponential random variables with unit mean.

Theorem 1. *Suppose ϕ is expressible as a finite sum of monotone functions and that $\int_0^1 \phi^4 < \infty$ and that the sequence w_k, $k \geq 1$, is square summable. Then T_n converges in law to T.*

The proof is given in section 3. In the special case $w_k = 1/k$ we have

$$\Pr\{T > t\} = 1 - \exp(-\exp(-t - \gamma)) \tag{2.3}$$

where

$$\gamma = \lim_{r\to\infty}\left(\sum_{k=1}^{r} k^{-1} - \log r\right) = 0.5772157\ldots$$

is Euler's constant. Thus, T has a (translated) type III extreme value distribution when $w_k = 1/k$. To prove this, fix $r > 1$ and let $e_{(1)} < \cdots < e_{(r)}$ denote the order statistics of $e_1, \ldots, e_r$. It is well known, see e.g. Karlin (1966, pp. 209-210), that

$$(e_k/k,\ k = 1, \ldots, r) =_{\mathcal{L}} (e_{(r-k+1)} - e_{(r-k)},\ k = 1, \ldots, r);\quad e_0 \equiv 0$$

where $=_{\mathcal{L}}$ denotes equality in distribution. Hence

$$\sum_{k=1}^{r} e_k/k =_{\mathcal{L}} e_{(r)}.$$

Since

$$\Pr\{e_{(r)} - \log(r) < t\} \to \exp(-\exp(-t)) \text{ as } r \to \infty,$$

the result (2.3) follows.

2.2. Power

We now present some results from a Monte Carlo study to compare the powers of some tests of H_0. Two situations are considered, namely where θ is a location parameter and where it is a scale parameter. The data are generated from a normal distribution. We show the results for sample sizes $n = 50$ and $n = 100$. Each comparison is based on 2,000 Monte Carlo trials. The critical level used was 0.05 in each case, "exact" critical values first having been found by Monte Carlo simulation.

Location parameter

The cusum test based on the statistic (1.2) is compared to the rank test based on (2.1)with $w_k = 1/k$ and normal scores $\phi(u) = \Phi^{-1}(u)$ in the following parameter configurations:

(a) X_i is $N(\theta_i, 1)$ with $\theta_i = 0$ for $i \le n/2$, $\theta_i = \zeta$ for $i > n/2$ and $\zeta = 0.5,\ 1.0$. This is an "at most one change" model;

(b) X_i is $N(\theta_i, 1)$ with $\theta_i = \zeta$ for $|i - n/2| \le n/5$, $\theta_i = 0$ otherwise and $\zeta = 0.5,\ 1.0$. This is often referred to in the literature as an "epidemic alternative";

Table I. Estimated powers of rank test (in brackets) and cusum test.

	$n = 50$		$n = 100$	
	$\zeta = 0.5$	$\zeta = 1.0$	$\zeta = 0.5$	$\zeta = 1.0$
(a)	0.24 (0.24)	0.84 (0.77)	0.50 (0.50)	0.99 (0.98)
(b)	0.12 (0.23)	0.45 (0.75)	0.17 (0.42)	0.79 (0.97)
(c)	0.09 (0.10)	0.20 (0.24)	0.11 (0.18)	0.34 (0.74)

Table II. Estimated powers of rank test (in brackets) and SIC test.

	$n = 50$		$n = 100$	
	$\zeta = 2.0$	$\zeta = 2.5$	$\zeta = 2.0$	$\zeta = 2.5$
(a)	0.78 (0.58)	0.95 (0.79)	0.97 (0.92)	1.00 (1.00)
(b)	0.45 (0.59)	0.77 (0.82)	0.82 (0.93)	0.99 (1.00)
(c)	0.22 (0.20)	0.38 (0.33)	0.41 (0.59)	0.72 (0.86)

(c) X_i is $N(\theta_i, 1)$ with $\theta_i = \zeta$ for $|i - [n/4]| \le n/10$ and $|i + 1 - [3n/4]| \le n/10$, $\theta_i = 0$ otherwise and $\zeta = 0.5,\ 1.0$.

On the basis of these limited simulation results it would seem that the rank test is superior to the cusum test for the epidemic-type alternatives and comparable to it in the single changepoint model.

Scale parameter

The Schwartz information criterion (SIC) test of Chen and Gupta is compared to the rank test based on (2.1) with $w_k = 1/k$ and normal scores $\phi(u) = \{\Phi^{-1}(u)\}^2$ in the following parameter configurations:

(a) X_i is $N(0, \theta_i^2)$ with $\theta_1 = 1$ for $i \le n/2$, $\theta_i = \zeta$ for $i > n/2$ and $\zeta = 1.5,\ 2.0,\ 2.5$.

(b) X_i is $N(0, \theta_i^2)$ with $\theta_i = \zeta$ for $|i - n/2| \le n/5$, $\theta_i = 1$ otherwise and $\zeta = 1.5,\ 2.0,\ 2.5$..

(c) X_i is $N(0, \theta_i^2)$ with $\theta_i = \zeta$ for $|i - n/4| \le n/10$ and $|i - 3n/4| \le n/10$, $\theta_i = 0$ otherwise and $\zeta = 1.5,\ 2.0,\ 2.5$.

The SIC test seems to be generally superior to the rank test in the single changepoint model, the reverse being true for the epidemic-type models.

The conclusions reached above are only tentative because of the limited scope of the Monte Carlo study. Nevertheless, the results indicate that it

could be unwise to use the cusum or SIC tests in a segmentation algorithm if one suspects that multiple up and down changes are present.

3. PROOF OF THEOREM 1

Lemma 1

Let the conditions of Theorem 1 be satisfied. Then

$$cov(I(\lambda_k), I(\lambda_{k'})) = \sigma^4 \times \mathcal{I}(k = k') + O(n^{-1})$$

where

$$\sigma^2 = \int_0^1 \phi^2$$

and $\mathcal{I}(\cdot)$ denotes the indicator function.

Proof

The proof consists of straightforward, but rather tedious, calculations. We merely indicate the main steps. To ease the writing we set (in this proof only)

$$e(x) = \exp(-2\pi i x).$$

Note that

$$I(\lambda_k) = n^{-1}\sum_s \phi^2(\frac{R_{ns}}{n+1}) + n^{-1}\sum\sum_{s\neq t} \phi(\frac{R_{ns}}{n+1})\phi(\frac{R_{nt}}{n+1})e(\lambda_k(s-t))$$

Then

$$cov\{I(\lambda_k), I(\lambda_{k'})\} = C_1 + C_2 + C_3 + C_4$$

where

$$\begin{aligned}
C_1 &= n^{-2}\sum_s\sum_t cov\{\phi^2(\tfrac{R_{ns}}{n+1}), \phi^2(\tfrac{R_{nt}}{n+1})\} \\
C_2 &= n^{-2}\sum_u\sum\sum_{s\neq t} cov\{\phi^2(\tfrac{R_{nu}}{n+1}), \phi(\tfrac{R_{ns}}{n+1})\phi(\tfrac{R_{nt}}{n+1})\}e(\lambda_k(s-t)) \\
C_3 &= n^{-2}\sum_u\sum\sum_{s\neq t} cov\{\phi^2(\tfrac{R_{nu}}{n+1}), \phi(\tfrac{R_{ns}}{n+1})\phi(\tfrac{R_{nt}}{n+1})\}e(\lambda_{k'}(s-t)) \\
C_4 &= n^{-2}\sum\sum_{s\neq t}\sum\sum_{u\neq v} cov\{\phi(\tfrac{R_{ns}}{n+1})\phi(\tfrac{R_{nt}}{n+1}), \phi(\tfrac{R_{nu}}{n+1})\phi(\tfrac{R_{nv}}{n+1})\} \\
&\quad \times e(\lambda_k(s-t) - \lambda_{k'}(u-v)).
\end{aligned}$$

To evaluate these terms we use the (easily derived) result that

$$\begin{aligned}
&cov\{f_1(\tfrac{R_{ns}}{n+1})f_2(\tfrac{R_{nt}}{n+1}), f_3(\tfrac{R_{nu}}{n+1})f_4(\tfrac{R_{nv}}{n+1})\} \\
&= \begin{cases} O(1) & \text{if } s = u, t = v \text{ or } s = v, t = u \\ O(n^{-1}) & \text{otherwise} \end{cases}
\end{aligned}$$

for any functions $f_i,\ i = 1, \ldots, 4$ satisfying

$$\int |f_1 f_2 f_3 f_4| < \infty.$$

We also use the fact that

$$\sum_s e(\lambda_k s) = 0.$$

It then follows easily that C_1, C_2 and C_3 are all $O(n^{-1})$. The summation in C_4 can be split into six separate summations:

$$\sum\sum_{s\neq t}\sum\sum_{u\neq v} = \sum\sum\sum_{u=s\neq t\neq v} + \text{three similar terms} + \sum\sum\sum\sum_{s\neq t\neq u\neq v} + \sum\sum_{s=u\neq t=v}.$$

The first five of these again evaluate to $O(n^{-1})$. The sixth one is

$$\begin{aligned}
&n^{-2}\sum\sum_{s\neq t} var\{\phi(\tfrac{R_{ns}}{n+1})\phi(\tfrac{R_{nt}}{n+1})\}e((\lambda_k - \lambda_{k'})(s-t)) \\
&= \{\mathcal{I}(k = k') - n^{-1}\}var\{\phi(\tfrac{R_{n1}}{n+1})\phi(\tfrac{R_{n2}}{n+1})\} \\
&= \mathcal{I}(k = k')\sigma^4 + O(n^{-1}).
\end{aligned}$$

The result follows.

Lemma 2

Let the conditions of Theorem 1 be satisfied and let $r,\ 1 \le r \le m$, be fixed. Then

$$(I(\lambda_1), \ldots, I(\lambda_r)) \to_{\mathcal{L}} \sigma^2 \times (e_1, \ldots, e_r).$$

Proof

Define, for $k = 1, \ldots, r$,

$$c(k, x) = \cos(2\pi k x),\ s(k, x) = \sin(2\pi k x)$$

and

$$C_{n,k} = n^{-\frac{1}{2}}\sum_{t=1}^{n}\cos(2\pi k\frac{t}{n})\phi(\frac{R_{nt}}{n+1}),\ S_{n,k} = n^{-\frac{1}{2}}\sum_{t=1}^{n}\sin(2\pi k\frac{t}{n})\phi(\frac{R_{nt}}{n+1})$$

and observe that

$$I(\lambda_k) = C_n^2 + S_n^2.$$

We use the Cramer-Wold device together with Theorem V.1.6a and Lemma V.1.6a of Hájek and Šidák (1967) to show the joint asymptotic normality of the random vector

$$V_n = (C_{n,1}, \ldots C_{n,r}, S_{n,1}, \ldots, S_{n.r}).$$

Let $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r$ be constants and set

$$\begin{aligned} U_n &= \textstyle\sum_{k=1}^{r}(\alpha_k C_{n,k} + \beta_k S_{n,k}) \\ &= \textstyle\sum_{t=1}^{n} b_{n,t}\phi(\frac{R_{nt}}{n+1}). \end{aligned}$$

where

$$b_{n,t} = n^{-\frac{1}{2}} \sum_{k=1}^{r} \{\alpha_k \cos(2\pi k \frac{t}{n}) + \beta_k \sin(2\pi k \frac{t}{n})\}.$$

Note that U_n has the form of a linear rank statistic. Now, using the Cauchy-Schwartz and c_r $(r = 2)$ inequalities,

$$\begin{aligned} b_{n,t}^2 &\leq n^{-1} r \textstyle\sum_{k=1}^{r}\{\alpha_k \cos(2\pi k \frac{t}{n}) + \beta_k \sin(2\pi k \frac{t}{n})\}^2 \\ &\leq 2n^{-1} r \textstyle\sum_{k=1}^{r}\{\alpha_k^2 \cos^2(2\pi k \frac{t}{n}) + \beta_k^2 \sin^2(2\pi k \frac{t}{n})\} \\ &\leq 2n^{-1} r \textstyle\sum_{k=1}^{r}\{\alpha_k^2 + \beta_k^2\} = O(n^{-1}) \end{aligned}$$

while use of the relations

$$\sum_{k=1}^{r} \cos^2(2\pi k \frac{t}{n}) = \sin^2(2\pi k \frac{t}{n}) = \frac{1}{2} \text{ and } \sum_{k=1}^{r} \cos(2\pi k \frac{t}{n}) \sin(2\pi k \frac{t}{n}) = 0$$

leads to

$$\sum_{t=1}^{n} b_{n,t}^2 = \frac{1}{2} \sum_{k=1}^{r} (\alpha_k^2 + \beta_k^2).$$

It follows that $V_n \to_L \frac{\sigma^2}{2}(Z_1, \ldots, Z_{2k})$ where the Z s are independent normal $(0, 1)$ random variables. Hence

$$(I(\lambda_1), \ldots, I(\lambda_r)) \to_{\mathcal{L}} \sigma^2 (\frac{Z_1^2 + Z_{r+1}^2}{2}, \ldots, \frac{Z_r^2 + Z_{2r}^2}{2}) =_{\mathcal{L}} \sigma^2 (e_1, \ldots, e_r).$$

This completes the proof of the lemma.

To complete the proof of Theorem 1 set

$$T_n = \sum_{k=1}^{m} \{I(\omega_k)/\sigma^2 - 1\} w_k,$$

$$T_{n,r} = \sum_{k=1}^{r} \{I(\omega_k)/\sigma^2 - 1\} w_k$$

and

$$T_r = \sum_{k=1}^{r} (e_k - 1) w_k.$$

where r $(< m)$ is fixed. It follows from Lemma 2 that $T_{n,r} \to_L T_r$ as $n \to \infty$ and from Proposition 13.15 in Breiman (1968), $T_r \to_L T$ as $r \to \infty$. Next,

from Lemma 1 and the Cauchy-Schwartz inequality, for some positive constant K,

$$\begin{aligned} var(T_n - T_{n,r}) &\leq \sigma^4 \textstyle\sum_{k=r+1}^{m} w_k^2 + \frac{K}{n}(\sum_{k=r+1}^{m} w_k)^2 \\ &\leq (\sigma^4 + K) \textstyle\sum_{k=r+1}^{m} w_k^2. \end{aligned}$$

Thus we see that

$$\lim_{r\to\infty} \limsup_{n\to\infty} var(T_n - T_{n,r}) = 0.$$

It now follows from Theorem 4.2 of Billingsley (1968) that $T_n \to_L T$.

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