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CHANGE POINTS WITH LINEAR TREND FOLLOWED BY ABRUPT CHANGE FOR THE EXPONENTIAL DISTRIBUTION

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We discuss testing procedures to detect if a random sequence of exponentially distributed random variables has been subjected to a linear trend change followed by an abrupt change. We propose three statistics and explore their distribution theories. As an illustration, we applied these tests to Stanford heart transplant data and airport inter arrival data.

Keywords: Change-points; Likelihood ratio statistics; Estimation; Consistency; Asymptotic distribution

1 INTRODUCTION

Consider n independent random variables $X_1, \ldots, X_n$ where X_i's have exponential distributions with density functions, $f(x_i, \theta_i) = 1/\theta_i \ e^{-x_i/\theta_i} \ i = 1, \ldots, n$. We will derive statistics to test if the θ_i's were subject to a linear trend change at an unknown period of time followed by an abrupt change. The hypotheses can be described in more formal terms as,

$$H_0\colon \theta_i = \theta, \quad i = 1, \ldots, n$$
$$H_A\colon \theta_i = \theta_i + \delta d_i(p, q)$$

where

$$d_i(p, q) = \begin{cases} 0 & i \le p \text{ or } q \le i \le n \\ \dfrac{(i-p)}{(q-p)} & p < i \le q \end{cases} \tag{1}$$

and p, q are the unknown change-points such that $1 \le p < q \le n$ and θ and δ are the unknown nuisance parameters such that $\theta, \delta > 0$. That is the mean function is constant up to some unknown time and then starts to increase linearly, and then drops abruptly to the

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original value at another unknown instant. Notice that if $p = 1$ and $q = n$, then we have the linear regression model. This model has numerous applications. For example, consider a situation where the death rate of a population increases linearly due to a disease. Once a drug for this disease is discovered the death rate will abruptly drop back to its original level. Although we know the time the drug was discovered, the effects of it will not be felt immediately. Hence this situation could be modeled by the above hypotheses. Also in neurophysiology, scientists study the behavior of neurons after the onset of a stimulus. Here the activity of the neuron changes abruptly and then reverts back to its original position after an unknown period. This too can be modeled by the change points with linear trend followed by an abrupt change.

Since Page (1954, 1955) discussed a procedure for detecting a change, many new promising directions of research have emerged. Many authors have considered the single change-point problem for the univariate and the multi-variate normal distributions. Chernoff and Zacks (1964) took a baysian approach to model the change points in sequence of normal means. Worsley (1986) constructed a confidence set for the change point for exponential famillies. Hsu (1979) proposed a test statistic for detecting a single shift change of the scale parameter in a sequence of gamma random variables. Ramanayake and Gupta (1997) discuss change-point problems with epidemic change for the exponential distribution. Change-point problems with linear-trend are discussed by Gupta and Ramanayake (1997). For related work on change point problems, see Gupta and Chen (1996), Chen and Gupta (1995, 1997).

In this paper we propose two test procedures based on the likelihood ratio type statistic, one by assuming that the change-points have an equal chance to take values in the set $\{(p, q) \in Z^2 : 1 \leq p < q \leq n\}$, and the other by using a different prior on (p, q). We discuss the asymptotic properties of these statistics. Next we derive Rao's efficient score statistic and discuss its asymptotic behavior. Then we compare these three statistics in terms of powers of the tests. Finally we apply these tests to Air traffic data and Stanford Heart transplant data.

2 LIKELIHOOD RATIO TYPE STATISTIC

First, we propose a test statistic based on the likelihood ratio to detect an abrupt linear trend change, in a sequence of independent exponential variables. Hsu (1979) proposed a similar statistic for detecting a single shift in the scale parameter in a sequence of independent gamma variables with fixed scale parameter. Suppose that the change-points (p, q) are fixed, then the likelihood ratio function under H_A that $\delta > 0$ to that under H_0, is given by,

$$R(\theta, \delta, p, q; \mathrm{x}) = \frac{\theta^{q-p}}{(\prod_{i=p+1}^{q} \theta_i)} \exp\left\{-\sum_{i=p+1}^{q} \frac{X_i}{\theta_i} + \sum_{i=p+1}^{q} \frac{X_i}{\theta}\right\}.$$

Thus the log likelihood ratio function under H_A to that of H_0, for fixed (p, q) is,

$$\ln\ R(\theta, \delta, p, q; \mathrm{x}) = -\sum_{i=p+1}^{q} \ln\left[1 + \frac{\delta}{\theta} d_i(p, q)\right] + \sum_{i=p+1}^{q} \frac{X_i \delta d_i(p, q)}{\theta[\theta + \delta d_i(p, q)]}.$$

Next we assume that (p, q) has an equal chance to fall at any possible points $p = 1, \ldots, n-1$ and $q = p+1, \ldots, n$. Then the likelihood function under H_A to that under H_0 as $\delta/\theta \to 0^+$ can be expressed as,

$$R(\theta, \delta, p, q; \mathrm{x}) = \frac{1}{N}\frac{\delta}{\theta}\sum_{q=2}^{n}\sum_{p=1}^{q-1}\left\{-\sum_{i=p+1}^{q} d_i(p,q) + \sum_{i=p+1}^{q}\frac{X_i}{\theta}d_i(p,q)\right\} + o_p\left(\frac{\delta}{\theta}\right)$$

where $N = n(n-1)/2$. Since the first term in $R(\theta, \delta, p, q; \mathrm{x})$ is a known constant, the equivalent test statistic for testing under H_A to that under H_0 as $\delta/\theta \to 0^+$ reduces to,

$$T_0 = \frac{1}{\theta}\sum_{q=2}^{n}\sum_{p=1}^{q-1}\left\{\sum_{i=p+1}^{q} X_i d_i(p,q)\right\}.$$

Next, if θ is unknown, we can replace θ by $\bar{X}$, the maximum likelihood estimator of θ under H_0. Now if we change the order of summation in the above expression, the test statistic for testing H_0 *vs.* H_A can be written as,

$$T^* = \frac{n}{\sum_{i=1}^{n} X_i}\sum_{i=2}^{n} X_i\left[\sum_{k=1}^{i-1} k\{\Psi(n-k+i) - \Psi(k)\}\right]$$

where, $\Psi(t)$ is the Digamma function defined as, $\Psi(x) = \mathrm{d}/\mathrm{d}x \; \ln\; \Gamma(x)$. Next, for simplicity in notation set, $c_i = \sum_{k=1}^{i-1} k\{\Psi(n-k+i) - \Psi(k)\}$ and, $M = 2/n\sum_{i=1}^{n} c_i = 1/6\;(n-1)(n-2)$. Now for convenience in computation, for testing H_A *vs.* H_0, we will use the statistic,

$$T = \frac{\sum_{i=1}^{n} c_i X_i}{M\sum_{i=1}^{n} X_i}. \tag{2}$$

2.1 Moments of the Statistic T Under H_0

Define, $Y_i = X_i/\sum_{j=1}^{n} X_j$, $i = 1, \ldots, n-1$. If the null hypothesis is true, then it can be shown that $(Y_1, \ldots, Y_{n-1}) \sim D_{n-1}(1/2, \ldots, 1/2)$ – a Dirichlet distribution. Hence the moments of $(Y_1, \ldots, Y_{n-1})$ can be written as (see *e.g.* Johnson and Kotz, 1972, Ch 40, Sec 5),

$$\mu_{r_1,\ldots,r_{n-1}} = E(Y_1^{r_1}Y_2^{r_2}\cdots Y_{n-1}^{r_{n-1}}) = \frac{\prod_{i=1}^{n-1} 1/2^{[r_i]}}{(n/2)^{\left[\sum_{i=1}^{n-1} r_i\right]}}$$

where $a^{[r]} = a(a+1)\cdots(a+r-1)$. Now in terms of Y_i's we can write T as, $T = 1/M\;\sum_{i=1}^{n} c_i Y_i$. Thus under the null hypothesis, $\mu_1(T) = \sum_{i=1}^{n} c_i/M\;E(Y_i) = 1/2$. Although the other moments of T cannot be written in closed form, they can be computed without much difficulty, for any fixed n. Table I gives the first two moments of T and the coefficient of skewness $\beta_1(T)$ and kurtosis $\beta_2(T)$, for some selected sample sizes.

It is clear from (2) that T takes values between 0 and 1. From Table I we see that the null distribution of T has very small positive skewness, which gets closer to zero with increasing n. Table I also indicates that the kurtosis ($\gamma_2 = \beta_2 - 3$), is positive and tends to zero as $n \to \infty$.

TABLE I Moments of the Statistic T for Some Selected Sample Sizes.

n	$\mu_1^n\ (T)$	$\mu_2^n\ (T)\ (10^{-3})$	$\beta_1^n\ (T)\ (10^{-2})$	$\beta_2^n\ (T)$
25	0.5	6.619	2.975	2.861
50	0.5	3.212	1.783	2.931
75	0.5	2.124	1.269	2.955
100	0.5	1.587	0.984	2.966
125	0.5	1.267	0.804	2.973
150	0.5	1.055	0.680	2.978
175	0.5	0.903	0.588	2.981
200	0.5	0.790	0.519	2.983

2.2 Null Distribution of Statistic T

Define, $T_1 = T - 0.5/\sqrt{Var(T)}$ to be the standardized statistic corresponding to T. Then by the Lyapounov Central Limit Theorem and Slutsky's Theorem, we get that the statistic T_1, under H_0 follows an asymptotic normal distribution as $n \to \infty$. The test statistic is based on the likelihood ratio, for small changes in the ratio δ/θ. Thus a test that rejects H_0 in favor of the H_A that $\delta > 0$, for large values of T_1, will give the locally most powerful one-sided test as $\delta/\theta \to 0^+$. Also a test that rejects H_0 for large $|T_1|$ is the locally most powerful unbiased test against the two-sided alternatives for small values of $|\delta/\theta|$.

For moderate sample sizes, we suggest using an Edgeworth expansion for the c.d.f. of statistic T_1. The three-term Edgeworth expansion for the c.d.f. of statistic T_1 (*e.g.* see Johnson and Kotz) is given by, $F_{T_1}(x) = \Phi(x) - \{1/6\ \sqrt{\beta_1(T)}(x^2 - 1) + 1/24\ (\beta_2(T) - 3)(x^3 - 3x) + 1/72\ \beta_1(T)(x^5 - 10x^3 + 15x)\}\phi(x)$, where $\Phi(x)$ and $\phi(x)$ are the distribution function and the density function of a standard normal random variable respectively. We used this Edgeworth expansion to obtain critical values for the standardized statistic T_1 for selected moderate sample sizes. These values are displayed in Table II.

As we can see from the values in Table II, it is clear that the normal approximation is sufficiently accurate for most situations.

2.3 Asymptotic Distribution of T_1 Under the Alternative Hypothesis

Under the alternative hypothesis, we know that $X_1, \ldots, X_n$ will follow independent exponential distributions with mean $\theta_i = \theta + \delta d_i(p, q)$. For convenience in notation we will suppress

TABLE II Approximate Critical Values of the T_1 Test.

n	$c_{0.01}$	$c_{0.025}$	$c_{0.05}$	$c_{0.10}$	$c_{0.25}$
25	2.493	2.065	1.699	1.290	0.645
50	2.445	2.035	1.686	1.291	0.655
75	2.423	2.021	1.679	1.291	0.659
100	2.410	2.013	1.675	1.290	0.662
125	2.401	2.007	1.672	1.289	0.663
150	2.394	2.003	1.669	1.289	0.665
175	2.289	2.000	1.668	1.288	0.665
200	2.385	1.997	1.666	1.288	0.666
∞	2.326	1.960	1.645	1.282	0.674

Note: c_γ is defined to be $\int_{c_\gamma}^{\infty} f(x)\,dx = \alpha$.

the indices (p, q) in $d_i(p, q)$, and write it as d_i. Now if we assume that H_A holds and that, $p/n \to \lambda_1$ and $q/n \to \lambda_2$ such that $0 < \lambda_1 < \lambda_2 < 1$. Then by the Lyapounov central limit theorem we get,

$$\frac{\left\{\sum_{i=1}^n c_i X_i - \sum_{i=1}^n c_i \theta_i\right\}}{\left[\sum_{i=1}^n c_i^2 \theta_i^2\right]^{1/2}} \xrightarrow{D} N(0, 1)$$

and by the weak law of large of numbers we have, $\bar{X}/E(\bar{X}) \xrightarrow{P} 1$, where $E(\bar{X}) = 1/n \sum_{i=1}^n \theta_i = \theta + 1/n\ \delta(n - (q - p + 1)/2)$. Thus from Slutsky's theorem, we have that,

$$\frac{T - \mu_n}{\sigma_n} \xrightarrow{D} N(0, 1),$$

where,

$$\mu_n = \frac{\sum_{i=1}^n c_i \theta_i}{M \sum_{i=1}^n \theta_i} = \frac{1/2 + \rho\ 1/Mn \sum_{i=1}^n c_i d_i}{1 + \rho\ 1/n \sum_{i=1}^n d_i}$$

and,

$$\sigma_n^2 = \frac{\sum_{i=1}^n c_i^2 \theta_i^2}{\left[M \sum_{i=1}^n \theta_i\right]^2} = \frac{\rho^2 \sum_{i=1}^n c_i^2 + 2\rho \sum_{i=1}^n c_i^2 d_i + \sum_{i=1}^n c_i^2 d_i^2}{M^2\left[n + \rho \sum_{i=1}^n d_i\right]^2},$$

where $\rho = \delta/\theta$. Thus under the null hypothesis we have that,

$$\frac{(T - 0.5)}{1/Mn\ \sqrt{\sum_{i=1}^n c_i^2}} \xrightarrow{D} N(0, 1).$$

Now the power of the test at level α can be written as,

$$\beta(\alpha) = 1 - \Phi\left\{\frac{\sqrt{\sum_{i=1}^n c_i^2}}{Mn} \frac{z_\alpha}{\sigma_n} - \frac{1}{\sigma_n}(\mu_n - 0.5)\right\}$$

where z_α is given by the equation, $\alpha = \int_{z_\alpha}^{\infty} \phi(x)\,\mathrm{d}x$. It is clear from the expression of $\beta(\alpha)$, that the power of the test depends on θ and δ only through $\rho = \delta/\theta$.

Figure A gives the approximate power curves for a sample size $n = 50$. The first plot gives the power curves for fixed $q(= 30)$ with values of p ranging from $(1, 29)$ for $\rho = 1(2)7$. The seconds plot shows that power curves for fixed $q(= 50)$ with values of ρ ranging from 0 to 5 for $q - p = 15, 25, 35$. We can see from these two plots that the power is an increasing function of ρ for any values of p and q. Moreover the power of the test is greater if the change occurs in the middle of the sequence. Notice that the rate of increase of power decreases with increasing p. The third plot shows the power curves for fixed $q - p(= 10)$ with values of ρ ranging from 0 to 20 for $p = 15, 25, 35$. The fourth plot shows the power curves for fixed duration of change $q - p(= 15)$ with values of ρ ranging from 0 to 20 for $p = 5(5)25$. Plots 3 and 4 both indicate that the power of the test is greater if the change occurs in the middle of the sequence and is smaller if the change occurs at either end of the sequence. These plots also indicate that the power of the test decreases with ρ, if the change point p is at 5.

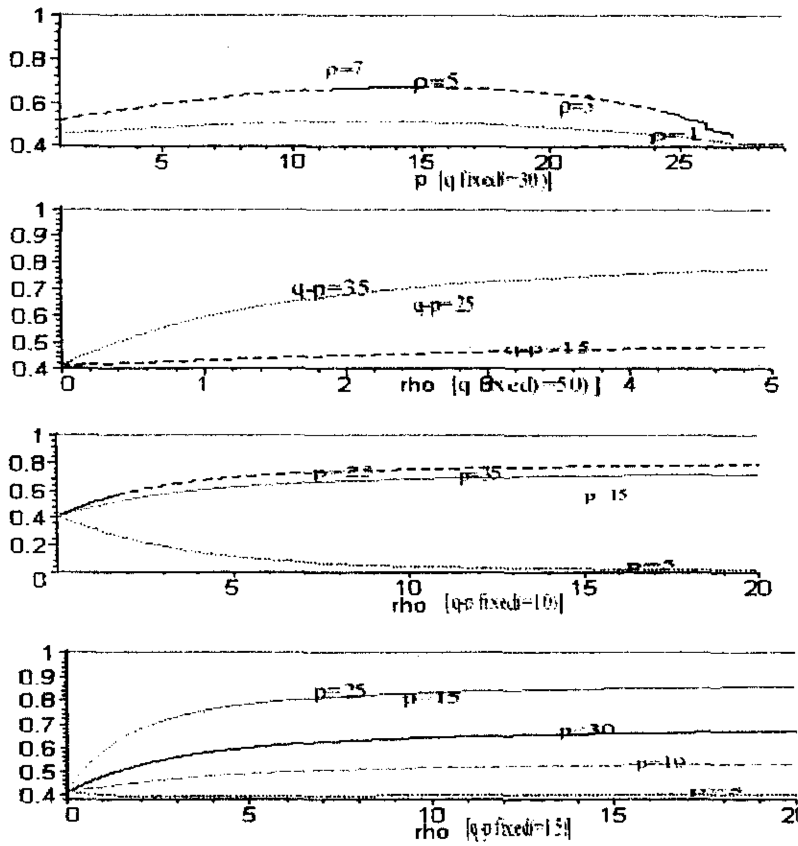

FIGURE A Power curves for statistic T_1.

3 ANOTHER LIKELIHOOD RATIO TYPE STATISTIC

Here we assume that (p, q) have a prior density,

$$f(p,q) \propto \frac{(q-p)(n-q+p)}{n^2}, \quad \text{for } 1 < p < q < n.$$

Then by a similar argument as for T in section 2, we can derive the corresponding the likelihood ratio type statistic for this prior for $\delta/\theta \to 0^+$ as,

$$T_{20} = n\frac{\sum_{i=1}^{n} i(i-1)(3n-i+2)(n+1-i)X_i}{M_1 \sum_{i=1}^{n} X_i},$$

where $M_1 = 2/n \ \sum_{i=1}^{n-1} i(i-1)(3n-i+2)(n+1-i) = 1/5\ (n-1)(2n+1)(n+2)(n+1)$.

3.1 Moments of the Statistic T_{20} Under H_0

Set $e_i = i(i-1)(3n-i+2)(n+1-i)/M_1$, then we can rewrite $T_{20} = n\sum_{i=1}^{n} X_i e_i / \sum_{i=1}^{n} X_i$. From section 2 we know that $(Y_1, \ldots, Y_{n-1}) \sim D_{n-1}\ (1/2, \ldots, 1/2)$. Thus we can find,

$$
\begin{aligned}
\mu_1(T_{20}) &= \frac{1}{2}\\
\mu_2(T_{20}) &= \frac{44n^4 + 25n^3 + 350n^2 + 495n + 486}{126(n+1)(n-1)(n+2)^2(2n+1)}\\
\mu_3(T_{20}) &= \frac{-2}{3003}(n-2)(373n^7 + 2381n^6 + 17737n^5 + 101924n^4 + 211567n^3\\
&\quad + 249149n^2 + 231123n + 258546)[(n-1)^2(2n+1)^2(n+2)^3(n+1)^2\\
&\quad \times (n+4)]^{-1}\\
\mu_4(T_{20}) &= 0.0915\, o(n^{-2})\\
\beta_1(T_{20}) &= 31.882\, o(n^{-4})\\
\beta_2(T_{20}) &= 3 - 4.161\, o(n^{-1})
\end{aligned}
$$

The null distribution is positively skewed and has negative kurtosis ($\gamma_2 = \beta_2 - 3$) but they both tends to 0 as $n \to \infty$.

3.2 Null Distribution of Statistic T_{20}

Define, $T_2 = (T_{20} - 0.5)/\sqrt{Var(T_{20})}$ to be the standardized statistic corresponding to T_{20}. Then as in section 2, we can show that T_2 is the locally most powerful one-sided test as $\delta/\theta \to 0^+$. And $|T_2|$ is the locally most powerful unbiased test against the two-sided alternatives for small values of $|\delta/\theta|$. Next, we use a three-term Edgeworth expansion for the c.d.f. of statistic T_2 to get, $F_{T_2}(x) = \Phi(x) - 1/24\ (\beta_2(T_2) - 3)(x^3 - 3x)\phi(x)$. This Edgeworth expansion was used to obtain critical values for the standardized statistic T_2 for selected moderate sample sizes. These values are displayed in Table III.

As we can see from the values in Table III, it is clear that the normal approximation is sufficiently accurate for most situations. Also notice that these critical values are very close to the critical values of statistic T_1, given in Table II.

TABLE III Approximate Critical Values of the T_2 Test.

n	$c_{0.01}$	$c_{0.025}$	$c_{0.05}$	$c_{0.10}$	$c_{0.25}$
25	2.490	2.064	1.699	1.290	0.646
50	2.436	2.028	1.681	1.289	0.656
75	2.413	2.014	1.674	1.288	0.661
100	2.400	2.006	1.670	1.288	0.663
125	2.391	2.000	1.667	1.287	0.665
150	2.385	1.997	1.665	1.287	0.666
175	2.380	1.993	1.666	1.287	0.667
200	2.376	1.991	1.666	1.287	0.667
∞	2.326	1.960	1.645	1.282	0.674

Note: c_α is defined to be $\int_{c_\alpha}^{\infty} f(x)\,dx = \alpha$.

3.3 Asymptotic Distribution of T_2 Under the Alternative Hypothesis

As in section 2 we can derive can show that under the alternative hypothesis, that for large n,

$$\frac{T_{20} - \mu_n^*}{\sigma_n^*} \xrightarrow{D} N(0, 1),$$

where,

$$\mu_n^* = \frac{1/2 + \rho \ \sum_{i=1}^n e_i d_i / Mn}{1 + \rho \ \sum_{i=1}^n d_i / n} \quad \text{and,} \quad \sigma_n^{*2} = \frac{\sum_{i=1}^n e_i^2 + 2\rho \sum_{i=1}^n e_i^2 d_i + (\rho)^2 \sum_{i=1}^n e_i^2 d_i^2}{M^2 \left[n + \rho \sum_{i=1}^n d_i\right]^2}.$$

We also can show that as $n \to \infty$, $\mu_n^* \to \mu^*$ and $n\sigma_n^{*2} \to \sigma^{*2}$, where,

$$\begin{aligned}\mu^* = \Bigg\{ & \frac{1}{2} + \rho\bigg(-\frac{1}{12}\lambda_1\lambda_2^4 - 2\lambda_2^4 + \frac{5}{12}\lambda_2^5 - \frac{1}{12}\lambda_2^3\lambda_1^2 - \frac{5}{8}\lambda_2\lambda_1^2 + \frac{1}{2}\lambda_2\lambda_1^3 - \frac{1}{12}\lambda_2\lambda_1^4 \\ & - \frac{1}{12}\lambda_2^3\lambda_1^3 - \frac{5}{8}\lambda_1^3 + \frac{1}{2}\lambda_1^4 - \frac{1}{12}\lambda_1^5 \bigg) \Bigg\} \left\{ 1 + \rho\left(\frac{\lambda_1 - \lambda_2}{2}\right) \right\}^{-1}\end{aligned}$$

and,

$$\begin{aligned}\sigma^{*2} = \Bigg\{ & \frac{5}{5544} \{ (-4158\lambda_1\lambda_2^4 - 4158\lambda_2^2\lambda_1^3 - 4158\lambda_2\lambda_1^4 + 20790\lambda_2^5 - 4158\lambda_1^5 \\ & - 5445\lambda_1^7 + 1540\lambda_1^8 + 7920\lambda_1^6 - 154\lambda_1^9 - 47520\lambda_2^6 + 38115\lambda_2^7 - 12320\lambda_2^8 \\ & + 1386\lambda_2^9 - 4158\lambda_2^3\lambda_1^2 + 7920\lambda_1^2\lambda_2^4 - 154\lambda_2^5\lambda_1^4 + 1540\lambda_2^5\lambda_1^3 + 1540\lambda_1\lambda_2^7 \\ & - 154\lambda_1\lambda_2^8 + 7920\lambda_2^3\lambda_1^3 - 154\lambda_1^2\lambda_2^7 - 154\lambda_2^2\lambda_1^7 + 7920\lambda_2^2\lambda_1^4 - 5445\lambda_2^3\lambda_1^4 \\ & + 1540\lambda_2^2\lambda_1^6 - 5445\lambda_2^2\lambda_1^5 + 7920\lambda_1\lambda_2^5 + 1540\lambda_1^2\lambda_2^6 + 1540\lambda_2^3\lambda_1^5 - 5445\lambda_1\lambda_2^6 \\ & - 5445\lambda_1^2\lambda_2^5 - 154\lambda_2^3\lambda_1^6 - 154\lambda_2^6\lambda_1^3 - 5445\lambda_2^4\lambda_1^3 + 7920\lambda_2\lambda_1^5 - 154\lambda_2^4\lambda_1^5 \\ & -5445\lambda_2\lambda_1^6 - 154\lambda_2\lambda_1^8 + 1540\lambda_2\lambda_1^7 + 1540\lambda_2^4\lambda_1^4)\rho + (-2970\lambda_1\lambda_2^4 \\ & - 1782\lambda_2^2\lambda_1^3 - 1188\lambda_2\lambda_1^4 + 8910\lambda_2^5 - 594\lambda_1^5 - 605\lambda_1^7 + 154\lambda_1^8 + 990\lambda_1^6 \\ & - 14\lambda_1^9 - 20790\lambda_2^6 + 16940\lambda_2^7 - 5544\lambda_2^8 + 630\lambda_2^9 - 2376\lambda_2^3\lambda_1^2 + 4950\lambda_1^2\lambda_2^4 \\ & - 84\lambda_2^5\lambda_1^4 + 921\lambda_2^5\lambda_1^3 + 1232\lambda_1\lambda_2^7 - 126\lambda_1\lambda_2^8 + 3960\lambda_2^3\lambda_1^3 - 112\lambda_1^2\lambda_2^7 \\ & - 42\lambda_2^2\lambda_1^7 + 2970\lambda_2^2\lambda_1^4 - 2420\lambda_2^3\lambda_1^4 + 462\lambda_2^2\lambda_1^6 - 1815\lambda_2^2\lambda_1^5 + 5940\lambda_1\lambda_2^5 \\ & + 1078\lambda_1^2\lambda_2^6 + 616\lambda_2^3\lambda_1^5 - 4235\lambda_1\lambda_2^6 - 3630\lambda_1^2\lambda_2^5 - 56\lambda_2^3\lambda_1^6 - 98\lambda_2^6\lambda_1^3 \\ & - 3025\lambda_2^4\lambda_1^3 + 1980\lambda_2\lambda_1^5 - 70\lambda_2^4\lambda_1^5 - 1210\lambda_2\lambda_1^6 - 28\lambda_2\lambda_1^8 + 308\lambda_2\lambda_1^7 \\ & + 770\lambda_2^4\lambda_1^4)\rho^2 + 374\} \Bigg\} \left\{ 1 + \rho\frac{(\lambda_1 - \lambda_2)}{2} \right\}^{-2}.\end{aligned}$$

Then we have that for large n,

$$\frac{\sqrt{n}(T_{20} - \mu^*)}{\sigma^*} \xrightarrow{D} N(0, 1).$$

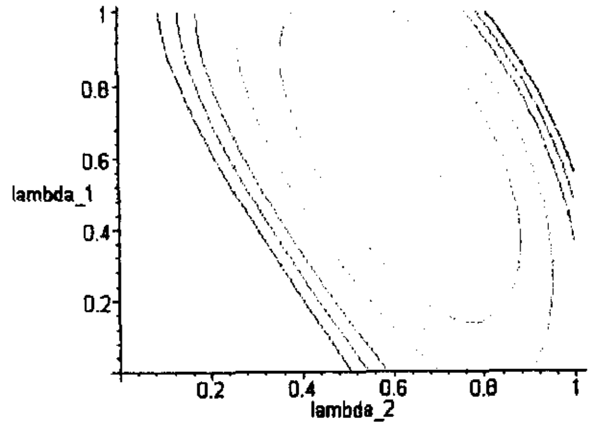

FIGURE B Plot of $\mu^* - 0.5$ *vs.* (λ_1, λ_2).

LEMMA 1 *The test based on T_2 is consistent for testing H_0 vs. H_A if,* $10\lambda_2^4 + 8\lambda_1\lambda_2^3 - 48\lambda_2^3 + 6\lambda_2^2\lambda_1^2 - 36\lambda_2^2\lambda_1 + 45\lambda_2^2 + 4\lambda_2\lambda_1^3 + 30\lambda_2\lambda_1 - 24\lambda_2\lambda_1^2 + 15\lambda_1^2 - 12\lambda_1^3 + 2\lambda_1^4 > 6.$

Proof Under the null hypothesis we have that, $\sqrt{n}(T_{20} - 0.5)/\sqrt{85/252} \xrightarrow{D} N(0, 1)$. Thus the power of the test based on T_3, at level α is given by,

$$\beta(\alpha) = 1 - \Phi\left\{\sqrt{\frac{85}{252}}\frac{z_\alpha}{\sigma^*} - \frac{\sqrt{n}}{\sigma^*}(\mu^* - 0.5)\right\} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

if $\mu^* - 0.5 > 0 \Longleftrightarrow 10\lambda_2^4 + 8\lambda_1\lambda_2^3 - 48\lambda_2^3 + 6\lambda_2^2\lambda_1^2 - 36\lambda_2^2\lambda_1 + 45\lambda_2^2 + 4\lambda_2\lambda_1^3 + 30\lambda_2\lambda_1 - 24\lambda_2\lambda_1^2 - 6 + 15\lambda_1^2 - 12\lambda_1^3 + 2\lambda_1^4 > 6$. Hence we get the result.

The contour plot for $\mu^* - 0.5 > 0$ is given in Figure B. The contours indicate the area where $\mu^* - 0.5 > 0$. From this plot it is can be seen that the test based on T_2 will be consistent if the change-points fall in the middle of the sequence.

4 SCORE STATISTIC

Notice that, for testing H_0 *vs.* H_A the maximum likelihood estimators cannot be computed in closed form. Hence we will derive Rao's efficient score statistic as an alternative. First we define,

$$\beta = \begin{bmatrix} \theta \\ \delta \end{bmatrix}, \quad h(t) = -\frac{1}{t} \text{ and } \mathbf{y}_i = [1 \quad d_i] \; i = 1, \dots, n,$$

where $d_i = d_i(p, q)$ if $i = p + 1, \dots, q$ and 0 elsewhere. Then notice that we can write, $\mu_i = h(\mathbf{y}_i\beta)$ and the log likelihood of x_i's as,

$$\ln L(\beta, x_i) = \{x_i h(\mathbf{y}_i\beta) + \ln h(\mathbf{y}_i\beta)\} \quad \text{for } i = 1, \dots, n.$$

Next let $\beta_0 = [\theta \quad 0]'$ and $\bar{\beta}_0$ be the restricted mle of β_0. Now for fixed (p, q), Rao's efficient score statistic for test $H_0 : \beta = \beta_0$ *vs.* $H_A : \beta = \beta_0$ can be written as,

$$T_3(p, q) = \frac{1}{n} U_n(\bar{\beta}_0)'[I_n(\bar{\beta}_0)]^{-1} U_n(\bar{\beta}_0),$$

where,

$$U_n(\bar{\beta}_0) = \frac{\partial}{\partial \beta} \ln L_n(\beta)|_{\beta=\bar{\beta}_0}$$

and

$$I_n(\bar{\beta}_0) = \frac{\partial}{\partial \beta \partial \beta'} \ln L_n(\beta)|_{\beta=\bar{\beta}_0}$$

Thus for fixed (p, q) we get the score statistic to be equal to,

$$T_3(p, q) = \left\{ \sum_{i=p+1}^{q} d_i(p, q) \frac{(X_i - \bar{X})}{\bar{X}\sqrt{v(p, q)}} \right\}^2$$

where, $v(p, q) = 1/12(q-p)\,(q-p+1)[4n(q-p) + 2n - 3(q-p)^2 - 3(q-p)]$. But since in this situation, the location of the change points (p, q) are unknown, so we can use,

$$T_3 = \max_{1<p<q<n} \left| \sum_{i=1}^{n} d_i(p, q) \frac{(X_i - \bar{X})d_i}{\bar{X}\sqrt{v(p, q)}} \right|,$$

to test H_0 against a two-sided alternative. On the other hand when we want to test H_0 against $H_A : \delta > 0$ we suggest,

$$T_3^+ = \max_{1<p<q<n} \sum_{i=1}^{n} d_i(p, q) \frac{(X_i - \bar{X})}{\bar{X}\sqrt{v(p, q)}},$$

and to test H_0 against $H_A : \delta < 0$ we suggest,

$$T_3^- = \min_{1<p<q<n} \sum_{i=1}^{n} d_i(p, q) \frac{(X_i - \bar{X})}{\bar{X}\sqrt{v(p, q)}}.$$

4.1 ASYMPTOTIC NULL DISTRIBUTION OF THE SCORE STATISTIC

Recall that the change points (p, q) are such that, $p/n \to \lambda_1$ and $q/n \to \lambda_2$, with $0 < \lambda_1 < \lambda_2 < 1$ when $n \to \infty$. Then we can show that as $n \to \infty$, $v(p,q)/n^2$, converges to $v_0(\lambda_1, \lambda_2) = 1/12(\lambda_2 - \lambda_1)(4 - 3\lambda_2 + 3\lambda_1)$. Next, notice that we can rewrite,

$$\begin{aligned} T_3^{1/2}(p,q) &= \left| \frac{1}{(q-p)\bar{X}\sqrt{v(p,q)}} \sum_{i=p+1}^{q} (i-p)(X_i - \bar{X}) \right| \\ &= \left| \frac{1}{(q-p)\sqrt{v(p,q)}} \sum_{i=p+1}^{q} \left\{ \sum_{j=1}^{q} \frac{(X_j - \bar{X})}{\bar{X}} - \sum_{j=1}^{i-1} \frac{(X_j - \bar{X})}{\bar{X}} \right\} \right| \end{aligned}$$

Now define a partial sum process on $[0, 1]$ to be

$$B_n(u) = \begin{cases} \dfrac{1}{\sqrt{n}} \displaystyle\sum_{j=1}^{i-1} \frac{(X_j - \bar{X})}{\bar{X}} & \text{if } \dfrac{1}{n} \le u < \dfrac{i+1}{n},\ 1 \le i < n \\ 0 & \text{if } u = 1 \text{ or if } 0 \le u < \dfrac{1}{n}. \end{cases}$$

Then under the null hypothesis, $\{B_n(u) : 0 \le u \le n\}$ converges weakly to a Brownian Bridge process $\{B(u) : 0 \le u \le 1\}$. Thus for fixed (p, q), under the null hypothesis of no change, $T_3/\sqrt{n}$ converges in distribution to

$$Ł := \sup_{0<\lambda_1<\lambda_2<1} \frac{1}{(\lambda_2 - \lambda_1)\sqrt{v_0(\lambda_1, \lambda_2)}} \left| (\lambda_2 - \lambda_1)B(\lambda_2) - \int_{\lambda_1}^{\lambda_2} B(u)\,\mathrm{d}u \right|.$$

LEMMA 2 *The critical points of the tests T_3, T_3^+, tend to infinity and T_3^- tend to negative infinity as $n \to \infty$.*

Proof First consider the statistic,

$$T_3^0 = \max_{1 \le p < n} \left| \sum_{i=1}^{n} \frac{(X_i - \bar{X})d_i}{\bar{X}\sqrt{v(p,n)}} \right|.$$

Note that T_3^0 is a special case of T_3 when $q = n$. Thus as $n \to \infty$, $T_3^0/\sqrt{n}$ converges in distribution to,

$$Ł^0 := \sup_{0<\lambda_1<1} \left| \int_{\lambda_1}^{1} \frac{B(u)\,\mathrm{d}u}{(1-\lambda_1)\sqrt{v_0(\lambda_1, 1)}} \right|,$$

where $v_0(\lambda_1, 1) = 1/12(1 - \lambda_1)(1 + 3\lambda_1)$. But we can write,

$$\begin{aligned} \frac{\int_{\lambda_1}^{1} B(u)\,\mathrm{d}u}{(1-\lambda_1)\sqrt{(1-\lambda_1)/12(1+3\lambda_1)}} &= \frac{1/(1-\lambda_1)\int_{\lambda_2}^{1} B(u)\,\mathrm{d}u}{B(1-\lambda_1)} \\ &\times \frac{B(1-\lambda_1)}{\sqrt{2(1-\lambda_1)\ln\ln(1/(1-\lambda_1))}} \frac{\sqrt{24(1-\lambda_1)\ln\ln(1/(1-\lambda_1))}}{\sqrt{(1-\lambda_1)(1+3\lambda_1)}}. \end{aligned} \tag{3}$$

Next consider the joint Gaussian process

$$\left\{\frac{1}{\sqrt{s(1-s)}}B(su), \frac{1}{\sqrt{s(1-s)}}B(s),\ 0 < u < 1\right\}.$$

Notice that this process converges in distribution as $s \to 1$ to $\{(X(u), Y), 0 < u < 1\}$, which has a distribution that does not depend on λ_1. Thus as $\lambda_1 \to 1$, the first-term in (3) converges in distribution to a non-degenerate distribution $1/Y \int_0^1 X(u)\,\mathrm{d}u$. By the iterated law of

TABLE IV Powers of T_1, T_2 and T_3 Test ($\alpha = 0.05$, right-tailed).

		$\delta = 3$			$\delta = 6$		
p	q	T_1	T_2	T_3	T_1	T_2	T_3
5	20	0.0024	0.0038	0.4630	0.0002	0.0002	0.8256
5	25	0.0452	0.0542	0.4480	0.0614	0.0656	0.7922
5	30	0.3740	0.3896	0.3926	0.6098	0.6186	0.7056
5	35	0.7496	0.7682	0.3380	0.9412	0.9470	0.5938
5	40	0.8914	0.9126	0.2724	0.9916	0.9940	0.4708
5	45	0.8616	0.9064	0.2488	0.9856	0.9936	0.3848
5	50	0.3932	0.4866	0.1824	0.5644	0.6680	0.2752
10	20	0.0098	0.0158	0.4366	0.0026	0.0046	0.7940
10	25	0.1228	0.1418	0.4634	0.2224	0.2392	0.8276
10	30	0.5412	0.5600	0.4386	0.8394	0.8418	0.7968
10	35	0.8498	0.8642	0.3938	0.9852	0.9866	0.7222
10	40	0.9344	0.9454	0.3520	0.9980	0.9988	0.6108
10	45	0.9004	0.9376	0.2812	0.9944	0.9984	0.4692
10	50	0.4118	0.5130	0.2306	0.5976	0.6950	0.3610
15	20	0.0304	0.0416	0.3574	0.0248	0.0342	0.6700
15	25	0.2064	0.2290	0.4418	0.4218	0.4392	0.8090
15	30	0.5772	0.5956	0.4550	0.8868	0.8908	0.8218
15	35	0.8506	0.8624	0.4392	0.9892	0.9898	0.7972
15	40	0.9354	0.9470	0.4030	0.9982	0.9988	0.7182
15	45	0.8920	0.9318	0.3470	0.9944	0.9980	0.5988
15	50	0.3624	0.4612	0.2892	0.5486	0.6600	0.4774
20	25	0.1724	0.1952	0.3342	0.3578	0.3764	0.6762
20	30	0.5256	0.5464	0.4406	0.8542	0.8588	0.7956
20	35	0.7952	0.8112	0.4512	0.9738	0.9770	0.8204
20	40	0.8930	0.9120	0.4518	0.9956	0.9962	0.8000
20	45	0.8198	0.8818	0.3904	0.9870	0.9956	0.7086
20	50	0.2676	0.3612	0.3386	0.4164	0.5260	0.6026
25	30	0.3138	0.3420	0.3432	0.6042	0.6238	0.6666
25	35	0.6390	0.6680	0.4360	0.9142	0.9224	0.7912
25	40	0.7686	0.8128	0.4702	0.9764	0.9842	0.8244
25	45	0.6554	0.7534	0.4524	0.9384	0.9704	0.7916
25	50	0.1336	0.2058	0.3896	0.2100	0.3136	0.7144
30	35	0.3546	0.3964	0.3380	0.6542	0.6886	0.6662
30	40	0.5428	0.6098	0.4330	0.8680	0.9000	0.7984
30	45	0.3864	0.5208	0.4698	0.7418	0.8520	0.8216
30	50	0.0464	0.0856	0.4406	0.0602	0.1172	0.8054
35	40	0.2458	0.3122	0.3462	0.5028	0.5850	0.6786
35	45	0.1438	0.2440	0.4438	0.2976	0.4896	0.7984
35	50	0.0096	0.0224	0.4524	0.0042	0.0140	0.8224
40	45	0.0392	0.0750	0.3502	0.0476	0.1012	0.6674
40	50	0.0024	0.0050	0.4336	0.0002	0.0010	0.7954

logarithm, the limsup of the second term in (3) converges to 1 a.s., as $\lambda_1 \to 1$, and the third term in (3) goes to infinity as $\lambda_1 \to 1$. Therefore we get,

$$\sup_{0<\lambda_1<1} \int_{\lambda_1}^{1} \left| \frac{B(u)\,\mathrm{d}u}{\lambda_1\sqrt{v_0(\lambda_1, 1)}} \right| \geq \limsup_{\lambda_1 \to 1} \int_{\lambda_1}^{1} \left| \frac{B(u)\,\mathrm{d}u}{(1-\lambda_1)\sqrt{v_0(\lambda_1, 1)}} \right|,$$

which tends to infinity a.s. Thus T_3^0 also goes to infinity a.s. as $n \to \infty$. Next note that in the definition of T_3 the set over which supremum is taken is a subset of the corresponding set for T_3^0. Thus T_3 also goes to infinity as $n \to \infty$. A similar argument was used by Suigira and Ogden (1994).

5 POWER COMPARISONS OF THE THREE TESTS

Table IV compares the powers of the tests based on the statistics T_1, T_2 and T_3 for 10,000 repetition Monte Carlo experiments. In order to keep the table to a reasonable size, only the case of sample size, $n = 50$ and $\theta = 1, 3$ with a significance level $\alpha = 0.05$ is reported. We see from Table IV that the powers of the statistics T_1 and T_2 are pretty close for all values δ, θ and (p, q). Notice that the powers of tests based on the statistics T_1 and T_2 are larger than the power of tests based on statistic T_3, when (p, q) fall in the middle of the sequence. On the other hand if (p, q) fall in the beginning or the end of the sequence the power of test based on T_3 is larger than either of T_1 or T_2. Power of all three tests are considerably small if (p, q) fall in the very beginning and at the very end of the sequence. Higher powers for all these three tests are achieved if (p, q) occur in the middle of the sequence. Also it is clear that the power increases with increasing δ for T_3. This holds true also for T_1 and T_2 when (p, q) fall in the middle of sequence.

Thus we recommend that one should use T_3 unless it is likely that the trend change has occurred in the middle of the sequence. In such a situation we recommend the use of statistic T_2. The only reason why T_2 is preferred over T_1 is because of its ease of computation.

6 ANALYSIS OF INTER ARRIVAL TIMES

To illustrate the test procedures, we use a set of aircraft arrival times collected from a low-altitude transitional control sector for the period from noon through 8 P.M. on April 30, 1969. This data set was taken from Hsu (1979). There are 212 inter arrival times within this period. Hsu (1979) has showed that the data is exponential and that the observations are independent. The hypotheses of interest here are, H_0: $\delta = 0$ *vs.* H_A: $\delta > 0$. The data is plotted in Figure C. The time series plot does not show a linear trend followed by an abrupt change in the sequence.

To examine this more rigorously we performed the three tests discussed in above sections to the data set, we get the results contained in Table V. From Table V, we can see that while the tests based T_1, T_2 and T_3 are well out side the significance bounds. Thus we can conclude that the arrivals of the aircrafts occurred at an approximately constant rate during the period of time considered.

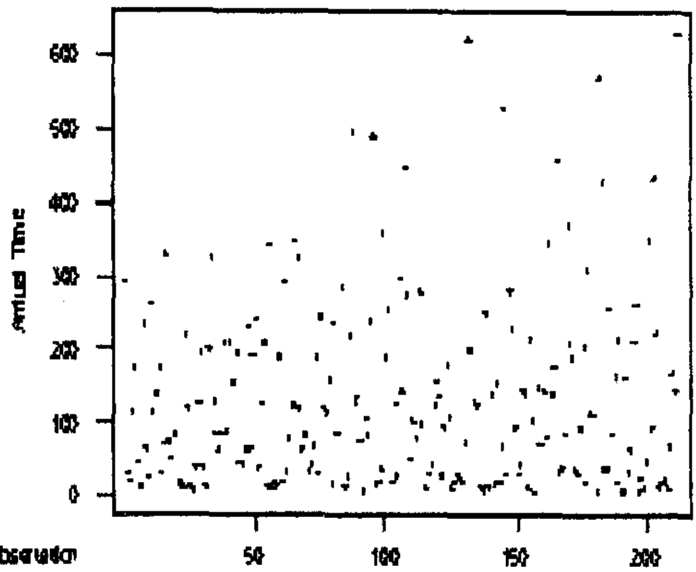

FIGURE C Plot of inter arrival times.

TABLE V Results of the Tests for Inter Arrival Times of Air Data.

Statistic	*Observed value*	*p-Value*
T_1	0.429	0.2203
T_2	0.442	0.2518
T_3	3.732	0.2378

7 ANALYSIS OF STANFORD HEART TRANSPLANT DATA

This data set was taken from "The Statistical Analysis of Failure Time Data" by Kalbfleisch and Prentice, Appendix I, pages 230–232. The average survival time for the 35 known age groups is considered. We plotted average survival times ordered by their magnitudes, against the expected order statistics of an exponential random variable with mean 1. Except for the largest observation, the sample points form a nearly straight line. Thus we can use an exponential model for this data. The hypotheses of interest here are, H_0: $\delta = 0$ *vs.* H_A: $\delta < 0$. The data is plotted in Figure D.

The plot does seem to indicate that there is a linear trend change in the sequence. To investigate this more rigorously we performed the test discussed in the previous sections for this data set. Table VI gives the critical values and the p-values for the three tests based on the three statistics T_1, T_2 and T_3. Thus we see that the p-values for the tests based on T_1 and T_2 are fairly large. But notice that the p-value of the test based on T_3 is somewhat small. The mle's of the change points in this case turn out to be 7 and 10. This translates to ages 29 and 33. This is not unexpected because the change did occur during the end of the sequence and in this situation T_3 was shown to be more powerful than

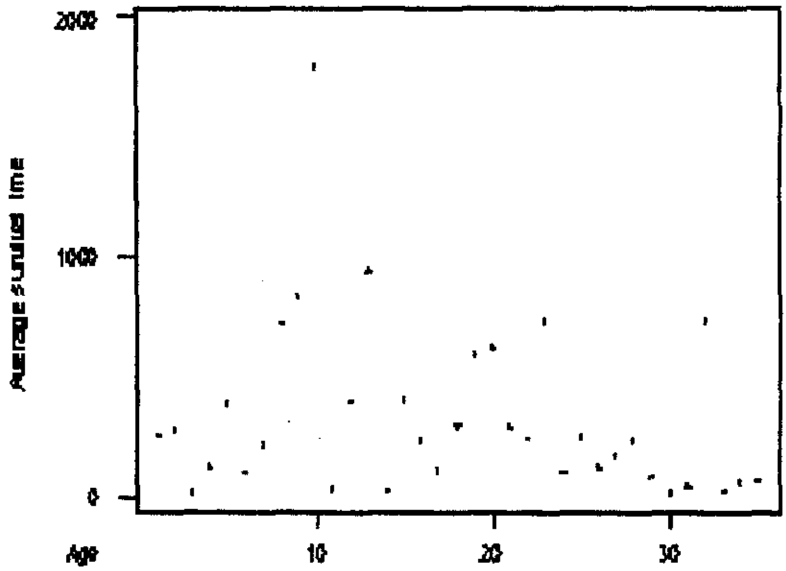

FIGURE D Plot of average survival times against the age.

TABLE VI Results of the Tests for Stanford Heart Transplant Data.

Statistic	*Observed value*	*p-Value*
T_1	0.047	0.4899
T_2	0.035	04932
T_3	4.78	0.0709

T_1 or T_2. We tested for exponentiality of data again after transforming the data according to the mle's and found that the exponentiality assumption holds true as well.

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