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TESTING FOR UNIT ROOTS IN MODELS WITH STRUCTURAL CHANGE

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This paper considers the unit root tests in models with structural change. Particular attention is given to their dependency on the limiting ratios of the subsample sizes between breaks. The dependency is analyzed in detail, and the invariant testing procedure based on a transformed model is developed. The required transformation is essentially identical to the generalized least-squares correction for heteroskedasticity. The limiting distributions of the new tests do not depend on the relative sizes of the subsamples and are shown to be simple mixtures of the limiting distributions of the corresponding tests from the independent unit root models without structural change.

1. INTRODUCTION

The evidence for the presence of unit roots in many economic time series has been accumulated at an accelerated rate since the work by Nelson and Plosser [5]. Besides some financial series like stock prices and exchange rates, it is now widely believed that many important macroeconomic time series such as consumption, GNP, and money supply are also well characterized as random walks or, more generally, as integrated processes. The unit root hypothesis has been investigated for such series by many authors using the tests by Dickey and Fuller [2,3] and Phillips [12] and their extensions.

The aforementioned macroeconomic time series and many others show a strong nonreversible, growing tendency. It seems therefore sensible to eliminate the "deterministic" component from these variables prior to investigating the stochastic nature that they reveal. The behavior of the time series after detrending, of course, is dependent upon the specification of the deterministic trend. Especially, trend specification is important when one seeks to test for unit roots, because the deterministic component can severely distort the test results if not properly taken care of. This is well demonstrated in Per-

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ron [10]. It is clear that an unexplained increasing deterministic time trend would support the presence of unit root in favor of stationarity.

The linear time trend has widely been used to model the growing tendency in many economic series, on the ground that it represents some constant growth rate. The unit root tests in models with linear time trend are studied in Dickey and Fuller [2,3] and Phillips and Perron [14], among many others. It appears in many cases that the linear trend specification of the deterministic component is appropriate and well justified. In many other cases, however, specifically when the time span of data is relatively long, the constant growth rate implied by a linear trend seems to be hard to justify. A natural extension of the linear time trend is the polynomial time trend. The model with the deterministic component represented by a time polynomial has indeed been considered by Park and Choi [7] and Ouliaris et al. [6].

There is yet another class of models whose deterministic trend cannot be represented effectively by a time polynomial of finite order: models with structural change. Such models, in their simplest form, include time dummy variables to permit structural changes in the deterministic component of the underlying time series. Though such a specification assuming the break points are known a priori may be subject to various criticism, as was raised by Zivot and Andrews [16], it often looks appealing. For many historical economic time series data, in fact, it is not uncommon that growth rate changes might be reasonably thought to have occurred at some fixed points. Research along this line has successfully been done by Perron [10]. Having tested for unit roots in models with dummy variables for structural change, he provided somewhat surprising results: Most U.S. macroeconomic time series reject the unit root hypothesis if we allow for a structural break due to the Great Depression or the oil price shock.

The purpose of this paper is twofold. First, we investigate the asymptotic distributions of the unit root test statistics in models with structural change. The dependency of the tests on the break points, which was observed earlier by Perron [10], is in particular analyzed in detail. The null distributions are shown to involve weighted sums of those for the independent unit root models without structural change, and the critical values depend symmetrically on the limiting ratios of the subsample sizes. The half of the critical values of the Phillips and Dickey–Fuller tests reported by Perron [10] for models with a single break, for instance, are therefore shown to be redundant.

Secondly and more importantly, we develop new tests that are invariant with respect to the break points. The tests are based on a transformed model. The transformation is, both in motivation and operationally, essentially identical to the generalized least-squares correction for heteroskedasticity. Our procedure uses weights that are inversely proportional to the relative ratios of the subsample sizes. This is parallel to the generalized least-squares transformation, which weighs observations disproportionately according to the reciprocals of error variances. The new testing procedure drastically decreases the dimension of the required tabulation of critical values, which is necessary for practical applications. Perron's [10] tabulation for the Phillips and Dickey–Fuller tests with the grid size 0.1 may serve well for models with a single break, if not entirely satisfactory from the theoretical point of view. Such an attempt to tabulate critical values for several other existing unit root tests, especially for models with multiple breaks, however, would quickly become unpracticable.

The paper is organized as follows. Section 2 presents the model with underlying assumptions. In Section 3 the limiting distributions of the unit root test statistics are obtained and analyzed. The tests are shown to depend on the relative sizes of the subsamples partitioned by the break. This dependency is further analyzed in Section 4, and new tests are developed. The new tests are shown to be invariant to the break point. Section 5 investigates the asymptotic powers of the tests under the local alternatives. For the simplicity of exposition, it is assumed that we have only one structural break in Sections 2–5. Section 6 extends the results in previous sections to allow for multiple breaks. Some concluding remarks follow in Section 7, and the proofs of the theorems are given in Appendix A. Finally, some critical values of the tests are tabulated in Appendix B.

2. THE MODEL AND ASSUMPTIONS

We consider the test of a unit root in the time series $\{x_t\}$, for which the stochastic and deterministic components, $\{x_t^s\}$ and $\{x_t^d\}$, are given, respectively, by

$$x_t^s = \alpha x_{t-1}^s + u_t, \tag{1}$$

with $\alpha = 1$, and

$$x_t^d = \sum_{k=1}^{p} \mu_k t^k + \sum_{k=1}^{p} \mu_k^+ t_m^k, \tag{2}$$

where

$$t_m^k = \begin{cases} 0 & \text{when } t \in T_1 \text{ for } T_1 = \{1, \ldots, m\}, \\ (t-m)^k & \text{when } t \in T_2 \text{ for } T_2 = \{m+1, \ldots, n\}. \end{cases}$$

We therefore assume that there is a structural shift in the data generating mechanism of $\{x_t\}$ at time $t = m + 1$. The constant term and the intercept dummy variable are not included in the specification of the deterministic component $\{x_t^d\}$ in (2), because they are dominated by the stochastic component $\{x_t^s\}$ and not identifiable in our asymptotic analysis. This does not imply that such terms are not permissible; they are allowed but simply not parametrized here. The initial condition for $\{x_t^s\}$ in (1) also does not affect the subsequent theory, and we let x_0^s be any random variable. The $(n+1)$ observations for $t = 0, \ldots, n$, are assumed to be available for $\{x_t\}$.

Throughout the paper we assume that the sequence $\{u_t\}$ in (1) has the mean zero and satisfies an invariance principle. More precisely, it is assumed that if we construct the stochastic process B_n by

$$B_n(r) = \frac{1}{\sqrt{n}} \sum_{t=1}^{[nr]} u_t$$

for $r \in [0,1]$, then

$$B_n \xrightarrow{\mathfrak{D}} B \equiv \omega W, \tag{3}$$

where W is the standard Brownian motion on [0,1] and

$$\omega^2 = \lim_{n\to\infty} \frac{1}{n} E\left(\sum_{t=1}^{n} u_t\right)^2.$$

The limit process B is thus a scaled Brownian motion. Also, we denote by

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} E(u_t^2)$$

and assume

$$\frac{1}{n} \sum_{t=1}^{n} u_t^2 \xrightarrow{\mathcal{P}} \sigma^2. \tag{4}$$

In (3) and (4) we assume $\sigma^2, \omega^2 > 0$.

Invariance principle (3) is known to apply to a large class of weakly dependent and possibly heterogenous processes (see Phillips [11] and the reference cited there for the explicit conditions under which (3) holds). In particular, (3) holds for general linear stationary processes including all practical ARMA models, as shown in Phillips and Ouliaris [13]. Note that when the sequence $\{u_t\}$ is a stationary process with spectral density $f(\lambda)$ and absolutely summable covariance function $\gamma(k)$, the parameters introduced in (3) and (4) reduce to $\sigma^2 = \gamma(0)$ and $\omega^2 = 2\pi f(0)$.

Our model therefore describes a very general time series, the stochastic component of which is integrated and has a unit root. As in Park and Choi [7] and Ouliaris et al. [6], we allow the deterministic part of the series to be driven by a time polynomial of an arbitrary order. Especially, a structural change is introduced and represented by a polynomial of time dummy variables as in Perron [10]. Formulations (1) and (2) thus generalize various unit root models considered earlier by many authors. Our approach is nonparametric and parallel to that of Phillips [11], Phillips and Perron [14], Park and Choi [7], Perron [10], and Ouliaris et al. [6].

3. TEST STATISTICS AND THEIR LIMITING DISTRIBUTIONS

The test of a unit root in $\{x_t\}$ is customarily based on the regression

$$x_t = \alpha x_{t-1} + \sum_{k=0}^{p} \nu_k t^k + \sum_{k=0}^{p} \nu_k^+ t_m^k + u_t. \tag{5}$$

The null hypothesis in regression (5) is, of course,

$$\alpha = 1, \tag{6}$$

which is tested to be against the alternative hypothesis $|\alpha| < 1$. For $\{x_t\}$ generated by (1) and (2), we have $\nu_k = \mu_{k+1}$ and $\nu_k^+ = \mu_{k+1}^+$ for $k = 0, \ldots, p-1$, and $\nu_p = \nu_p^+ = 0$. The terms t^p and t_m^p are therefore redundant under the null hypothesis (6). Their inclusion in regression (5), however, is necessary to allow for the same deterministic trend under the alternative hypothesis.

Following Dickey and Fuller [2,3], we first consider two statistics: $n(\hat{\alpha} - 1)$ and $t(\alpha)$, where $\hat{\alpha}$ is the least-squares estimate of α in (5) and $t(\alpha)$ denotes the standard t-statistic for hypothesis (6). The next theorem gives the limiting null distributions of the Dickey–Fuller statistics. We assume $m/n \to c$ as n tends to infinity.

THEOREM 1. *Suppose* $0 < c < 1$. *Then*

$$n(\hat{\alpha} - 1) \xrightarrow{\mathfrak{D}} \left(\omega^2 \int_0^1 V^2\right)^{-1} \left(\omega^2 \int_0^1 V\,dW + \lambda\right),$$

$$t(\alpha) \xrightarrow{\mathfrak{D}} \left(\sigma\omega \int_0^1 V^2\right)^{-1/2} \left(\omega^2 \int_0^1 V\,dW + \lambda\right),$$

where W is the standard Brownian motion, $\lambda = (\omega^2 - \sigma^2)/2$, *and*

$$V = W - \int_0^1 Wh' \left(\int_0^1 hh'\right)^{-1} h,$$

$$h = (f_0, \ldots, f_p, f_0^+, \ldots, f_p^+)',$$

where in turn

$$f_k(r) = r^k, \qquad 0 \le r \le 1,$$

$$f_k^+(r) = \begin{cases} (r - c)^k & \textit{if } c \le r \le 1, \\ 0, & \textit{otherwise}. \end{cases}$$

Similarly as in Phillips [11] and Phillips and Perron [14] the limiting distributions of the Dickey–Fuller tests are shown in Theorem 1 to depend on various nuisance parameters for general integrated processes. There is a special case that the tests are invariant with respect to the parameters of the stochastic component of $\{x_t\}$. When $\{u_t\}$ in (1) is a martingale difference sequence, as is primarily assumed in Dickey and Fuller [2,3], we have

$\omega^2 = \sigma^2$ and $\lambda = 0$. The results in Theorem 1 are simplified correspondingly, and the limiting distributions do not include any parameter of the stochastic component $\{x_t\}$.

It is interesting to note that the stochastic process V can simply be defined as the Hilbert space projection residual in $L^2[0,1]$, the set of square integrable functions on $[0,1]$ with the inner product $(x,y) = \int_0^1 xy$. More precisely, for each realization, $V(r)$ is the residual function from the projection of $W(r)$ on the subspace generated by the set of functions $\{f_k, f_k^+\}_{k=0}^p$. We can easily see that

$$\frac{1}{n}\sum_{t=1}^{n}\frac{t^k}{n^k} \xrightarrow{\mathcal{L}^2} \int_0^1 f_k \quad \text{and} \quad \frac{1}{n}\sum_{t=1}^{n}\frac{t_m^k}{n^k} \xrightarrow{\mathcal{L}^2} \int_0^1 f_k^+.$$

In fact, the results in Theorem 1 may be regarded as a direct consequence of the fact that the projection operation is preserved under the passage into the asymptotics for the regressions with integrated processes like (5). This was first pointed out by Park and Phillips [8,9].

As we have already mentioned, the Dickey–Fuller tests are nuisance parameter dependent and not applicable for a more general process. The Phillips [12] tests can, however, be easily extended to our model with structural change. We define

$$\tau_1 = n(\hat{\alpha} - 1) - \frac{n^2\hat{\lambda}}{\sum_1^n \hat{x}_{t-1}^{s2}}, \tag{7}$$

$$\tau_2 = \frac{\hat{\sigma}}{\hat{\omega}}\, t(\alpha) - \frac{n\hat{\lambda}}{\left(\sum_1^n \hat{x}_{t-1}^{s2}\right)^{1/2}}, \tag{8}$$

where $\hat{\lambda} = (\hat{\omega}^2 - \hat{\sigma}^2)/2$, $\hat{\omega}^2$ and $\hat{\sigma}^2$ are consistent estimates of ω^2 and σ^2, respectively, and $\{\hat{x}_{t-1}^s\}$ is the residual from the least-squares regression of $\{x_{t-1}\}$ on t^k and t_m^k for $k = 0, \ldots, p$. The reader is referred to Andrews [1] and the reference cited there for the consistent estimation of ω^2. A consistent estimate of σ^2 is given simply by the usual variance estimate. The estimates $\hat{\omega}^2$ and $\hat{\sigma}^2$ are, of course, to be obtained from the least-squares residuals from regression (5).

If $\{u_t\}$ in (1) is assumed to be an ARMA process, then we may also consider the regression

$$x_t = \alpha x_{t-1} + \sum_{k=0}^{p} \nu_k t^k + \sum_{k=0}^{p} \nu_k^+ t_m^k + \sum_{k=1}^{s} \alpha_k \Delta x_{t-k} + u_t \tag{9}$$

following Said and Dickey [15]. It is possible to show that the usual t-statistic for α in regression (9) can be used to test for a unit root for general ARMA models, when the number s of the lagged differences is increased at the controlled rate $o(n^{1/3})$. The limiting distribution of the t-statistic in (9) for null hypothesis (6) is identical to that of τ_2 in (8). This can be shown as in Phillips and Ouliaris [13]. To simplify the exposition, we will not consider the Said–Dickey statistic explicitly in the rest of the paper. For all the methods developed later in this paper, the Phillips-type nonparametric correction in (8) can be replaced by the Said–Dickey approach based on the regression augmented with differenced lags such as (9).

It is simple to deduce the following corollary.

COROLLARY 2. *Suppose* $0 < c < 1$. *Then*

$$\tau_1 \xrightarrow{\mathfrak{D}} \left(\int_0^1 V^2\right)^{-1} \int_0^1 V\,dW,$$

$$\tau_2 \xrightarrow{\mathfrak{D}} \left(\int_0^1 V^2\right)^{-1/2} \int_0^1 V\,dW,$$

where notation is defined in Theorem 1.

The τ_1 and τ_2 statistics are therefore invariant across a wide class of data generating processes for the stochastic component of $\{x_t\}$. Their limiting distributions are free of nuisance parameters not just for models generated by martingale differences but also for general integrated processes.

The limiting distributions in Corollary 2, however, are dependent on c, the limiting proportion of the samples before the break. For the tests to be applicable, it is therefore necessary to find the critical values for each of the relevant values of c. From a practical point of view, the dependency of the tests on the break point is certainly undesirable, if not critical. The complete tabulation for $c \in [0,1]$ is, of course, impossible even for models with a single break. This may not be a serious problem for such simple models, because the tabulation of the distributions with a fine enough grid would serve well for any practical purpose. This is certainly feasible. Perron [10] indeed tabulates the distributions for $c = 0.1, \dots, 0.9$, the cases of $p = 0$ and 1.

Such a tabulation, nevertheless, quickly becomes overly burdensome, as we allow for higher-order time polynomials and, in particular, multiple breaks. Fortunately, it is possible to develop unit root tests that do not depend on the break points as long as they are known a priori. We shall subsequently introduce new tests for the unit root hypothesis and show how the procedure can be readily extended to models with more than one break. To motivate the construction of such tests, we will first investigate the limiting distributions given in Corollary 2 in further detail.

4. BREAK POINT DEPENDENCY AND INVARIANT TESTS

In this section we develop tests for unit roots whose critical values do not depend on the proportion of prebreak sample in regression (5). First, we look more closely at the dependency on c of the limiting distributions in Corollary 2.

LEMMA 3. *Let the process V be defined as in Theorem* 1. *Then*

$$V(r) = \begin{cases} \sqrt{c}V_1\left(\dfrac{r}{c}\right), & 0 \le r \le c, \\ \sqrt{1-c}V_2\left(\dfrac{r-c}{1-c}\right), & c < r \le 1, \end{cases}$$

where

$$V_k = W_k - \int_0^1 W_k f' \left(\int_0^1 ff' \right)^{-1} f, \qquad k = 1,2,$$

$$f = (f_0, \dots, f_p)' = (1, r, \dots, r^p)',$$

and W_1 and W_2 are independent standard Brownian motions defined on $[0,1]$.

In Lemma 3 we represent the stochastic process V introduced in Theorem 1 in terms of two independent functionals of Brownian motion. The result follows from the Hilbert space projection theory and the fact that Brownian motion has independent increments. We can easily deduce from Lemma 3 the next proposition.

PROPOSITION 4. *Suppose* $0 < c < 1$. *Then*

$$\tau_1 \xrightarrow{\mathfrak{D}} \left(c^2 \int_0^1 V_1^2 + (1-c)^2 \int_0^1 V_2^2 \right)^{-1} \left(c \int_0^1 V_1\, dW_1 + (1-c) \int_0^1 V_2\, dW_2 \right),$$

$$\tau_2 \xrightarrow{\mathfrak{D}} \left(c^2 \int_0^1 V_1^2 + (1-c)^2 \int_0^1 V_2^2 \right)^{-1/2} \left(c \int_0^1 V_1\, dW_1 + (1-c) \int_0^1 V_2\, dW_2 \right),$$

where notation is defined in Lemma 3.

The results in Proposition 4 are very helpful in understanding the dependency on c of the limiting distributions of the statistics τ_1 and τ_2 in (7) and (8). Interestingly, they just involve the weighted sums of the corresponding terms for the two independent unit root models without structural change (see Phillips [11], Phillips and Perron [14], and Ouliaris et al. [6]. Notice also that the limiting distributions of τ_1 and τ_2 remain the same even if we interchange c and $(1-c)$. This can be easily seen since the roles of c and $(1-c)$ are symmetric and interchangeable in our representation of the limiting distributions in Proposition 4. Asymmetry of the critical values obtained by Perron [10] is thus simply due to the sampling variations.

We now consider the regression

$$\Delta x_t = \beta x_{t-1}^* + \sum_{k=0}^{p} \nu_k t^k + \sum_{k=0}^{p} \nu_k^+ t_m^k + u_t, \tag{10}$$

where

$$x_{t-1}^* = \begin{cases} (n/m)x_{t-1}, & t-1 \in T_1, \\ (n/(n-m))x_{t-1}, & t-1 \in T_2. \end{cases}$$

As in (5), $\nu_k = \mu_{k+1}$ and $\nu_k^+ = \mu_{k+1}^+$ for $k = 0, \ldots, p-1$, and $\nu_p = \nu_p^+ = 0$. For the model given by (1) and (2), we have $\beta = 0$.

For the unit root hypothesis, we may test

$$\beta = 0 \tag{11}$$

in regression (10). Similarly as in (7) and (8), we define

$$\tau_1^* = n\hat{\beta} - \frac{2n^2\hat{\lambda}}{\sum_1^n \hat{x}_{t-1}^{s*2}}, \tag{12}$$

$$\tau_2^* = \frac{\hat{\sigma}}{\hat{\omega}} t(\beta) - \frac{2n\hat{\lambda}}{\left(\sum_1^n \hat{x}_{t-1}^{s*2}\right)^{1/2}}, \tag{13}$$

where $\{\hat{x}_{t-1}^{s*}\}$ is the residual from the least-squares regression of $\{x_{t-1}^*\}$ on t^k and t_m^k for $k = 0, \ldots, p$, $t(\beta)$ is the standard t-statistic for hypothesis (11) in regression (10), and other notation is defined as in (7) and (8). Consistent estimates of the parameters can be obtained using the residuals from either (5) or (10). Under null hypothesis (10), the transformation for $\{x_{t-1}^*\}$ in (10) clearly would not affect the limiting behavior of the residuals. We may consider regression (10) augmented by Δx_{t-k}'s to use the approach by Said and Dickey [15].

The motivation for the new statistics τ_1^* and τ_2^* in (12) and (13) is simple and straightforward from the results in Proposition 4. The following theorem shows that the asymptotic distributions of these statistics do not depend on c.

THEOREM 5. *Suppose* $0 < c < 1$. *Then*

$$\tau_1^* \xrightarrow{\mathfrak{D}} \left(\int_0^1 V_1^2 + \int_0^1 V_2^2\right)^{-1} \left(\int_0^1 V_1\,dW_1 + \int_0^1 V_2\,dW_2\right),$$

$$\tau_2^* \xrightarrow{\mathfrak{D}} \left(\int_0^1 V_1^2 + \int_0^1 V_2^2\right)^{-1/2} \left(\int_0^1 V_1\,dW_1 + \int_0^1 V_2\,dW_2\right),$$

where notation is defined in Proposition 4.

As we have shown earlier in Proposition 4, the limiting distributions of the conventional τ_1 and τ_2 are mixtures of the corresponding statistics from the two independent unit root models without structural changes. The statistics τ_1^* and τ_2^* based on the transformed regression in (10) makes appropriate adjustments for the weights c and $(1 - c)$. It is interesting to note that the transformation in (10) is quite similar to the generalized least-squares correction for heteroskedasticity. The rationale behind the adjustment is intuitively clear. In the transformation for regression (10), we give the weights n/m and $n/(n - m)$ for the two subsamples. The weights are inversely proportional, respectively, to the sizes of pre- and postbreak samples relative to the sample size. Therefore, we effectively give a smaller weight to a longer subsample, which introduces more variation to the test statistics. The transformation in (10) is in this sense also very similar to the heterogeneity correction made by the generalized least squares.

Notice that the limiting distributions of τ_1^* and τ_2^* are, respectively, the same as those of $\tau_1/2$ and τ_2 with $c = 1 - c = \frac{1}{2}$. Therefore, the critical values in Perron [10] (for $\lambda = \frac{1}{2}$ in his notation) can be used for τ_1^* and τ_2^* tests, in the case of $p = 0$ and $p = 1$. We provide some additional critical values of these statistics in Appendix B.

5. ASYMPTOTIC LOCAL POWER

To investigate the asymptotic powers of the tests introduced in previous sections, we consider the sequence of local alternatives given by

$$\alpha = \exp\left(-\frac{\delta}{n}\right) \approx 1 - \frac{\delta}{n} \tag{14}$$

as in Phillips [12]. The following theorem can easily be deduced from Phillips [12] and our earlier results.

THEOREM 6. *Suppose* $0 < c < 1$. *Then under the local alternatives* (14)

$$\tau_1 \xrightarrow{\mathfrak{D}} -\delta + M(c,\delta)^{-1}N(c,\delta),$$

$$\tau_2 \xrightarrow{\mathfrak{D}} -\delta M(c,\delta)^{1/2} + M(c,\delta)^{-1/2}N(c,\delta),$$

where

$$M(c,\delta) = c^2 \int_0^1 K_1(c\delta)^2 + (1-c)^2 \int_0^1 K_2((1-c)\delta)^2,$$

$$N(c,\delta) = c \int_0^1 K_1(c\delta)\,dW_1 + (1-c) \int_0^1 K_2((1-c)\delta)\,dW_2,$$

where in turn

$$(K_k(\delta))(r) = J_k(r,\delta) - \int_0^1 J_k(s,\delta)f(s)'\,ds\left(\int_0^1 f(s)f(s)'\,ds\right)^{-1} f(r),$$
$$k = 1,2,$$

$$J_k(r,\delta) = \int_0^r e^{(r-s)\delta}\,dW_k(s), \qquad k = 1,2,$$

using notation in Lemma 3.

The asymptotic local powers of τ_1 and τ_2 are thus dependent on the break point c in a symmetric fashion as are their asymptotic sizes. It is important to note that the limiting distributions given in Theorem 6 are *not* the weighted sums of those for the two independent unit root models under the local alternatives of (14). This is in contrast to the results in Proposition 4 for the null distributions. The latter distributions would be given similarly as in Theorem 6 with

$$c^2 \int_0^1 K_1(\delta)^2 + (1-c)^2 \int_0^1 K_2(\delta)^2$$

$$c \int_0^1 K_1(\delta)\,dW_1 + (1-c)\int_0^1 K_2(\delta)\,dW_2$$

in place of $M(c,\delta)$ and $N(c,\delta)$. Allowing for a structural change in testing for a unit root therefore not only effectively separates samples but also entails a deteriorating effect on the power by shrinking the local alternative parameter δ to $c\delta$ and $(1-c)\delta$ for the divided two subsamples.

The asymptotics for the statistics τ_1^* and τ_2^* under the local alternatives of (14) can also be easily worked out.

THEOREM 7. *Suppose* $0 < c < 1$. *Then under the local alternatives of* (14)

$$\tau_1^* \xrightarrow{\mathcal{D}} -\delta M_*(c,\delta)^{-1}L(c,\delta) + M_*(c,\delta)^{-1}N_*(c,\delta),$$
$$\tau_2^* \xrightarrow{\mathcal{D}} -\delta M_*(c,\delta)^{-1/2}L(c,\delta) + M_*(c,\delta)^{-1/2}N_*(c,\delta),$$

where

$$L(c,\delta) = c\int_0^1 K_1(c\delta)^2 + (1-c)\int_0^1 K_2((1-c)\delta)^2$$

$$M_*(c,\delta) = \int_0^1 K_1(c\delta)^2 + \int_0^1 K_2((1-c)\delta)^2$$

$$N_*(c,\delta) = \int_0^1 K_1(c\delta)\,dW_1 + \int_0^1 K_2((1-c)\delta)\,dW_2$$

using notation in Theorem 6.

The asymptotic local powers of τ_1^* and τ_2^*, as well as those of τ_1 and τ_2, depend on c. This implies, of course, that the adjustments used to construct the statistics τ_1^* and τ_2^* do not remove the break point dependency under the local alternatives of (14). Both the old statistics τ's and the new statistics τ^*'s thus may well be expected to have differing powers dependent on c. Unfortunately, however, it does not seem possible to draw any decisive conclusion on the power comparison of τ's and τ^*'s from the results in Theorems 6 and 7. When $c = \frac{1}{2}$, the asymptotic distributions in Theorems 6 and 7 become identical. As we mentioned earlier, τ_1^* and τ_2^* are identical to $\tau_1/2$ and τ_2, respectively, in this case.

6. EXTENSIONS TO MULTIPLE BREAKS

Given the previous results for models with a single break, it is simple to extend our methodology to models with multiple breaks. We assume that the deterministic component of the time series $\{x_t\}$ is now generated as

$$x_t^d = \sum_{k=1}^{p} \mu_k t^k + \sum_{i=1}^{p} \sum_{j=1}^{q} \mu_{ij}^+ t_{nj}^i \tag{15}$$

in place of (2), where $0 = n_0 < n_1 < \cdots < n_q < n_{q+1} = n$, $T_k = \{n_{k-1}+1, \ldots, n_k\}$ for $k = 1,\ldots,q+1$, and

$$t_{nj}^i = \begin{cases} 0 & \text{when } t \in T_k \text{ for } k \le j, \\ (t-n_j)^i & \text{when } t \in T_k \text{ for } k \ge j+1. \end{cases}$$

Model (15) allows for multiple breaks and extends (2). Here the q structural shifts are assumed to occur at $t = n_k + 1$, $k = 1,\ldots,q$. It is assumed as before that

$$\frac{n_k}{n} \to c_k, \qquad k = 1,\ldots,q$$

as n tends to infinity.

Now we consider the regression

$$\Delta x_t = \beta x_{t-1}^* + \sum_{k=0}^{p} \nu_k t^k + \sum_{i=0}^{p} \sum_{j=1}^{q} \nu_{ij}^+ t_{nj}^i + u_t, \tag{16}$$

where

$$x_{t-1}^* = (n/(n_k - n_{k-1}))x_{t-1}, \qquad t \in T_k,$$

for $k = 1,\ldots,q+1$. Regression (16) is completely analogous to (10). Define τ_1^* and τ_2^* similarly as (7) and (8), based on regression (16). That is,

$$\tau_1^* = n\hat{\beta} - \frac{n^2(q+1)\hat{\lambda}}{\sum_1^n \hat{x}_{t-1}^{s*2}}, \tag{17}$$

$$\tau_2^* = \frac{\hat{\sigma}}{\hat{\omega}}\, t(\beta) - \frac{n(q+1)\hat{\lambda}}{\left(\sum_1^n \hat{x}_{t-1}^{s*2}\right)^{1/2}}, \tag{18}$$

where $\{\hat{x}_{t-1}^{s*}\}$ is the residual from the least-squares regression of $\{x_{t-1}^*\}$ on t^k for $k = 0, \ldots, p$, and t_{nj}^i for $i = 0, \ldots, p$, and $j = 1, \ldots, q$. Other notation is defined from regression (16) similarly as in (7) and (8).

The limiting distributions of τ_1^* and τ_2^* are given by the following theorem.

THEOREM 8. *Suppose* $0 < c < 1$. *Then*

$$\tau_1^* \xrightarrow{\mathfrak{D}} \left(\sum_{k=1}^{q+1} \int_0^1 V_k^2\right)^{-1} \left(\sum_{k=1}^{q+1} \int_0^1 V_k\, dW_k\right),$$

$$\tau_2^* \xrightarrow{\mathfrak{D}} \left(\sum_{k=1}^{q+1} \int_0^1 V_k^2\right)^{-1/2} \left(\sum_{k=1}^{q+1} \int_0^1 V_k\, dW_k\right),$$

where W_k, $k = 1, \ldots, q+1$, *are* $(q+1)$ *independent standard Brownian motions and* V_k *is defined from* W_k *similarly as in Proposition* 4.

Theorem 8 shows that the asymptotic distributions are invariant with respect not only to the parameters of the stochastic component but also to the break points in the deterministic component of $\{x_t\}$. The distributions depend only on p, the order of maintained time trend, and q, the number of structural changes. We provide in Appendix B the critical values of the statistics τ_1^* and τ_2^* for $p = 0,1,2,3$, and $q = 1,2$.

7. CONCLUDING REMARKS

The unit root models with structural change in deterministic trends were investigated in this paper. The dependency on the break points of the conventional unit root tests in such models was analyzed in detail, and the testing procedure that is invariant with respect to the break points has been developed. The new tests rely on a transformation of regression, which was motivated similarly to the generalized least-squares correction for heteroskedasticity.

The theory and methodology developed in the paper to solve the test dependency on the break points are not confined to any specific estimators. Although we have explicitly considered only some special class of tests based on the unit root regression, this is just for illustrative purposes. The invariant tests can easily be derived for all the existing approaches, using similar transformations. For instance, we may simply transform a given series using weights inversely proportional to the relative sizes of the subsamples to make the variable addition test by Park and Choi [7] invariant. All the existing unit root test statistics have nonstandard limiting distributions that are dependent, in general, on trend specifications and, in particular, on the break points for trends with breaks. All their asymptotic distributions, however, can be represented as functionals of Brownian motion, and for this reason our procedure utilizing the property of independent increments of Brownian motion naturally extends to them.

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APPENDIX A

Proof of Theorem 1. We set $x_0^s = 0$ for simplicity, write

$$x_t^s = \sum_1^t u_k,$$

and define

$$d_t = (n^{-1/2}, \ldots, n^{-(2p+1)/2} t^p, n^{-1/2} t_m^0, \ldots, n^{-(2p+1)/2} t_m^p)'.$$

Let

$$\hat{x}_{t-1}^s = x_{t-1}^s - \left(\sum_1^n x_{t-1}^s d_t'\right)\left(\sum_1^n d_t d_t'\right)^{-1} d_t. \tag{A.1}$$

Then we have

$$n(\hat{\alpha} - 1) = \left(\frac{1}{n^2}\sum_1^n \hat{x}_{t-1}^{s\,2}\right)^{-1} \frac{1}{n}\sum_1^n \hat{x}_{t-1}^s u_t,$$

$$t(\alpha) = \left(\hat{\sigma}^2 \frac{1}{n^2}\sum_1^n \hat{x}_{t-1}^{s\,2}\right)^{-1/2} \frac{1}{n}\sum_1^n \hat{x}_{t-1}^s u_t.$$

Define

$$f_{nk}(r) = \left(\frac{[nr]}{n}\right)^k \quad \text{and} \quad f_{nk}^+(r) = \left(\frac{[(n-m)r]^+}{n}\right)^k$$

for $r \in [0,1]$, where $[w]$ denotes the largest integer not exceeding w, and $[w]^+ = \max(0,[w])$. Then $f_{nk} \to f_k$ and $f_{nk}^+ \to f_k^+$ uniformly for $k = 1, \ldots, p$. Let

$$d_{kt} = n^{-(2k+1)/2} t^k \quad \text{and} \quad d_{kt}^+ = n^{-(2k+1)/2} t_m^k.$$

It follows immediately that

$$\sum_1^n d_{it} d_{jt}^+ = \int_0^1 f_{ni} f_{nj}^+ + o(1) \to \int_0^1 f_i f_j^+ \tag{A.2}$$

for $i,j = 1, \ldots, p$, and

$$\frac{1}{n}\sum_1^n d_{kt} x_t^s = \int_0^1 f_{nk} B_n + o_p(1) \xrightarrow{\mathfrak{D}} \int_0^1 f_k B$$

$$\frac{1}{n}\sum_1^n d_{kt}^+ x_t^s = \int_0^1 f_{nk}^+ B_n + o_p(1) \xrightarrow{\mathfrak{D}} \int_0^1 f_k^+ B \tag{A.3}$$

for $k = 1, \ldots, p$. Also, we let

$$g_{nk}(r) = n\left(\left(\frac{[nr]+1}{n}\right)^k - \left(\frac{[nr]}{n}\right)^k\right),$$

$$g_{nk}^+(r) = n\left(\left(\frac{[(n-m)r]^+ + 1}{n}\right)^k - \left(\frac{[(n-m)r]^+}{n}\right)^k\right). \tag{A.4}$$

Then $g_{nk} \to g_k$ and $g_{nk}^+ \to g_k^+$ uniformly, where $g_k(r) = (d/dr) f_k(r)$ and $g_k^+ = g_k(r-c)$ if $r > c$, and 0 otherwise. It can be shown using summation and integration by parts that

$$\sum_1^n d_{kt} u_t = B_n(1) - \int_0^1 g_{nk} B_n \xrightarrow{\mathfrak{D}} B(1) - \int_0^1 g_k B = \int_0^1 f_k \, dB$$

$$\sum_1^n d_{kt}^+ u_t = B_n(1) - \int_0^1 g_{nk}^+ B_n \xrightarrow{\mathfrak{D}} B(1) - \int_0^1 g_k^+ B = \int_0^1 f_k^+ \, dB \tag{A.5}$$

for $k = 1, \dots, p$. Finally, we have

$$\sum_1^n x_{t-1}^s u_t \xrightarrow{\mathfrak{D}} \int_0^1 B \, dB + \lambda,$$

$$\sum_1^n x_{t-1}^{s\,2} \xrightarrow{\mathfrak{D}} \int_0^1 B^2, \tag{A.6}$$

as shown in Phillips [11]. The theorem follows from (A.2)–(A.6) with $B = \omega W$. ■

Proof of Corollary 2. The stated result follows immediately from

$$\frac{1}{n^2} \sum_1^n \hat{x}_{t-1}^{s\,2} \xrightarrow{\mathfrak{D}} \omega^2 \int_0^1 V^2,$$

which was shown in the proof of Theorem 1. ■

Proof of Lemma 3. Let $f_k^1(r) = r^k$ if $0 \le r \le c$, and 0 otherwise, and redefine $f_k^2(r) = f_k^+(r)$. Let

$f_1^* = (f_1^1, \dots, f_p^1)'$ and $f_2^* = (f_1^2, \dots, f_p^2)'$,

and subsequently define

$f_* = (f_1^{*\prime}, f_2^{*\prime})'$.

Because the subspace of $L^2[0,1]$ generated by f_* is identical to that generated by h, we have

$$V = W - \int_0^1 W f_*' \left(\int_0^1 f_* f_*' \right)^{-1} f_*.$$

We have on the interval $[0, c]$,

$$V = W - \int_0^1 W f_1^{*\prime} \left(\int_0^1 f_1^* f_1^{*\prime} \right)^{-1} f_1^*.$$

If we define

$$W_1(r) = \frac{1}{\sqrt{c}} W(cr) \quad \text{and} \quad h_1(r) = \frac{1}{\sqrt{c}} f_1^*(cr),$$

it can then be easily deduced for $0 \le r \le c$ that

$$\begin{aligned} V(r) &= \sqrt{c} W_1\left(\frac{r}{c}\right) - \int_0^1 W_1 h_1' \left(\int_0^1 h_1 h_1' \right)^{-1} \sqrt{c} h_1\left(\frac{r}{c}\right) \\ &= \sqrt{c} W_1\left(\frac{r}{c}\right) - \sqrt{c} \int_0^1 W_1 f' \left(\int_0^1 f f' \right)^{-1} f\left(\frac{r}{c}\right) \\ &= \sqrt{c} V_1\left(\frac{r}{c}\right). \end{aligned} \tag{A.7}$$

Similarly, on the interval $(c,1]$,

$$V = W - \int_0^1 Wf_2^{*\prime}\left(\int_0^1 f_2^* f_2^{*\prime}\right)^{-1} f_2^*.$$

Therefore, if we let

$$W_2(r) = \frac{1}{\sqrt{1-c}}\,(W(c+(1-c)r) - W(c)),$$

$$h_2(r) = \frac{1}{\sqrt{1-c}}\,f_2^*(c+(1-c)r),$$

then for $c < r \leq 1$

$$\begin{aligned} V(r) &= \sqrt{1-c}\,W_2\left(\frac{r-c}{1-c}\right) - \int_0^1 W_2 h_2'\left(\int_0^1 h_2 h_2'\right)^{-1}\sqrt{1-c}\,h_2\left(\frac{r-c}{1-c}\right) \\ &= \sqrt{1-c}\,W_2\left(\frac{r-c}{1-c}\right) - \sqrt{1-c}\int_0^1 W_2 f'\left(\int_0^1 ff'\right)^{-1} f\left(\frac{r-c}{1-c}\right) \\ &= \sqrt{1-c}\,V_2\left(\frac{r-c}{1-c}\right). \end{aligned} \tag{A.8}$$

It is easy to see that the stochastic processes W_1 and W_2 are Brownian motions on $[0,1]$. The independence of W_1 and W_2 follows from the fact that a Brownian motion has independent increments (see, e.g., Doob [4]). ■

Proof of Proposition 4. We have from (A.7) and (A.8)

$$\begin{aligned} \int_0^1 V^2 &= \int_0^c V^2 + \int_c^1 V^2 \\ &= c\int_0^c V_1\left(\frac{r}{c}\right)^2 dr + (1-c)\int_c^1 V_2\left(\frac{r-c}{1-c}\right)^2 dr \\ &= c^2\int_0^1 V_1^2 + (1-c)^2\int_0^1 V_2^2. \end{aligned} \tag{A.9}$$

Also, it follows that

$$\begin{aligned} \int_0^1 V\,dW &= \int_0^c V\,dW + \int_c^1 V\,dW \\ &= \int_0^c V_1\left(\frac{r}{c}\right)dW_1\left(\frac{r}{c}\right) + \int_c^1 V_2\left(\frac{r-c}{1-c}\right)dW_2\left(\frac{r-c}{1-c}\right) \\ &= c\int_0^1 V_1\,dW_1 + (1-c)\int_0^1 V_2\,dW_2. \end{aligned} \tag{A.10}$$

The stated result is now immediate from Corollary 2. ■

Proof of Theorem 5. Let $\hat{x}_{t-1}^s$ be defined as (A.1), $a_n = n/m$ and $b_n = n/(n-m)$. We have

$$n\hat{\beta} = \left(\frac{1}{n^2}\left(a_n^2 \sum_1^m \hat{x}_{t-1}^{s\,2} + b_n^2 \sum_{m+1}^n \hat{x}_{t-1}^{s\,2}\right)\right)^{-1} \frac{1}{n}\left(a_n \sum_1^m \hat{x}_{t-1}^s u_t + b_n \sum_{m+1}^n \hat{x}_{t-1}^s u_t\right)$$

$$\xrightarrow{\mathfrak{D}} \left(\frac{\omega^2}{c^2}\int_0^c V^2 + \frac{\omega^2}{(1-c)^2}\int_c^1 V^2\right)^{-1}$$

$$\times \left(\left(\frac{\omega^2}{c}\int_0^c V\,dW + \lambda\right) + \left(\frac{\omega^2}{1-c}\int_c^1 V\,dW + \lambda\right)\right)$$

$$= \left(\omega^2\left(\int_0^1 V_1^2 + \int_0^1 V_2^2\right)\right)^{-1}\left(\omega^2\left(\int_0^1 V_1\,dW_1 + \int_0^1 V_2\,dW_2\right) + 2\lambda\right),$$

which follows from (A.9) and (A.10). In particular, if $x_t^s = \sum_1^t u_k$ as in the proof of Theorem 1, then

$$\frac{1}{n}\sum_1^m x_{t-1}^s u_t = \frac{1}{2}\left(\frac{1}{n}\left(\sum_1^m u_t\right)^2 - \frac{1}{n}\sum_1^m u_t^2\right)$$

$$= \tfrac{1}{2}(B_n(c)^2 - c\sigma^2) + o_p(1)$$

$$\xrightarrow{\mathfrak{D}} \tfrac{1}{2}(B(c)^2 - c\sigma^2)$$

$$= \omega^2\left(\int_0^c W\,dW + c\lambda\right)$$

using Ito's formula, which yields $dW^2 = 2W\,dW + dr$. We also note that

$$\frac{1}{n^2}\sum_1^n x_{t-1}^{s\,2} \xrightarrow{\mathfrak{D}} \omega^2\left(\int_0^1 V_1^2 + \int_0^1 V_2^2\right),$$

and the proof for τ_1^* is complete. The proof for τ_2^* is entirely analogous and omitted. ■

Proof of Theorem 6. As shown in Phillips [11], we have

$$\frac{1}{\sqrt{n}} x_{[nr]}^s \xrightarrow{\mathfrak{D}} \omega J(r,\delta)$$

under the local alternatives of (14), where

$$J(r,\delta) = \int_0^r e^{(r-s)\delta}\,dW(s).$$

Notice that

$$\frac{1}{\sqrt{c}} J(cr,\delta) = \frac{1}{\sqrt{c}}\int_0^{cr} e^{(cr-s)\delta}\,dW(s)$$

$$= \int_0^r e^{(r-s)c\delta}\,dW_1(s)$$

$$= J_1(r, c\delta)$$

and

$$\frac{1}{\sqrt{1-c}} J(c+(1-c)r,\delta) = \frac{1}{\sqrt{1-c}} \int_0^{c+(1-c)r} e^{(c+(1-c)r-s)\delta}\, dW(s)$$
$$= \int_0^r e^{(r-s)(1-c)\delta}\, dW_2(s)$$
$$= J_2(r,(1-c)\delta).$$

The rest of the proof is essentially identical to those for Lemma 3 and Proposition 4. ∎

Proof of Theorem 7. The proof is parallel to those of Lemma 3 and Theorem 5, given the results in Theorem 6. ∎

Proof of Theorem 8. It is easy to see that all the arguments used to prove the previous results are also applicable to the case of multiple breaks. The proof is therefore omitted. ∎

APPENDIX B

The critical values were estimated based on 500 observations and 50,000 iterations. The computations were done using GAUSS386.

		$q=1$		$q=2$	
p	Size	τ_1^*	τ_2^*	τ_1^*	τ_2^*
0	0.010	−13.0104	−3.9000	−10.3697	−4.2805
	0.025	−11.0341	−3.5851	−8.9309	−4.9813
	0.050	−9.3807	−3.3341	−7.8248	−3.7199
	0.100	−7.7925	−3.0237	−6.6190	−3.4164
	0.150	−6.8498	−2.8226	−5.9331	−3.2086
	0.200	−6.1548	−2.6707	−5.4013	−3.0517
1	0.010	−20.4220	−4.6798	−17.3567	−5.2789
	0.025	−18.0130	−4.4025	−15.6212	−5.0024
	0.050	−16.1212	−4.1529	−14.1524	−4.7504
	0.100	−14.0485	−3.8690	−12.6366	−4.4758
	0.150	−12.7991	−3.6841	−11.6070	−4.2863
	0.200	−11.8475	−3.5358	−10.8592	−4.1403
2	0.010	−26.8195	−5.2846	−23.1889	−6.0197
	0.025	−23.9114	−4.9962	−21.0999	−5.7388
	0.050	−21.7587	−4.7516	−19.4661	−5.4982

continued

Appendix B continued

		$q = 1$		$q = 2$	
p	Size	τ_1^*	τ_2^*	τ_1^*	τ_2^*
	0.100	−19.4375	−4.4822	−17.6675	−5.2307
	0.150	−17.8696	−4.2994	−16.5188	−5.0463
	0.200	−16.7632	−4.1554	−15.6608	−4.9076
3	0.010	−32.4731	−5.7924	−28.6213	−6.6406
	0.025	−29.4556	−5.5177	−26.4259	−6.3776
	0.050	−27.0620	−5.2736	−24.5663	−6.1346
	0.100	−24.4780	−4.9996	−22.4815	−5.8638
	0.150	−22.7461	−4.8152	−21.1862	−5.6814
	0.200	−21.4828	−4.6723	−20.1912	−5.5406

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