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A THRESHOLD AR(1) MODEL

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Abstract

We consider the model $Z_t = \phi_1 Z_{t-1}^+ + \phi_2 Z_{t-1}^- + a_t$ where ϕ_1, ϕ_2 are real coefficients, not necessarily equal, and the a_t's are a sequence of i.i.d. random variables with mean 0. Necessary and sufficient conditions on the ϕ's are given for stationarity of the process. Least squares estimators of the ϕ's are derived and, under mild regularity conditions, are shown to be consistent and asymptotically normal. An hypothesis test is given to differentiate between an AR(1) (the case $\phi_1 = \phi_2$) and this threshold model. The asymptotic behavior of the test statistic is derived. Small-sample behavior of the estimators and the hypothesis test are studied via simulated data.

NON-LINEAR TIME SERIES; TAR MODELS; AUTOREGRESSIVE MODELS; MARKOV CHAINS

1. Introduction

The study of non-linear time series models has recently received a great deal of attention (e.g. see Jones (1978), Priestley (1980), and Tong and Lim (1980)). One class of non-linear models which appears to be particularly useful is the class of threshold autoregressive (TAR) models introduced by Tong (1978) and discussed comprehensively in Tong and Lim (1980). Several examples are given by these authors which show that TAR models provide better fits than linear models. In addition, TAR models are shown to exhibit strictly non-linear behavior (e.g. limit cycles) which linear models cannot duplicate.

In Tong and Lim (1980), the problem of model identification and model fitting was considered and the methods of Klimko and Nelson (1978) were suggested to obtain sampling properties for the parameter estimators. In addition, only sufficient conditions were established for the ergodicity of the TAR model.

The present paper deals with these issues for the simplest of the TAR models, namely

$$Z_t = \phi_1 Z_{t-1}^+ + \phi_2 Z_{t-1}^- + a_t, \qquad t = 1, 2, \cdots, \tag{1.1}$$

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where $x^+ = \max(x, 0)$ and $x^- = \min(x, 0)$. Equivalently, (1.1) may be written

$$Z_t = h(Z_{t-1}) + a_t, \qquad t = 1, 2, \cdots \tag{1.2}$$

where

$$h(Z_{t-1}) = [\phi_1 I(Z_{t-1} > 0) + \phi_2 I(Z_{t-1} \leqq 0)]Z_{t-1}$$

and $I(A)$ is the indicator function for the set A. In both (1.1) and (1.2) above, we take ϕ_1 and ϕ_2 to be real constants and assume that $\{a_t; t \geqq 1\}$ is a sequence of independent, identically distributed (i.i.d) random variables, each having a strictly positive density, $f(\cdot)$, on $\mathbb{R}$, and mean 0.

Figures 1.1–1.3 show realizations of 250 observations from series (1.1), with (ϕ_1, ϕ_2) taking values $(0.9, 0.5)$, $(0.1, -10.0)$ and $(-0.4, -2.0)$ respectively. In all three realizations, the distribution of the $\{a_t\}$ is taken to be N(0, 1).

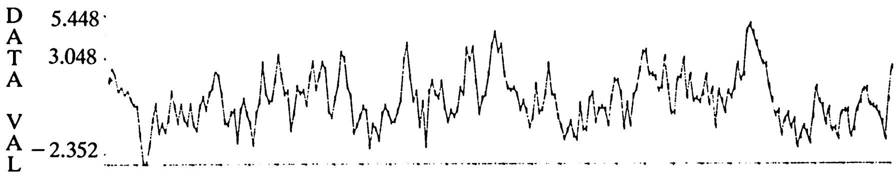

Figure 1.1
$\phi_1 = 0.9, \ \phi_2 = 0.5$

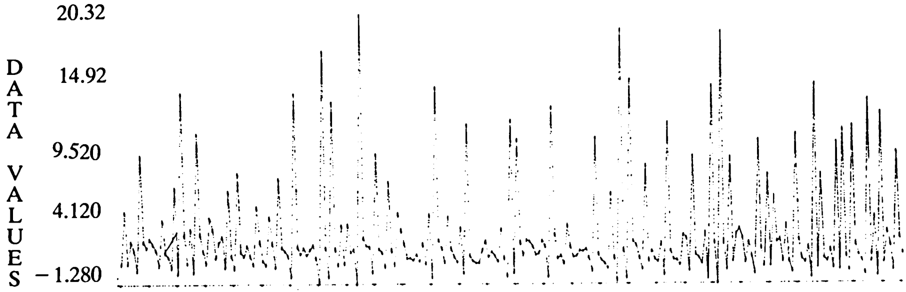

Figure 1.2
$\phi_1 = 0.1, \ \phi_2 = -10.0$

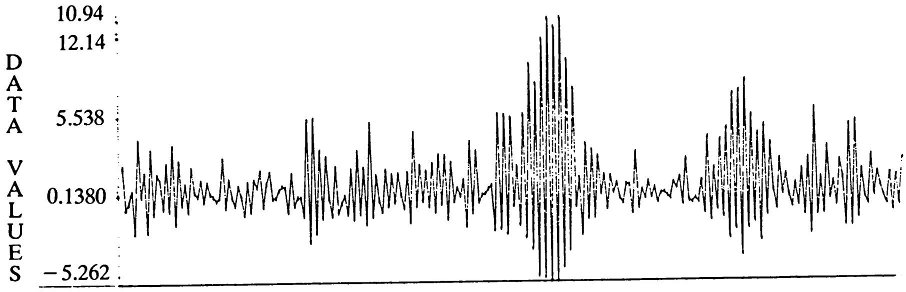

Figure 1.3
$\phi_1 = -0.4, \ \phi_2 = -2.0$

We note that even if the a_t's are symmetrically distributed, the marginal stationary distribution for Z_t (when it exists) will be symmetric about 0 if and only if $\phi_1 = \phi_2$. Hence, we could also refer to (1.1) as an asymmetric autoregressive model. The analogous asymmetric moving average model was considered by Wecker (1977) in an attempt to describe the behavior of industrial prices.

In Section 2, we obtain necessary and sufficient conditions on ϕ_1 and ϕ_2 for the process defined by (1.1) to be ergodic. These conditions are seen to be much broader than the sufficient conditions obtained by Tong and Lim (1980) for the TAR and by Jones (1978) for the non-linear autoregressive process (1.2) (see also Remark 2 in Section 2).

In Section 3, we assume that $E(|a_t|^{2+\xi})<\infty$, for some $\xi > 0$. This allows us to establish the consistency of the least squares estimators for ϕ_1 and ϕ_2 as well as for the estimator for $\sigma^2 = E(a_t^2)$. In addition, a central limit theorem is shown to hold for the estimators of ϕ_1 and ϕ_2. An hypothesis test, to test whether $\phi_1 = \phi_2$, is developed in Section 4. The asymptotic distribution of the associated test statistic is also obtained. Finally, in Section 5, we study, via simulation, the general behavior of model (1.1) and the small-sample performance of the parameter estimators and of the hypothesis test when the error terms, $\{a_t\}$, are normally distributed.

2. Ergodicity

We note that $\{Z_t; t \geqq 0\}$, as defined in (1.1) is a Markov chain with state space $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra on the real numbers $\mathbb{R}$. The transition density is given by

$$p(x, y) = f(y - \phi_1 x^+ - \phi_2 x^-). \tag{2.1}$$

If μ is Lebesgue measure on $\mathbb{R}$, then $\{Z_t; t \geqq 0\}$ is μ-irreducible and aperiodic (see Orey (1971) for the relevant definitions).

The following theorem gives necessary and sufficient conditions on the parameters ϕ_1 and ϕ_2 for the process $\{Z_t\}$ to be ergodic.

Theorem 2.1. *The process* $\{Z_t, t \geqq 0\}$, *defined by* (1.1), *is ergodic if and only if* ϕ_1 *and* ϕ_2 *satisfy*

$$\phi_1 < 1, \qquad \phi_2 < 1 \quad \text{and} \quad \phi_1\phi_2 < 1. \tag{2.2}$$

The region of ergodicity described by (2.2) is illustrated in Figure 2.1 below.

The proof of Theorem 2.1 is divided into the following four lemmas, the first of which proves the sufficiency of Condition (2.2).

Lemma 2.1. *If* ϕ_1 *and* ϕ_2 *satisfy* (2.2), *then the process* $\{Z_t; t \geqq 0\}$ *is ergodic.*

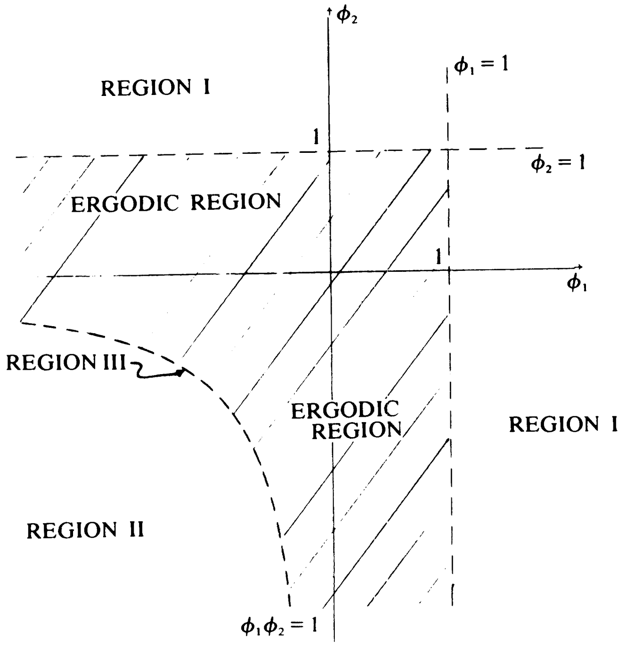

Figure 2.1

Proof. Defining the transition function for the Markov chain $\{Z_t\}$ by

$$P(x, A) = \int_A p(x, y)dy, \qquad x \in \mathbb{R}, \quad A \in \mathcal{B},$$

then it is easy to show that the transition law $\{P(x, \cdot)\}$ is strongly continuous (see Tweedie (1975), p. 393). The result of the lemma will follow from Theorem 4.2 of Tweedie (1975) if we can find a compact set $K \subset \mathcal{B}$, having positive Lebesgue measure, and a non-negative measurable function g on $\mathbb{R}$ such that

(i) $\int_{\mathbb{R}} p(x, y)g(y)dy \leqq g(x) - 1$, $x \notin K$,

(ii) $\int_{\mathbb{R}} p(x, y)g(y)dy = \lambda(x) \leqq R < \infty$, $x \in K$, for some fixed $R > 0$.

From (2.2) it is possible to find positive constants a and b such that $1 > \phi_1 > -(ab^{-1})$ and $1 > \phi_2 > -(ba^{-1})$. Then, by choosing

$$g(x) = \begin{cases} ax, & x > 0 \\ b\,|\,x\,|, & x \leqq 0, \end{cases}$$

it can be shown that there is an $M > 0$ such that Conditions (i) and (ii) hold for $K = [-M, M]$.

The proof of the necessity of (2.2) in Theorem 2.1 is divided into three lemmas. These lemmas, labelled 2.2–2.4 below, prove the non-ergodicity of the process $\{Z_t; t \geqq 0\}$ for values of ϕ_1 and ϕ_2 which lie in regions I, II, and III (shown in Figure 2.1) respectively.

Lemma 2.2. *If $\phi_1 \geqq 1$ or $\phi_2 \geqq 1$ (Region* I *in Figure* 2.1) *then the process* $\{Z_t\}$ *is not ergodic.*

Proof. Without loss of generality we consider the case $\phi_1 \geqq 1$. Assume first that $\phi_1 > 1$. Then for $Z_{t-1} > 0$, $E(Z_t \mid Z_{t-1}) = \phi_1 Z_{t-1}$. Thus for any $1 < \eta < \phi_1$ and $Z_{t-1} > 0$,

$$P(Z_t \leqq 2^{-1}(\eta+1)Z_{t-1} \mid Z_{t-1}) \leqq 2E(|a_1|)[(\eta-1)Z_{t-1}]^{-1} \tag{2.3}$$

by Markov's inequality. Choose $M > 0$ such that $c = 2E(|a_1|)[(\eta-1)M]^{-1} < 1$. Then, whenever $Z_1 > M$, (2.3) implies that

$$P(Z_2 > 2^{-1}(\eta+1)Z_1 \mid Z_1) \geqq 1 - c$$

which in turn implies that

$$\begin{aligned} &P(Z_3 > 2^{-1}(\eta+1)Z_2,\ Z_2 > 2^{-1}(\eta+1)Z_1 \mid Z_1) \\ &\quad \geqq (1 - c[2^{-1}(\eta+1)]^{-1})(1-c) = (1-c\beta)(1-c), \end{aligned}$$

where $\beta = 2(\eta+1)^{-1} < 1$. Continuing in this manner, we have, whenever $Z_1 > M$,

$$\begin{aligned} P(Z_{l+1} > 2^{-1}(\eta+1)Z_l,\ l = 1, \cdots, t \mid Z_1) &\geqq \prod_{i=1}^{t} (1 - c\beta^{i-1}) \\ &\geqq (1-c)^{1/(1-\beta)} \end{aligned}$$

for all t. Consequently for any $Z_0 \in \mathbb{R}$,

$$P(Z_t \to \infty \mid Z_0) \geqq (1-c)^{1/(1-\beta)} P(Z_1 > M \mid Z_0) > 0.$$

Hence, $\{Z_t\}$ is not ergodic for $\phi_1 > 1$.

If $\phi_1 = 1$, then, given $Z_1 > 0$, $Z_t = Z_{t-1} + a_t$, $t > 1$, is a random walk until the first time $Z_t < 0$. However, for such a random walk, $E(T \mid Z_1) = \infty$ where $T = \inf\{t > 0 : Z_t \in (-\infty, 0)\}$. Since $P(Z_1 > 0 \mid Z_0) > 0$ for any $Z_0 \in \mathbb{R}$, Theorem 7 of Tweedie (1974) implies that $\{Z_t\}$ is not ergodic.

Lemma 2.3. *If $\phi_1 < 0$ and $\phi_1\phi_2 > 1$ (Region* II *in Figure* 2.1), *then the process* $\{Z_t\}$ *is not ergodic.*

Proof. Again, without loss of generality we consider only the case $\phi_1 < -1$ and $\phi_1\phi_2 > 1$. The proof is similar to that for Lemma 2.2 except that we show that the Markov chain $\{Z_{2t}; t \geqq 0\}$ has the property that, for any $Z_0 \in \mathbb{R}$,

$$P(Z_{2t} \to \infty \mid Z_0) > 0. \tag{2.4}$$

In particular, it can be shown that for $1 < \eta < \phi_1\phi_2$ there is an $M > 0$ such that $Z_{t-2} > M$ implies

$$E(Z_t \mid Z_{t-2}) > \eta Z_{t-2}, \qquad t \geqq 2. \tag{2.5}$$

In addition, whenever $Z_{t-2}>0$, $E(|Z_t-E(Z_t\,|\,Z_{t-2})|\,|\,Z_{t-2})\leqq \xi<\infty$ for $t\geqq 2$. By choosing M large enough so that (2.5) holds and $2\xi[(\eta-1)M]^{-1}<1$, an argument similar to that used in the proof of Lemma 2.1 implies that (2.4) holds. Since the chain $\{Z_{2t}\}$ is not ergodic, neither is $\{Z_t\}$.

Lemma 2.4. *If* $\phi_1<0$ *and* $\phi_1\phi_2=1$ (*Region* III *in Figure* 2.1), *then the process* $\{Z_t\}$ *is not ergodic.*

Proof. For definiteness, let $\phi_1\leqq -1$, $-1\leqq \phi_2<0$, and $\phi_1\phi_2=1$. Then, for $Z_{t-2}<0$ and $t\geqq 2$,

$$\begin{aligned} Z_t &= a_t+(Z_{t-2}+\phi_1 a_{t-1})I(Z_{t-2}+\phi_1 a_{t-1}<0)\\ &\quad +\phi_1^{-2}(Z_{t-2}+\phi_1 a_{t-1})I(Z_{t-2}+\phi_1 a_{t-1}\geqq 0) \qquad (2.6)\\ &\leqq Z_{t-2}+\phi_1 a_{t-1}+a_t \quad \text{a.s.}\end{aligned}$$

But, for $t=3,5,7,\cdots$, the right side of (2.6) can be written as $Z_{t-2}+\gamma_t$, where $\{\gamma_{2j+1};j\geqq 1\}$ is an i.i.d. sequence of zero-mean random variables. Define the random walk $\{Y_t;t\geqq 1\}$ by $Y_1=Z_1$ and

$$Y_t=Y_{t-1}+\gamma_{2t-1},\qquad t>1.$$

Consider the stopping times

$$T(Z)=\inf\{t>0:Z_{2t+1}\in(0,\infty)\}$$

and

$$T(Y)=\inf\{t>0:Y_{t+1}\in(0,\infty)\}.$$

Now (2.6) implies that $\{T(Y)>n\}\subset\{T(Z)>n\}$ whenever $Z_1<0$. Since the process $\{Y_t;t\leqq T(Y)\}$ is a random walk and $E(\gamma_1)=0$,

$$E(T(Z)\,|\,Z_1)\geqq E(T(Y)\,|\,Z_1)=\infty$$

whenever $Z_1<0$. The result follows.

Remarks.

1. In terms of the Markov chain $\{Z_t\}$, the results of Lemmas 2.1–2.4 imply the following:

(a) The chain is positive recurrent when (2.2) holds.

(b) The chain is transient for ϕ_1 and ϕ_2 in the interior of Regions I or II (see Figure 2.1).

(c) For ϕ_1 and ϕ_2 in Region III or on the boundary of Region I (Figure 2.1), Lemmas 2.4 and 2.2 show that the process is not positive recurrent; we conjecture that it is null recurrent in these cases.

2. Jones (1978) obtained a sufficient condition for the ergodicity of models of the form (1.2) with $h(\cdot)$ continuous and a_t absolutely continuous. Subsequently,

Tong and Lim (1980) applied this condition to their TAR. In both cases, sufficient conditions for ergodicity are derived from Corollary 5.2 in Tweedie (1975). However, in our case, (2.2) represents a much larger region of ergodicity than the region $|\phi_1| < 1$ and $|\phi_2| < 1$ which would result if Corollary 5.2 in Tweedie (1975) were applied to our model (1.1).

3. The sufficiency of (2.2) may be easily shown to apply for any error term distribution such that $E(|a_t|^\alpha) < \infty$, some $0 < \alpha < 1$, by taking

$$g(x) = \begin{cases} ax^\alpha, & x > 0 \\ b|x|^\alpha, & x \leqq 0 \end{cases}$$

in Lemma 2.1.

When (2.2) holds, Theorem 2.1 implies the existence of an invariant probability distribution for $\{Z_t\}$. Additionally, we obtain the following result, which will be used in Sections 3 and 4 and the proof of which appears in Appendix 1.

Theorem 2.2. *Assume* $E(|a_t|^{2+\xi}) < \infty$ *for some* $\xi > 0$. *Then if* ϕ_1 *and* ϕ_2 *satisfy* (2.2), *the invariant probability distribution for the chain* $\{Z_t\}$ *has a finite second moment.*

Remark. By using methods similar to those in the proof of Theorem 2.2, we can show that if $E(|a_t|^{b+\xi}) < \infty$, for some $\xi > 0$ and any $0 \leqq b \leqq \infty$, then the invariant probability distribution for the chain $\{Z_t\}$ has moments of order b or less.

We are now able to consider estimators for the parameters ϕ_1 and ϕ_2 and the properties of these estimators.

3. Estimation of model parameters

In this section, we assume that the error sequence $\{a_t\}$ has a finite absolute moment of order $2 + \xi$, for some $\xi > 0$, so that the stationary distribution for $\{Z_t\}$ has a finite second moment. Let σ^2 denote the common variance of the error terms. In what follows we shall also take Z to be a random variable, having as its distribution the invariant probability distribution for $\{Z_t\}$, and will denote $(Z^\pm)^k$ and $(Z_t^\pm)^k$ by $Z^{\pm k}$ and $Z_t^{\pm k}$. The least squares estimators for the parameters ϕ_1 and ϕ_2 are

$$\hat{\phi}_1 = \sum_{t=1}^{n} Z_t Z_{t-1}^+ \Big/ \sum_{t=1}^{n} Z_{t-1}^{+2} \tag{3.1}$$

$$\hat{\phi}_2 = \sum_{t=1}^{n} Z_t Z_{t-1}^- \Big/ \sum_{t=1}^{n} Z_{t-1}^{-2} \tag{3.2}$$

and the corresponding natural estimator of σ^2 is

$$\hat{\sigma}^2 = n^{-1} \sum_{t=1}^{n} (Z_t - \hat{\phi}_1 Z_{t-1}^+ - \hat{\phi}_2 Z_{t-1}^-)^2. \tag{3.3}$$

We note that $\hat{\phi}_1$, $\hat{\phi}_2$ and $\hat{\sigma}^2$ are also the maximum likelihood estimators for $\hat{\phi}_1$, $\hat{\phi}_2$, and $\hat{\sigma}^2$, respectively, under the assumption of a normal error distribution.

The next two theorems establish consistency and asymptotic normality, for the estimators in (3.1)–(3.3), when the process $\{Z_t\}$ is ergodic.

Theorem 3.1. *If ϕ_1 and ϕ_2 satisfy* (2.2) *then $\hat{\phi}_1$, $\hat{\phi}_2$, and $\hat{\sigma}^2$ are consistent estimators of ϕ_1, ϕ_2, and σ^2, respectively.*

Proof. We first note that (3.1) and (3.2) may be rewritten as

$$\hat{\phi}_1 = \phi_1 + \sum_{t=1}^{n} Z_{t-1}^{+} a_t \Big/ \sum_{t=1}^{n} Z_{t-1}^{+2} \tag{3.4}$$

and

$$\hat{\phi}_2 = \phi_2 + \sum_{t=1}^{n} Z_{t-1}^{-} a_t \Big/ \sum_{t=1}^{n} Z_{t-1}^{-2}. \tag{3.5}$$

Since the Markov chain $\{Z_t\}$ is ergodic, it is strong mixing. This implies that $\{Z_t^{+k}\}$ is strong mixing for any power k as is $\{Z_{t-1}^{+} a_t\}$. Hence these latter processes are also ergodic. Using Hannan (1970) (Theorem 2, p. 203) we obtain, as $n \to \infty$,

$$n^{-1} \sum_{t=1}^{n} Z_t^{+2} \to E(Z^{+2}) \quad \text{a.s.}$$

and

$$n^{-1} \sum_{t=1}^{n} Z_{t-1}^{+} a_t \to E(Z^{+}) E(a_1) = 0 \quad \text{a.s.}$$

so that $\hat{\phi}_1 \to \phi_1$ a.s. as $n \to \infty$. Similarly, $\hat{\phi}_2 \to \phi_2$ a.s. as $n \to \infty$.

From (3.3), we obtain

$$\hat{\sigma}^2 = n^{-1} \sum_{t=1}^{n} a_t^2 - (\hat{\phi}_1 - \phi_1) n^{-1} \sum_{t=1}^{n} Z_{t-1}^{+} a_t - (\hat{\phi}_2 - \phi_2) n^{-1} \sum_{t=1}^{n} Z_{t-1}^{-} a_t$$

and so, by the above results, $\hat{\sigma}^2 \to \sigma^2$ a.s. as $n \to \infty$.

Theorem 3.2. *If ϕ_1 and ϕ_2 satisfy* (2.2), *then*

$$\lim_{n \to \infty} P((nE(Z^{+2}))^{1/2}(\hat{\phi}_1 - \phi_1) \leqq x\sigma, \quad (nE(Z^{-2}))^{1/2}(\hat{\phi}_2 - \phi_2) \leqq y\sigma) = \Phi(x)\Phi(y),$$

where $\Phi(\cdot)$ is the standard normal distribution function.

Proof. Let ξ_1 and $\xi_2 \in \mathbb{R}$ and consider

$$\xi_1(nE(Z^{+2}))^{1/2}(\hat{\phi}_1 - \phi_1) + \xi_2(nE(Z^{-2}))^{1/2}(\hat{\phi}_2 - \phi_2).$$

This is asymptotically equivalent to

$$n^{-1/2}\sum_{t=1}^{n}(\psi_1 Z_{t-1}^{+}a_t + \psi_2 Z_{t-1}^{-}a_t), \tag{3.6}$$

where $\psi_1 = \xi_1/E(Z^{+2})^{1/2}$ and $\psi_2 = \xi_2/E(Z^{-2})^{1/2}$. However,

$$\{\psi_1 Z_{t-1}^{+}a_t + \psi_2 Z_{t-1}^{-}a_t, t \geqq 1\}$$

is a martingale difference sequence satisfying the conditions of Theorem 23.1 in Billingsley (1968). Consequently, (3.6) converges in law to an $N(0, \sigma_0^2)$ distribution where $\sigma_0^2 = \sigma^2(\xi_1^2 + \xi_2^2)$. This implies the result.

4. A hypothesis test for the equality of ϕ_1 and ϕ_2

Suppose we wish to test the null hypothesis

$$H_0 : \phi_1 = \phi_2 = \phi$$

versus the alternative hypothesis

$$H_1 : \phi_1 \neq \phi_2.$$

Assuming the error terms $\{a_t\}$ to be normally distributed, a rejection region for the likelihood ratio test is given by

$$\lambda = [\hat{\sigma}^2/\hat{\sigma}_R^2]^{(n-1)/2} < c, \qquad c > 0, \tag{4.1}$$

where

$$\hat{\sigma}_R^2 = n^{-1}\sum_{t=1}^{n}(Z_t - \hat{\phi}Z_{t-1})^2 \quad \text{with} \quad \hat{\phi} = \sum_{t=1}^{n} Z_t Z_{t-1} \Big/ \sum_{t=1}^{n} Z_{t-1}^2$$

and $\hat{\sigma}^2$ is given by (3.3).

Asymptotically, the distribution of $-2\ln\lambda$ is independent of the distribution of the $\{a_t\}$ provided that $E(|a_t|^{2+\xi}) < \infty$, for some $\xi > 0$, as the following shows.

From (4.1) we have that

$$\begin{aligned} -2\ln\lambda = &-(n-1)\ln\left[1 - (\hat{\phi} - \phi)^2 \sum_{t=1}^{n} Z_{t-1}^2 \Big/ \sum_{t=1}^{n} a_t^2\right] \\ &-(n-1)\ln\left[1 - (\hat{\phi}_1 - \phi_1)^2 \sum_{t=1}^{n} Z_{t-1}^{+2} \Big/ \sum_{t=1}^{n} a_t^2\right. \\ &\left.- (\hat{\phi}_2 - \phi_2)^2 \sum_{t=1}^{n} Z_{t-1}^{-2} \Big/ \sum_{t=1}^{n} a_t^2\right]. \end{aligned} \tag{4.2}$$

Now, by Theorem 3.2, under H_0, as $n \to \infty$

$$[\sqrt{n}(\hat{\phi}_1 - \phi_1), \sqrt{n}(\hat{\phi}_2 - \phi_2)] \xrightarrow{\mathcal{D}} [Y_1, Y_2]$$

where Y_1, Y_2 are i.i.d. $N(0, 2\sigma^2/E(Z^2))$ random variables and $\xrightarrow{\mathcal{D}}$ is convergence in distribution. This implies that

$$\begin{aligned}\sqrt{n}(\hat{\phi}-\phi) &= \sqrt{n}(\hat{\phi}_1-\phi_1)\sum_{t=1}^{n} Z_{t-1}^{+2} \Big/ \sum_{t=1}^{n} Z_{t-1}^{2} \\ &\quad + \sqrt{n}(\hat{\phi}_2-\phi_2)\sum_{t=1}^{n} Z_{t-1}^{-2} \Big/ \sum_{t=1}^{n} Z_{t-1}^{2} \\ &\xrightarrow{\mathcal{D}} (Y_1+Y_2)/2.\end{aligned}$$

From (4.2), a Taylor series expansion shows that $-2\ln\lambda$ is asymptotically equivalent, under H_0, to

$$\begin{aligned}&n(\hat{\phi}_1-\phi_1)^2\sum_{t=1}^{n} Z_{t-1}^{+2} \Big/ \sum_{t=1}^{n} a_t^2 + n(\hat{\phi}_2-\phi_2)^2\sum_{t=1}^{n} Z_{t-1}^{-2} \Big/ \sum_{t=1}^{n} a_t^2 \\ &\qquad - n(\hat{\phi}-\phi)^2\sum_{t=1}^{n} Z_{t-1}^{2} \Big/ \sum_{t=1}^{n} a_t^2.\end{aligned} \tag{4.3}$$

This last quantity converges in law to $(Y_1-Y_2)^2E(Z^2)/4\sigma^2$ which has a χ^2 distribution with one degree of freedom. Hence we conclude that $P(-2\ln\lambda \leqq x)$ converges in law to a χ^2 distribution function with one degree of freedom, as $n\to\infty$, under the null hypothesis.

5. Small-sample properties

Simulations of model (1.1) were run to determine the small-sample properties of the estimators defined in Section 3 and the hypothesis test defined in Section 4. In all simulations, the distribution of the $\{a_t\}$ was taken to be N(0, 1). For ϕ_1, ϕ_2 in the interior of Regions I and II of Figure 2.1, the transient nature of the process was readily apparent as diverging sample values caused machine overflows in every instance tested. For ϕ_1, ϕ_2 in Region III, the hyperbolic boundary of the region of ergodicity, or on the boundary of Region I, no such overflows occurred. This behavior is similar to that of an AR(1) process when $|\phi|>1$ and $|\phi|=1$, respectively.

Selected results from the simulation are tabulated in Tables 5.1 and 5.2. The parameter values for ϕ_1 and ϕ_2 represented in these tables lie in the region of ergodicity and in Region III of Figure 2.1. For each ϕ_1, ϕ_2 pair, 1000 simulations were run, first with 50 observations (Table 5.1) and again with 100 observations (Table 5.2). The sample values generated by each simulation were used to obtain estimators for ϕ_1, ϕ_2 and σ^2 via (3.1)–(3.3) and $-2\ln\lambda$ was calculated from (4.1). The quantities recorded in Tables 5.1 and 5.2 (see Appendix 2 for the computational formulas) are:

$\mathrm{M}(\hat{\phi}_j)$ — average $\hat{\phi}_j$ for 1000 simulations, $j=1,2$,
$\mathrm{SE}(\hat{\phi}_j)$ — standard error of $\hat{\phi}_j$ for 1000 simulations, $j=1,2$,

TABLE 5.1
Simulation results based on 1000 simulations of 50 observations each

ϕ_1	ϕ_2	$M(\hat{\phi}_1)$	$SE(\hat{\phi}_1)$	$M(\hat{\phi}_2)$	$SE(\hat{\phi}_2)$	$MESE(\hat{\phi}_1)$	$SESE(\hat{\phi}_1)$	$MESE(\hat{\phi}_2)$	$SESE(\hat{\phi}_2)$	$M(\hat{\sigma})$	$SE(\hat{\sigma})$	P01	P05
0.9	0.5	0.838	0.127	−0.132	12.119	0.090	0.039	0.665	9.447	0.954	0.192	0.069	0.248
0.9	−0.5	0.860	0.085	−0.943	3.909	0.072	0.021	0.777	3.585	0.959	0.197	0.678	0.861
0.9	−10.0	0.894	0.034	−10.134	1.817	0.033	0.010	1.224	1.842	0.952	0.202	0.971	0.984
0.1	0.5	0.039	0.259	0.451	0.182	0.237	0.051	0.159	0.031	0.953	0.197	0.101	0.248
0.1	−0.5	0.087	0.167	−0.510	0.248	0.166	0.018	0.232	0.034	0.965	0.202	0.283	0.516
0.1	−10.0	0.101	0.031	−9.992	0.317	0.029	0.005	0.294	0.056	0.962	0.204	1.000	1.000
−0.1	0.5	−0.144	0.268	0.462	0.160	0.259	0.059	0.151	0.025	0.967	0.199	0.228	0.470
−0.1	−0.5	−0.102	0.175	−0.508	0.224	0.171	0.018	0.213	0.027	0.961	0.204	0.117	0.289
* −0.1	−10.0	−0.098	0.007	−9.996	0.052	0.003	0.003	0.030	0.032	0.959	0.197	1.000	1.000
−0.9	0.5	−0.930	0.328	0.474	0.135	0.307	0.068	0.125	0.015	0.968	0.196	0.943	0.984
−0.9	−0.5	−0.880	0.169	−0.484	0.143	0.163	0.024	0.134	0.018	0.959	0.193	0.211	0.461
0.9	0.9	0.714	0.562	0.673	1.113	0.190	0.448	0.206	0.746	0.950	0.199	0.010	0.067
0.5	0.5	0.436	0.202	0.440	0.209	0.183	0.047	0.184	0.049	0.967	0.196	0.014	0.067
−0.5	−0.5	−0.499	0.182	−0.492	0.168	0.173	0.020	0.173	0.021	0.956	0.195	0.009	0.047
−0.9	−0.9	−0.877	0.105	−0.878	0.101	0.092	0.024	0.093	0.024	0.958	0.204	0.013	0.065
* −0.2	−5.0	−0.196	0.011	−4.997	0.044	0.006	0.006	0.028	0.029	0.951	0.202	1.000	1.000
−0.2	−4.9	−0.190	0.019	−4.891	0.077	0.013	0.008	0.064	0.038	0.966	0.193	1.000	1.000
−0.19	−5.0	−0.180	0.022	−4.991	0.096	0.016	0.008	0.081	0.039	0.958	0.203	1.000	1.000

* Simulations for ϕ_1 and ϕ_2 in Region III, Figure 2.1

TABLE 5.2
Simulation results based on 1000 simulations of 100 observations each

ϕ_1	ϕ_2	M($\hat{\phi}_1$)	SE($\hat{\phi}_1$)	M($\hat{\phi}_2$)	SE($\hat{\phi}_2$)	MESE($\hat{\phi}_1$)	SESE($\hat{\phi}_1$)	MESE($\hat{\phi}_2$)	SESE($\hat{\phi}_2$)	M($\hat{\sigma}$)	SE($\hat{\sigma}$)	P01	P05
0.9	0.5	0.866	0.074	0.416	0.244	0.058	0.018	0.185	0.079	0.978	0.139	0.290	0.636
0.9	−0.5	0.881	0.054	−0.559	0.427	0.048	0.010	0.319	0.145	0.977	0.140	0.965	0.990
0.9	−10.0	0.898	0.022	−10.032	0.779	0.022	0.005	0.649	0.361	0.980	0.143	1.000	1.000
0.1	0.5	0.073	0.171	0.475	0.122	0.164	0.022	0.111	0.014	0.983	0.139	0.263	0.500
0.1	−0.5	0.085	0.115	−0.510	0.176	0.117	0.008	0.164	0.016	0.977	0.142	0.610	0.824
0.1	−10.0	0.101	0.021	−10.008	0.212	0.020	0.002	0.204	0.027	0.982	0.137	1.000	1.000
−0.1	0.5	−0.116	0.186	0.485	0.111	0.178	0.024	0.106	0.012	0.976	0.134	0.598	0.836
−0.1	−0.5	−0.103	0.116	−0.503	0.154	0.122	0.009	0.150	0.013	0.982	0.143	0.267	0.521
* −0.1	−10.0	−0.099	0.003	−10.000	0.025	0.002	0.002	0.018	0.016	0.977	0.139	1.000	1.000
−0.9	0.5	−0.913	0.211	0.490	0.091	0.212	0.027	0.088	0.007	0.979	0.141	0.999	1.000
−0.9	−0.5	−0.891	0.114	−0.491	0.099	0.115	0.012	0.094	0.009	0.972	0.135	0.536	0.763
0.9	0.9	0.830	0.168	0.838	0.144	0.086	0.074	0.083	0.057	0.982	0.137	0.011	0.066
0.5	0.5	0.463	0.139	0.464	0.135	0.127	0.022	0.125	0.021	0.980	0.140	0.015	0.062
−0.5	−0.5	−0.495	0.120	−0.502	0.125	0.122	0.010	0.123	0.010	0.984	0.139	0.011	0.064
−0.9	−0.9	−0.885	0.070	−0.884	0.068	0.065	0.013	0.065	0.013	0.985	0.145	0.011	0.050
* −0.2	−5.0	−0.197	0.007	−4.997	0.025	0.004	0.003	0.018	0.017	0.982	0.144	1.000	1.000
−0.2	−4.9	−0.194	0.011	−4.892	0.045	0.008	0.004	0.039	0.019	0.982	0.143	1.000	1.000
−0.19	−5.0	−0.184	0.014	−4.995	0.055	0.010	0.004	0.052	0.020	0.977	0.142	1.000	1.000

* Simulations for ϕ_1 and ϕ_2 in Region III, Figure 2.1

MESE($\hat{\phi}_j$) — average estimated standard error of $\hat{\phi}_j$ for 1000 simulations, $j = 1, 2$,
SESE($\hat{\phi}_j$) — standard error of the estimated standard error of $\hat{\phi}_j$ for 1000 simulations, $j = 1, 2$,
M($\hat{\sigma}$) — average $\hat{\sigma}$ for 1000 simulations
SE($\hat{\sigma}$) — standard error of $\hat{\sigma}$ for 1000 simulations,
P01(P05) — sample power of the hypothesis test at the 0.01 (0.05) level of significance.

In the course of 1000 simulations of series of length 50 with $\phi_1 = 0.9$, at least one series consisted of only non-negative values. In theory, this presents no problem as one would fit an AR(1) to such a series using $\hat{\phi}_1$ to estimate the AR coefficient ($\hat{\phi}_2$ would be set to 0). However, for the purpose of simulation, this causes MESE($\hat{\phi}_2$) and SESE($\hat{\phi}_2$) to be infinite. To avoid this problem, our results in Table 5.1, corresponding to simulations with $\phi_1 = 0.9$, were obtained by using 1000 simulations for which there were both positive and negative observations. We note that this situation did not occur for the longer series represented in Table 5.2.

Among the properties observed in Tables 5.1 and 5.2 are the following:

1. In general, $\hat{\phi}_1$ and $\hat{\phi}_2$ exhibit better overall performance when ϕ_1 and ϕ_2 are both negative. This may be a result of a fairly even distribution of positive and negative observations occurring for the negative values of ϕ_1 and ϕ_2 tested.

2. As $\phi_1(\phi_2)$ approaches 1, while $\phi_2(\phi_1)$ remains fixed, the parameter estimates for $\phi_2(\phi_1)$ become unstable. This is not true as the parameter values approach the hyperbolic lower boundary of the region of ergodicity. In fact, as the last three entries in each table indicate, the behavior of the parameter estimators $\hat{\phi}_1$ and $\hat{\phi}_2$ is better when ϕ_1 and ϕ_2 are on that lower boundary than when ϕ_1 and ϕ_2 are near the boundary but in the ergodic region. One possible explanation for this phenomenon may be that, for relatively short series, values of ϕ_1 and ϕ_2 in Region III of Figure 2.1 result in a greater separation of negative and positive observations, a situation which generally seems to result in more accurate estimation.

3. On average, the expected standard error, MESE($\cdot$), tends to underestimate the corresponding sample standard error, SE($\cdot$).

4. The performance of $\hat{\sigma}$ is consistent throughout and does not seem to be affected by the performance of $\hat{\phi}_1$ or $\hat{\phi}_2$.

6. Concluding remarks

Extensions of these results to the more general SETAR $(l; 1, \cdots, 1)$ model (see Tong and Lim (1980)), for both the known and unknown threshold case, are currently being investigated.

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Appendix 1

Proof of Theorem 2.2. Due to symmetry, we shall assume that $\phi_1 < \phi_2$ without loss of generality. This implies that $|\phi_2| < 1$ and hence that there exists a $k_0 \geqq 1$ such that

$$k_0 = \min\{k \geqq 1 : |\phi_1 \phi_2^{k-1}| < 1\}.$$

Note that for $\phi_1 \leqq -1$ and $-1 < \phi_2 \leqq 0$, $k_0 = 2$ by hypothesis and so $k > 2$ only if $\phi_1 \leqq -1$ and $0 < \phi_2 < 1$. Let $k^* = k_0 - 1$, $\alpha = \max(|\phi_1|, |\phi_2|)$ and $\eta = 2^{k^*} - 1$. Then

$$\text{(A.1.1)}\quad |Z_t| < \alpha |\phi_2|^{k^*} |Z_{t-k_0}| + |Z_{t-k_0}| \sum_{j=1}^{\eta} \delta_j I(\varepsilon_j > \beta_j |Z_{t-k_0}|) + \sum_{j=1}^{k_0} \gamma_j |a_{t-k_0+j}|,$$

where $\{\delta_j ; 1 \leqq j \leqq \eta\}$, $\{\beta_j ; 1 \leqq j \leqq \eta\}$ and $\{\gamma_j ; 1 \leqq j \leqq k_0\}$ are sequences of positive constants depending only on ϕ_1, ϕ_2 and k_0 while, for $1 \leqq j \leqq \eta$, ε_j is a linear combination of $\{a_{t-k_0+l} ; 1 \leqq l \leqq k_0\}$ and as such has a finite absolute moment of order $2 + \xi$ and has zero mean and is independent of Z_{t-k_0}. We also note that, by definition, $0 \leqq \alpha |\phi_2|^{k^*} < 1$. From (A.1.1), we have

$$\text{(A.1.2)}\qquad E(|Z_t| \mid Z_{t-k_0}) \leqq \alpha |\phi_2|^{k^*} |Z_{t-k_0}| + M_1,$$

where, by Markov's inequality,

$$E(|Z_{t-k_0}| I(\varepsilon_j > \beta_j |Z_{t-k_0}|) \mid Z_{t-k_0}) \leqq \beta_j^{-1} E(|\varepsilon_j|), \quad 1 \leqq j \leqq \eta,$$

and

$$E(|a_{t-k_0+j}| \mid Z_{t-k_0}) \leqq A, \qquad 1 \leqq j \leqq k_0$$

so that we may take

$$M_1 \geqq \sum_{j=1}^{k_0} \gamma_j A + \sum_{j=1}^{\eta} \delta_j \beta_j^{-1} E(|\varepsilon_j|).$$

However, (A.1.2) implies that, if $R_1 = \max_{0 \leq l \leq k_0} E(|Z_l| \mid Z_0)$, then

$$\text{(A.1.3)}\qquad E(|Z_t| \mid Z_0) \leqq [\alpha |\phi_2|^{k^*}]^{\mu} R_1 + M_1 \sum_{m=0}^{\mu-1} [\alpha |\phi_2|^{k^*}]^m,$$

where $\mu = [t/k_0]$ and $[\cdot]$ is the greatest integer function. But, the right side of (A.1.3) is bounded independently of t. Hence, there is a constant $B_1 > 0$ such that

$$E(|Z_t| \mid Z_0) \leqq B_1 < \infty, \qquad \forall t.$$

For any integer $\nu \geqq 2$, we use (A.1.1) to obtain

$$\text{(A.1.4)} \qquad \begin{aligned} |Z_t|^\nu \leqq [\alpha\,|\phi_2|^{k^*}]^\nu\,|Z_{t-k_0}|^\nu + |Z_{t-k_0}|^\nu \sum_{j=1}^{\eta} \delta_j^\nu I(\varepsilon_j > \beta_j\,|Z_{t-k_0}|) \\ + f_1(|Z_{t-k_0}|, |a_{t-k_0+1}|, \cdots, |a_t|), \end{aligned}$$

where $f_1(\cdot, \cdots, \cdot)$ is a polynomial in which the powers of $|Z_{t-k_0}|$ do not exceed $\nu - 1$ and all indicator functions are replaced by their upper bounds of 1. Using (A.1.4), with $\nu = 2$, we obtain

$$\text{(A.1.5)} \qquad E(|Z_t|^2 \,|\, Z_{t-k_0}) \leqq [\alpha\,|\phi_2|^{k^*}]^2\,|Z_{t-k_0}|^2 + f_2(|Z_{t-k_0}|) + M_2,$$

where $f_2(\cdot)$ is a polynomial of degree 1 or less and

$$E(|Z_{t-k_0}|^2 I(\varepsilon_j > \beta_j\,|Z_{t-k_0}|)\,|\,Z_{t-k_0}) \leqq \beta_j^{-2} E(\varepsilon_j^2)$$

so that we may take

$$M_2 \geqq \sum_{j=1}^{\eta} \delta_j^2 \beta_j^{-2} E(\varepsilon_j^2).$$

In addition,

$$E(f_2(|Z_{t-k_0}|)\,|\,Z_0) \leqq f_2(B_1) \qquad \forall t$$

where $f_2(B_1)$ is $f_2(x)$ evaluated at $x = B_1$. Hence, letting $M = M_2 + f_2(B_1)$ and $R_2 = \max_{0 \leq l \leq k_0} E(|Z_l|^2 \,|\, Z_0)$,

$$E(|Z_t|^2 \,|\, Z_0) \leqq [\alpha\,|\phi_2|^{k^*}]^{2\mu} R_2 + M \sum_{m=0}^{\mu-1} [\alpha\,|\phi_2|^{k^*}]^{2m}.$$

Again, the right-hand side is bounded independently of t so that

$$\text{(A.1.6)} \qquad E(|Z_t|^2 \,|\, Z_0) \leqq B_2 < \infty$$

for some $B_2 > 0$ and all t.

Now choose an integer l such that $l^{-1} < \xi < 1$. Let $\nu = 2l + 1$ so that $\nu/l = 2 + l^{-1} < 2 + \xi$. Using (A.1.4) and the fact that, for $0 < \alpha < 1$, $|\Sigma_{i=1}^n c_i|^\alpha \leqq \Sigma_{i=1}^n |c_i|^\alpha$, we obtain

$$\begin{aligned} |Z|^\rho \leqq [\alpha\,|\phi_2|^{k^*}]^\rho\,|Z_{t-k_0}|^\rho + f_1^{(l)}(|Z_{t-k_0}|, |a_{t-k_0+1}|, \cdots, |a_t|) \\ + |Z_{t-k_0}|^\rho \sum_{j=1}^{\eta} \delta_j^\rho I(\varepsilon_j > \beta_j\,|Z_{t-k_0}|), \end{aligned}$$

where $\rho = \nu/l$ and $f_1^{(l)}(\cdot)$ is a polynomial in fractional powers $\{m/l; m = 1, \cdots, 2l\}$ of $|Z_{t-k_0}|$ and $\{m/l; m = 1, \cdots, 2l+1\}$ of the $|a|$'s. Consequently,

$$E(|Z_t|^\rho \,|\, Z_{t-k_0}) \leqq [\alpha\,|\phi_2|^{k^*}]^\rho\,|Z_{t-k_0}|^\rho + M_2^{(l)} + f_2^{(l)}(|Z_{t-k_0}|)$$

where $f_2^{(l)}$ is a polynomial in fractional powers $\{m/l; m = 1, \cdots, 2l\}$ of $|Z_{t-k_0}|$ and, since

$$E(|Z_{t-k_0}|^p I(\varepsilon_j > \beta_j |Z_{t-k_0}|) | Z_{t-k_0}) \leqq \beta_j^{-p} E(|\varepsilon_j|^p),$$

we may take

$$M_2^{(l)} \geqq \sum_{j=1}^{\eta} \delta_j^p \beta_j^{-p} E(|\varepsilon_j|^p).$$

We now use (A.1.6) to obtain $E(f_2^{(l)}(|Z_{t-k_0}|)|Z_0) < B_2^{(l)} < \infty$ for some $0 < B_2^{(l)} < \infty$ and all t. Hence, taking $M^{(l)} = M_2^{(l)} + B_2^{(l)}$ and $R_\nu^{(l)} = \max_{0 \leq m \leq k_0} E(|Z_m|^p \mid Z_0)$, we can again bound $E(|Z_t|^p \mid Z_0)$ independently of t as we did to obtain (A.1.6).

Using Feller (1971), pp. 251–252, along with the ergodicity of $\{Z_t\}$, we conclude that

$$E(|Z_t|^2 \mid Z_0) \to E(|Z|^2) < \infty$$

for Z a random variable having the invariant probability distribution for $\{Z_t\}$.

Appendix 2 Computational formulas for Tables 5.1 and 5.2

Let $\{Z_{ki}; k = 1, \cdots, n\}$ be the observations generated for the ith simulation, $i = 1, \cdots, 1000$, where $n = 50$ (Table 5.1) or $n = 100$ (Table 5.2). In addition, for the ith simulation, let $\hat{\sigma}_i$ denote the estimator for σ and $\hat{\phi}_{ji}$ denote the estimator for ϕ_j, $j = 1, 2$.

From Theorem 3.2, the asymptotic standard errors of $\hat{\phi}_{1i}$ and $\hat{\phi}_{2i}$ are $\sigma(nEZ^{+2})^{-1/2}$ and $\sigma(nEZ^{-2})^{-1/2}$ respectively. Hence, for the ith simulation, we estimate these standard errors by

$$\text{(A.2.1)} \qquad \mathrm{ESE}(\hat{\phi}_{1i}) = \hat{\sigma}_i \left(\sum_{k=1}^{n} Z_{ki}^{+2} \right)^{-1/2},$$

$$\text{(A.2.2)} \qquad \mathrm{ESE}(\hat{\phi}_{2i}) = \hat{\sigma}_i \left(\sum_{k=1}^{n} Z_{ki}^{-2} \right)^{-1/2}.$$

The quantities in Tables 5.1 and 5.2 are then computed as follows:

$$\text{(A.2.3)} \qquad \mathrm{M}(\hat{\phi}_j) = (1000)^{-1} \sum_{i=1}^{1000} \hat{\phi}_{ji}, \qquad j = 1, 2,$$

$$\text{(A.2.4)} \qquad \mathrm{SE}(\hat{\phi}_j) = \left[(1000)^{-1} \sum_{i=1}^{1000} (\hat{\phi}_{ji} - \mathrm{M}(\hat{\phi}_j))^2 \right]^{1/2}, \qquad j = 1, 2,$$

$$\text{(A.2.5)} \qquad \mathrm{MESE}(\hat{\phi}_j) = (1000)^{-1} \sum_{i=1}^{1000} \mathrm{ESE}(\hat{\phi}_{ji}), \qquad j = 1, 2,$$

$$\text{(A.2.6)} \qquad \mathrm{SESE}(\hat{\phi}_j) = \left[(1000)^{-1} \sum_{i=1}^{1000} (\mathrm{ESE}(\hat{\phi}_{ji}) - \mathrm{MESE}(\hat{\phi}_{ji}))^2 \right]^{1/2}, \quad j = 1, 2,$$

$$\text{(A.2.7)} \qquad \mathrm{M}(\hat{\sigma}) = (1000)^{-1} \sum_{i=1}^{1000} \hat{\sigma}_i$$

and

$$\text{(A.2.8)} \qquad \mathrm{SE}(\hat{\sigma}) = \left[(1000)^{-1} \sum_{i=1}^{1000} (\hat{\sigma}_i - \mathrm{M}(\hat{\sigma}))^2 \right]^{1/2}.$$

Finally, the quantity P01 (P05) is the proportion of the 1000 simulations for which the value of $-2\ln\lambda$ (see Section 4) exceeds $\chi^2_{1,0.01}$ ($\chi^2_{1,0.05}$).

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