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INDIVIDUALS CONTROL SCHEMES FOR MONITORING THE MEAN AND VARIANCE OF PROCESSES SUBJECT TO DRIFTS

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ABSTRACT

This paper investigates control chart schemes for detecting drifts in the process mean μ and/or process standard deviation σ when individual observations are sampled. Drifts may be due to causes such as gradual deterioration of equipment, catalyst aging, waste accumulation, or human causes, such as operator fatigue or close supervision. The standard Shewhart X chart and moving range (MR) chart are evaluated, as well as several types of exponentially weighted moving average (EWMA) charts and combinations of charts involving these EWMA charts. We show that the combinations of the EWMA charts detect slow-rate and moderate-rate drifts much faster than the combined X and MR charts. We also show that varying the sampling interval adaptively

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as a function of the process data results in notable reductions in the detection delay of drifts in μ and/or σ.

Key Words: Exponentially weighted moving average control charts; Shewhart control charts; Statistical process control; Steady state; Variable sampling interval

1. INTRODUCTION

Statistical process control (SPC) refers to statistical methods used extensively to monitor and improve the quality and productivity of manufacturing processes and service operations. SPC primarily involves the implementation of *control charts*, which are used to monitor a process to detect special causes of process variation that may result in lower-quality process output. The practical applications of control charts now extend far beyond manufacturing, into engineering, environmental science, genetics, epidemiology, medicine, finance, and even law enforcement and athletics (see Gibbons [1], Lai [2], Montgomery [3], and Ryan [4]). The most commonly used types of control charts are the *Shewhart* charts, proposed by Shewhart [5], the *exponentially weighted moving average* (EWMA) charts, originating in the work of Roberts [6], and the *cumulative sum* (CUSUM) charts, initially investigated by Page [7].

In many applications, it is assumed that the process variable X being measured is normally distributed with mean μ and standard deviation σ. In this case, a special cause that affects the distribution of X could change μ, σ, or both μ and σ. The standard practice for monitoring in this situation is to concurrently use a pair of control charts, one for detecting changes in μ and the other for detecting changes in σ.

The statistics plotted on the control charts are usually based on samples (or subgroups) of n observations that are taken at regular sampling intervals. For example, a sample of $n = 4$ observations might be taken every hour. The traditional pair of control charts for such cases is the Shewhart $\bar{X}$ chart, on which the sample means are plotted, and the Shewhart R chart, upon which the sample ranges are plotted.

In many practical situations, however, it may not be feasible to take samples larger than one, so control charts must be based on individual observations ($n = 1$) rather than samples of $n > 1$. In such situations, sampling may be expensive, destructive, or time consuming. Also, repeat process measurements may differ only because of laboratory or analysis error, as in many chemical and process industries. The traditional pair of control charts for these situations is the Shewhart X chart, on which the individual observations are plotted, and the Shewhart *moving range* (MR) chart, upon which the ranges of successive individual observations are plotted.

Shewhart control charts are simple to understand and very widely used for on-line process monitoring, but they have the disadvantage that they are not effective for the detection of small changes in process parameters. The ability to detect small parameter changes can be significantly improved by using a control chart on which a statistic is plotted that incorporates information from past samples in addition to the information in the current sample. For example, on the EWMA chart, a weighted average of current and past sample statistics is plotted.

Another approach to improving the ability to detect small process changes is to use a *variable sampling interval* (VSI) control chart instead of a traditional *fixed sampling interval*(FSI) control chart. In a VSI chart, the sampling interval is varied as a function of the control statistic. A short sampling interval is used whenever there is evidence from the process data that a parameter may have changed, and a long sampling interval is used when there is no such evidence. If the evidence of a process change is sufficiently strong, then a VSI control chart signals in the same way as a traditional FSI chart.

Reynolds and Stoumbos [8] investigated the performance of a number of FSI and VSI Shewhart and EWMA control charts for monitoring μ and σ when the effect of a special cause is to produce a sustained *shift* (or simply, a shift) in one or both of these parameters. With a shift, it is assumed that once a special cause has shifted a parameter to a new value, the parameter stays at the new value until a control chart detects this shift and rectifying action is taken. For practical applications, Reynolds and Stoumbos [8] recommended several pairs of control charts that provide very good performance for detecting a wide range of magnitudes of shifts in μ and/or σ.

Stoumbos and Reynolds [9] investigated the robustness of Shewhart and EWMA control chart schemes for monitoring μ and σ when the actual distribution of the process observations may not be the assumed normal distribution. For practical implementation, they recommended a pair of EWMA charts that is robust to non-normal distributions.

The objective of this paper is to investigate the performances of the FSI as well as the VSI control schemes recommended by Reynolds and Stoumbos [8] and Stoumbos and Reynolds [9] for the situation in which individual observations are sampled from the process, and the effect of a special cause is to produce a *drift* in μ and/or σ. With a drift, it is assumed that once a special cause initiates the drift of a process parameter away from its in-control value, the parameter continues to drift away at a constant rate until a control chart detects this drift.

Drifts are usually attributed to the wearing out or deterioration of tools or of some other critical process components. In chemical processes, they often occur because of the settling or separation of the components of a mixture. Drifts can also be attributed to human causes, such as operator fatigue or close supervision.

More specifically, for the problem of monitoring μ, drifts may be the result of improving operator skills, dirt build-up in fixtures, tool wear, changes in temperature or humidity, or aging equipment. For the problem of monitoring σ, an upward or increasing drift may be due to gradual decline in material quality, operator fatigue, gradual loosening of a fixture or a tool, or dulling of a tool. A downward or decreasing drift in σ is often the result of improved operator skill, improved work methods, better materials, or improved or more frequent maintenance.

Most all evaluations of the performance of control charts in the SPC literature have investigated process shifts, and only very few papers have considered process drifts. In particular, Davis and Woodall [10] evaluated the performance of the simple Shewhart chart when various trend rules are used to detect a drift in μ. Gan [11] investigated the performance of the EWMA chart when there is a drift in μ. Aerne, Champ, and Rigdon [12] evaluated the performances of Shewhart, CUSUM, and EWMA charts also for the case of a potential drift in μ. However, none of these papers investigated the much more complex and generally more realistic problem of simultaneously monitoring μ and σ. The investigation of this control problem requires the evaluation of the joint statistical performance of multiple control charts collectively used to monitor μ and σ.

The balance of this paper is structured as follows. In Section 2, various individuals control charts for monitoring μ and σ are presented. Statistical measures of control chart performance and methods for their evaluations are discussed in Sections 3 and 4, respectively. In Sections 5 and 6, the performances of individuals control schemes for monitoring μ and σ are compared when the effect of a special cause is to produce an upward or a downward drift in either μ, σ, or both. Then, the VSI feature and its effect on the performances of these charts for the problem of detecting drifts are investigated in Sections 7 and 8. Finally, some concluding remarks are presented in Section 9.

2. CONTROL CHARTS FOR MONITORING THE MEAN AND VARIANCE

Let μ_0 and σ_0 denote the in-control values for μ and σ, respectively. Suppose that μ_0 and σ_0 have been estimated with negligible error during a preliminary phase of parameter estimation by a large enough sample. Also suppose that the time intervals between successive observations are sufficiently large so that successive observations are essentially independent. The objective of process monitoring is the detection of any special cause that changes μ from μ_0 and/or changes σ from σ_0. When there is a change in σ, it is usually assumed that the primary interest is in detecting increases in σ, because an increase corresponds to deterioration in quality. However, detecting a decrease in σ may also be important in some cases, so this problem is also considered in this paper.

Let X_k represent the observation taken at sampling point k. The Shewhart X chart is based on plotting X_k versus k, and a signal is generated at sampling point k if X_k falls outside of control limits constructed at

$$\mu_0 \pm h_X \sigma_0.$$

In most practical situations, the chart parameter h_X is taken to be 3 to yield the traditional "three-sigma" control limits.

The Shewhart MR chart is based on plotting the control statistic

$$R_k = |X_k - X_{k-1}|, \quad k = 2, 3, \ldots,$$

and a signal is given that σ has increased if R_k exceeds the upper control limit

$$h_R^+ \sigma_0.$$

In most practical applications, the control limit is taken to be three standard deviations of R_k above the mean of R_k, but here we choose h_R^+ to yield specified statistical properties when the process is in control. A lower control limit at

$$h_R^- \sigma_0$$

can be introduced when it is desirable to detect decreases in σ.

For the problem of detecting shifts in μ and/or σ, a number of papers (Reynolds and Stoumbos [8], Stoumbos and Reynolds [9], Nelson [13], Roes, Does, Schurink [14], Rigdon, Cruthis, and Champ [15], Albin, Kang, and Shea [16], and Amin and Ethridge [17]) have shown that there is essentially no advantage to using the MR chart with the X chart. For a given false-alarm rate, using the X chart alone is much better for detecting shifts in μ and only slightly worse for detecting shifts in σ. The MR chart is considered in this paper because it is the traditional control chart for monitoring σ, and it will be informative to determine whether the conclusions reached for process shifts also apply for the problem of controlling process drifts.

The EWMA chart for detecting changes in μ is based on the control statistic

$$Y_k = (1 - \lambda) Y_{k-1} + \lambda X_k, \quad k = 1, 2, \ldots,$$

where λ is a smoothing parameter satisfying $0 < \lambda \leq 1$, and the starting value is usually taken to be $Y_0 = \mu_0$. When the parameter λ is chosen to be small, the EWMA chart will be much more effective than the Shewhart X chart at detecting small and medium-sized shifts in μ (see, for example, Lucas and Saccucci [18]). When λ is large, the EWMA chart will be effective at detecting large shifts, and when $\lambda = 1$, this chart reduces to the X chart. A signal is given at sample k if Y_k

falls outside of control limits constructed at

$$\mu_0 \pm h_Y \sigma_0 \sqrt{\lambda/(2-\lambda)},$$

where $\sigma_0 \sqrt{\lambda/(2-\lambda)}$ is the asymptotic in-control standard deviation of Y_k. The EWMA chart that is based on the statistic Y_k will be referred to as the EWMA$_Y$ chart.

If it is important to detect small and medium-sized changes in σ, then it is sensible to consider an EWMA chart of the squared deviations from target (see, for example, Domangue and Patch [19], Shamma and Amin [20], and MacGregor and Harris [21]). It is assumed here that the primary interest is in detecting an increase in σ, so a one-sided EWMA chart will be considered first. The EWMA control statistic for this chart is

$$S_k = (1-\lambda)\max\{S_{k-1}, \sigma_0^2\} + \lambda(X_k - \mu_0)^2, \quad k = 1, 2, \ldots,$$

where λ is the smoothing parameter, and the starting value is usually $S_0 = \sigma_0^2$. This one-sided statistic is defined so that if S_{k-1} is below σ_0^2 then there is a reset back to σ_0^2 before computing S_k. A signal is given if S_k falls above a control limit constructed at

$$\sigma_0^2 + h_S \sigma_0^2 \sqrt{2\lambda/(2-\lambda)}.$$

Note that when the process is in control, $\sqrt{2}\sigma_0^2$ is the standard deviation of $(X_k - \mu_0)^2$, but $\sigma_0^2\sqrt{2\lambda/(2-\lambda)}$ is not the asymptotic standard deviation of S_k because the reset is used. However, the term $\sigma_0^2\sqrt{2\lambda/(2-\lambda)}$ is used in defining the control limit, so that the form of the limit for S_k corresponds to the form for Y_k. The EWMA chart that is based on the statistic S_k will be referred to as the EWMA$_S$ chart.

The combination of the EWMA$_Y$ and EWMA$_S$ control charts relies on the one-sided EWMA$_S$ chart to detect increases in σ. This combination can easily be modified to make the EWMA$_S$ chart two-sided by eliminating the reset at σ_0^2 in the definition of the EWMA$_S$ statistic and then adding a lower control limit. Gan [22] discusses two-sided EWMA charts for monitoring μ and σ, but considered only the case of $n > 1$. Instead of using a single two-sided EWMA chart, we use two individual one-sided EWMA charts run in parallel, because the two-chart schemes are robust to potential inertial problems in the EWMA statistics that can increase the delay in detecting process changes (see Reynolds and Stoumbos [8], Stoumbos and Reynolds [9], and Yashchin [23]).

For detecting a decrease in σ, the one-sided EWMA chart based on the squared deviations from target would use the control statistic

$$W_k = (1-\lambda)\min\{W_{k-1}, \sigma_0^2\} + \lambda(X_k - \mu_0)^2, \quad k = 1, 2, \ldots,$$

where λ is the usual smoothing parameter, and the starting value is typically chosen to be $W_0 = \sigma_0^2$. This one-sided EWMA statistic is defined so that if W_{k-1} is above σ_0^2, then there is a reset back to σ_0^2 before computing W_k. This chart signals if W_k falls below a control limit constructed at

$$\sigma_0^2 - h_W \sigma_0^2 \sqrt{2\lambda/(2-\lambda)}.$$

The term $\sigma_0^2 \sqrt{2\lambda/(2-\lambda)}$ is used in defining this control limit so that the form of control limit for W_k corresponds to the form for Y_k and S_k. The EWMA chart based on the statistic W_k will be represented as the EWMA$_W$ chart. When the EWMA$_W$ chart is used with the EWMA$_S$ chart, the two charts can be plotted together so that they have the appearance of a single two-sided chart.

The EWMA$_S$ chart is the logical control chart to use under the assumption of normality. However, when this assumption is violated, the convergence rate to normality for the EWMA$_S$ statistic, which is a weighted average of squared deviations, is notably slower than that for a statistic, such as the EWMA$_Y$ statistic, which is a weighted average of observations that are in original units. As an alternative to the EWMA$_S$ chart, when normality is doubtful, an upper one-sided EWMA chart for detecting increases in σ can be defined using the *absolute deviations from target*. The EWMA control statistic for this chart can be expressed as

$$V_k = (1-\lambda)\max\{V_{k-1}, \sqrt{2/\pi}\sigma_0\} + \lambda|X_k - \mu_0|, \quad k = 1, 2, \ldots,$$

where $0 < \lambda \leq 1$ is the smoothing parameter, and the starting value would usually be taken to be $V_0 = \sqrt{2/\pi}\sigma_0$. A signal is given if V_k exceeds a control limit constructed at

$$\sqrt{2/\pi}\sigma_0 + h_V \sqrt{1-(2/\pi)}\sigma_0 \sqrt{\lambda/(2-\lambda)}.$$

Note that when the process is in control, $\sqrt{1-(2/\pi)}\sigma_0$ is the standard deviation of $|X_k - \mu_0|$, but $\sqrt{1-(2/\pi)}\sigma_0\sqrt{\lambda/(2-\lambda)}$ is not the asymptotic standard deviation of V_k because the reset is used. However, the term $\sqrt{1-(2/\pi)}\sigma_0\sqrt{\lambda/(2-\lambda)}$ is used to define the control limit so that the form of the limit for V_k corresponds to the forms for Y_k, S_k, and W_k above. The EWMA chart that is based on the control statistic V_k will be referred to as the EWMA$_V$ chart.

The standard method for monitoring μ and σ, which is to use the Shewhart X chart to detect changes in μ and the Shewhart MR chart to detect changes is σ, is quite ineffective. There are many combinations of Shewhart and EWMA control charts that offer significant gains in performance for the problem of monitoring μ and σ. Various combinations of these charts will be evaluated for the case in which the effect of a special cause is to produce a drift in μ and/or σ, following the discussions on measures of performance and methods for their evaluation in the next two sections.

3. STATISTICAL MEASURES OF PERFORMANCE

Different control charts for process monitoring can be evaluated by comparing the expected amount of time that each one requires to detect various process changes. A fair comparison among charts can be made when all of the charts have the same expected frequency of false alarms and the same in-control average sampling rate. Thus, it is necessary to develop measures for the expected detection delay, the expected frequency of false alarms, and the in-control average sampling rate.

Define the *average time to signal* (ATS) to be the expected time from the start of process monitoring until a signal is given. Define the *average number of observations to signal* (ANOS) to be the expected number of observations from the start of process monitoring until a signal is given. When a VSI control chart is being used, the sampling intervals are not constant, so it is not possible to determine the ATS directly from the ANOS, or vice versa. The average false-alarm rate per unit time can be expressed as 1/ATS, where the ATS is computed when the process is in a state of statistical control. The average sampling rate per unit time is given by the ratio ANOS/ATS.

For example, if for a Shewhart X chart with $h_X = 3$, one observation is taken every 2 hours, then for this chart, the in-control ANOS would be 370.4, the in-control ATS 740.8 hours, and the false-alarm rate one every 740.8 hours or $1/740.8 = 0.00135$ per hour. The average in-control sampling rate would be $\mathrm{ANOS}/\mathrm{ATS} = 370.4/740.8 = 0.5$ observations per hour.

The ATS is a measure of the time required to detect a parameter change when this change is present at the time that monitoring starts. However, a process change may occur at some random time in the future after monitoring has started. In this case, the appropriate measure of detection time is the expected time from the change to the signal by the control chart. For EWMA charts, this expected time will depend on the value of the control statistic at the time that the parameter change occurs. When the change occurs some time after monitoring has started, the effect of the value of the control statistic at the time of the shift can be modeled by using the *steady-state* ATS (SSATS). The SSATS is computed assuming that the control statistic has reached its steady-state or stationary distribution by the time that the change occurs. The SSATS also allows for the possibility that the change can occur within a sampling interval between successive observations. As in past work (see, for example, Reynolds [24], Stoumbos and Reynolds [25], and Stoumbos, Mittenthal, and Runger [26]), it will be assumed that when a process change occurs in a given sampling interval, the position of the change within this interval is uniformly distributed over the interval. Methods for evaluating the ANOS, ATS, and SSATS for the control charts investigated here are discussed in the next section.

4. METHODS FOR EVALUATING THE PERFORMANCE MEASURES

For an individual Shewhart X chart, the ANOS, ATS, and SSATS can simply be expressed in terms of probabilities involving the normal distribution. For more complex charts, such as the EWMA charts discussed in Section 2, methods that are based on modeling the control statistic as a Markov process, such as the *integral equation method* or the *Markov chain method* can usually be used. The Markov chain method offers flexibility for evaluating certain performance measures that are impossible to compute with the integral equation method. When applicable, however, the integral equation method is usually preferred to the Markov chain method because it provides higher accuracy for the same computational effort.

When two control charts, such as the EWMA_Y and EWMA_S charts, are implemented concurrently, it can be very difficult or impossible to apply the integral equation or even the Markov chain method. Thus, we resorted to using simulation for some of the control schemes that were based on combinations of multiple charts. In general, for all control schemes investigated in this paper other than the Shewhart X charts, either the integral equation method, the Markov chain method, or simulation (using 100,000 runs) was appropriately used to evaluate their ANOS, ATS, and SSATS, in order to ensure high accuracy of the results (see Reynolds and Stoumbos [8] for a more detailed discussion on the choice of method for evaluating the ANOS, ATS, and SSATS of various individuals control charts). The standard deviation of the time to signal is approximately equal to the ATS (see Reynolds [24] and Stoumbos and Reynolds [25]), so the standard deviation of the simulated results is approximately equal to $\mathrm{ATS}/\sqrt{100,000}$. Thus, the simulation results presented here are very accurate for meaningful comparisons.

The evaluation of the SSATS of control charts depends on the assumption made regarding the behavior of the control statistic before the process goes out of control. In particular, the distribution of the state of the Markov process at the sample immediately before the drift starts must be identified. In using the Markov process methods, it was assumed that the distribution of the control statistic immediately before the drift starts is the stationary distribution conditional on no false alarms (see, for example, Reynolds [27], Stoumbos and Reynolds ([28,29]), or Reynolds and Arnold [30]). For the cases in which simulation was used, 100 initial in-control observations were generated before the parameter drift was initiated. If a false alarm occurred in these initial observations, this sequence of observations was discarded and a new sequence was generated.

The evaluation of the SSATS also depends on the assumption made about the time that the drift starts. Analogous to previous work, it was assumed here that the drift can start anywhere within a sampling interval, and the distribution of the starting point within an interval is uniformly distributed over the interval (see Reynolds and Stoumbos [8] and numerous references therein). When simulation was used, for each run, a starting point was chosen randomly within the following sampling interval, after the initial 100 in-control observations were generated.

In the following two sections, the performances of individuals control schemes for monitoring μ and σ are compared when the effect of a special cause is to produce an upward or a downward drift in μ and/or σ. The VSI feature and its impact on the performances of these control schemes for the problem of detecting process drifts are investigated in Sections 7 and 8.

5. DETECTING DRIFTS IN *M* AND/OR UPWARD DRIFTS IN σ

A parameter drift is typically the result of a special cause that progressively affects the quality characteristics of a product and causes the values of the control statistic that are plotted on a control chart to gradually move up or down, systematically in the same direction. As discussed in the Introduction, most all evaluations of the performance of control charts in the SPC literature have investigated parameter shifts, and only very few papers have considered parameter drifts. Moreover, to our knowledge, no papers have considered the control problem of simultaneously monitoring μ and σ for potential drifts. This problem is important from a practical viewpoint (see Stoumbos et al., [31], Woodall and Montgomery [32], and references therein) and will be investigated next.

The in-control ANOS of the control charts being compared here was taken to be 370.4, the value corresponding to the Shewhart X chart with the standard “three-sigma” limits ($h_X = 3.0$). When combinations of charts are being considered, the control limits have been adjusted so that all of the charts in the combination have the same individual in-control ANOS, and the joint in-control ANOS is 370.4. For simplicity in explaining the comparisons that are presented, we take the sampling interval, d, of the traditional FSI control charts to be $d = 1$ hour and measure time in hours. This means that the in-control ATS is 370.4 hours. The numerical ATS and SSATS values given here apply to other cases, if the unit of time is taken to be the length of the time interval between successive observations taken for the FSI charts.

The drift in μ and/or σ is defined as follows. Let $\mu(t)$ be the value of μ at a time point t hours after a special cause starts a drift in the process. It is assumed that the drift is at a constant rate so that

$$\mu(t) = \mu_0 + r_\mu \sigma_0 t,$$

where r_μ is the rate of drift per hour in units of σ_0. Similarly, let $\sigma(t)$ be the value of σ at a time point t hours after the start of the drift, and assume that

$$\sigma(t) = \sigma_0 + r_\sigma \sigma_0 t,$$

where r_σ is the rate of drift per hour in units of σ_0. The performance of the control charts considered here depends on $\mu(t)$ only through $|r_\mu|$, so the direction of the drift in μ does not matter. However, for drifts in σ, the performance for upward drifts is different than performance for downward drifts. When r_σ is negative, the above formula for $\sigma(t)$ implies that $\sigma(t)$ will eventually become negative if t is large enough. The results presented here for the case of $r_\sigma < 0$ are based on bounding $\sigma(t)$ so that $\sigma(t) \geq 0.1\sigma_0$ holds for all t.

Column 3 of Table 1 provides ATS and SSATS values for the X chart with three-sigma limits. The ATS values are given for the in-control case, and the SSATS values are given for various drift rates represented by positive values of r_μ and/or r_σ. The in-control case in Table 1 corresponds to the row with $r_\mu = r_\sigma = 0$. The next nine rows of Table 1 correspond to drifts in μ, and the following nine rows to upward drifts in σ. The remaining nine rows correspond to various drifts in both μ and σ.

Column 4 of Table 1 provides ATS and SSATS values for the X chart and MR chart combination. The next two columns give ATS and SSATS values for the X chart and EWMA_Y chart combination, where the EWMA_Y chart has $\lambda = 0.1$ or 0.2. The next two columns give ATS and SSATS values for the combination of the EWMA_Y and EWMA_S charts, where both charts have $\lambda = 0.1$ or both have $\lambda = 0.2$. The last three columns give ATS and SSATS values for the EWMA_Y and EWMA_V chart combination, where both charts have $\lambda = 0.05$, 0.1, or 0.2. The small value of $\lambda = 0.05$ was included for the pair of EWMA_Y and EWMA_V charts, because the EWMA_V chart is especially recommended when robustness to non-normal distributions is required, and smaller values of λ make an EWMA chart more robust (see Stoumbos and Reynolds [9]). Note that if two or more EWMA charts are used concurrently, it is not necessary that they use the same value of λ, but, for simplicity, the same value of λ was used in each of the combined EWMA charts being compared here.

Compared to the X chart alone, combining the X chart with the MR chart provides a slight reduction in the expected time required to detect upward drifts in σ, but there is an increase in the time required to detect drifts in μ. Thus, the conclusion for drifts is the same as the conclusion for shifts reached in other

Table 1. ATS ($r_\mu = r_\sigma = 0$) and SSATS ($r_\mu > 0$ and/or $r_\sigma > 0$) Values with Parameter Values for Matched FSI Shewhart Charts and FSI EWMA Charts

Drift Rates				X Chart & EWMA$_Y$		EWMA$_Y$ & EWMA$_S$		EWMA$_Y$ & EWMA$_V$		
r_μ	r_σ	X Chart	X Chart & MR Chart	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.2$
0.000	0.000	372.0	370.8	370.8	371.2	370.7	370.3	371.0	370.0	370.4
0.010	0.000	89.6	98.8	53.2	58.4	52.9	58.2	51.0	53.1	58.3
0.025	0.000	49.2	54.0	30.4	32.3	30.4	32.2	30.4	30.4	32.2
0.050	0.000	30.5	33.1	20.0	20.6	20.0	20.6	20.5	20.0	20.5
0.100	0.000	18.4	19.8	13.3	13.2	13.2	13.2	14.0	13.3	13.2
0.250	0.000	9.3	9.9	7.8	7.4	7.7	7.4	8.3	7.8	7.4
0.500	0.000	5.5	5.8	5.1	4.9	5.0	4.8	5.5	5.1	4.9
1.000	0.000	3.3	3.4	3.3	3.2	3.2	3.1	3.7	3.4	3.2
2.000	0.000	2.0	2.0	2.0	2.0	2.1	2.0	2.5	2.3	2.1
4.000	0.000	1.2	1.3	1.3	1.3	1.4	1.3	1.7	1.5	1.4
0.000	0.010	44.0	42.8	45.3	44.3	39.2	40.5	39.4	39.4	40.1
0.000	0.025	25.7	25.1	26.6	25.9	23.2	23.8	23.9	23.5	23.6
0.000	0.050	17.0	16.6	17.5	17.2	15.6	15.8	16.5	16.0	15.8
0.000	0.100	11.2	11.0	11.6	11.3	10.6	10.5	11.5	10.9	10.6
0.000	0.250	6.5	6.4	6.8	6.6	6.4	6.3	7.2	6.7	6.4
0.000	0.500	4.4	4.3	4.5	4.5	4.4	4.3	5.1	4.7	4.4
0.000	1.000	3.0	3.0	3.1	3.1	3.1	3.0	3.6	3.3	3.1
0.000	2.000	2.1	2.1	2.2	2.2	2.2	2.1	2.6	2.4	2.2
0.000	4.000	1.5	1.5	1.6	1.6	1.6	1.6	1.9	1.8	1.6
0.050	0.050	15.4	15.4	15.0	14.7	13.9	14.0	14.9	14.3	14.0

0.050	0.250	6.5	6.4	6.7	6.6	6.3	6.2	7.1	6.7	6.4
0.050	1.000	3.0	3.0	3.1	3.1	3.1	3.0	3.6	3.4	3.1
0.250	0.050	8.2	8.6	7.5	7.2	7.4	7.1	8.0	7.5	7.2
0.250	0.250	5.7	5.7	5.8	5.7	5.6	5.5	6.3	5.9	5.6
0.250	1.000	3.0	3.0	3.1	3.0	3.0	3.0	3.6	3.3	3.1
1.000	0.050	3.2	3.4	3.3	3.1	3.2	3.1	3.7	3.4	3.2
1.000	0.250	3.1	3.2	3.2	3.1	3.2	3.1	3.6	3.4	3.2
1.000	1.000	2.6	2.6	2.7	2.6	2.7	2.6	3.2	2.9	2.7
h_X		3.000	3.161	3.190	3.183	—	—	—	—	—
h_R^+		—	4.445	—	—	—	—	—	—	—
h_Y		—	—	2.938	3.067	2.924	3.053	2.738	2.921	3.049
h_S		—	—	—	—	3.883	4.617	—	—	—
h_V		—	—	—	—	—	—	2.902	3.199	3.477

papers: the slight benefit gained from combining the MR chart with the X chart seems to be outweighed by the disadvantage just mentioned. If someone wants to use a Shewhart chart, it is generally better to use the X chart alone, without the MR chart.

Combining the EWMA$_Y$ chart with the X chart (see columns 5 and 6) improves the ability to detect slow drifts in μ, but slightly increases the time required to detect upward drifts in σ. Improved performance for slow drifts in either μ or σ can be obtained by pairing up the EWMA$_Y$ and EWMA$_S$ charts. That is, the overall best pair of control charts for detecting drifts in μ and/or σ for normally distributed process data appears to be the combination of the EWMA$_Y$ and EWMA$_S$ charts. The combination of the X chart with the EWMA$_Y$ chart may be slightly simpler to plot and interpret, but it involves a modest loss of SSATS performance compared to the best alternative under consideration. The combination of the EWMA$_Y$ and EWMA$_V$ charts has been recently shown to be quite robust to non-normal distributions (see Stoumbos and Reynolds [9]), but there is a slight penalty in performance for normally distributed process data. In the next section, the performances of individuals control schemes for monitoring μ and σ are compared when a special cause can produce a downward drift in σ.

6. DETECTING DOWNWARD DRIFTS IN σ

Reynolds and Stoumbos [8] investigated the performance of the combination of the EWMA$_Y$, EWMA$_S$, and EWMA$_W$ charts for shifts and found that a small decrease in σ is harder to detect if it is also accompanied by a small change in μ. They found that choosing a small value of λ in the charts improves the ability to detect this situation. Thus, for this combination of charts, we also include a relatively small value of λ.

Table 2 provides SSATS values for the combination of the EWMA$_Y$, EWMA$_S$, and EWMA$_W$ charts for $\lambda = 0.05$, 0.1, and 0.2, and for various drifts in μ and/or σ. The first row of values in Table 2 consists of ATS values corresponding to the in-control case.

The next 18 rows in Table 2 give SSATS values for drifts in μ or upward drifts in σ, corresponding to the respective rows in Table 1. Comparing the SSATS values in these rows to the corresponding values for the combination of the EWMA$_Y$ and EWMA$_S$ charts in Table 1 shows that adding the EWMA$_W$ chart produces a small increase in the time required by these latter two charts to detect drifts in μ and/or upward drifts in σ. This occurs, of course, because to maintain a joint average false-alarm rate of one in 370.4, while adding the EWMA$_W$ chart to the combination of the EWMA$_Y$ and EWMA$_S$ charts, requires a slight increase in the control limits of these latter two charts.

Table 2. ATS ($r_\mu = r_\sigma = 0$) and SSATS ($r_\mu > 0$ and/or $r_\sigma \neq 0$) Values with Parameter Values for Matched FSI Shewhart Charts and FSI EWMA Charts

Drift Rates		EWMA$_Y$ & EWMA$_S$ & EWMA$_W$			
r_μ	r_σ	$\lambda = 0.05$	$\lambda = 0.1$	$\lambda = 0.2$	X Chart & Two-Sided MR Chart
0.000	0.000	370.0	371.2	369.3	370.1
0.010	0.000	53.1	55.9	62.0	98.8
0.025	0.000	31.5	31.6	34.0	54.2
0.050	0.000	21.3	20.7	21.5	33.2
0.100	0.000	14.3	13.7	13.7	19.9
0.250	0.000	8.4	8.0	7.7	9.9
0.500	0.000	5.4	5.2	5.0	5.8
1.000	0.000	3.5	3.3	3.2	3.4
2.000	0.000	2.2	2.1	2.1	2.1
4.000	0.000	1.4	1.4	1.3	1.3
0.000	0.010	40.7	42.1	44.1	45.7
0.000	0.025	24.3	24.6	25.5	26.4
0.000	0.050	16.4	16.4	16.7	17.4
0.000	0.100	11.2	11.0	11.1	11.4
0.000	0.250	6.8	6.6	6.5	6.6
0.000	0.500	4.7	4.6	4.5	4.5
0.000	1.000	3.3	3.2	3.1	3.0
0.000	2.000	2.3	2.3	2.2	2.1
0.000	4.000	1.7	1.6	1.6	1.6
0.000	−0.010	41.0	42.2	45.7	151.5
0.000	−0.025	24.4	23.6	24.1	87.8
0.000	−0.050	17.3	15.8	15.1	52.5
0.000	−0.100	13.5	11.5	10.2	30.2
0.025	−0.010	32.2	33.4	38.5	104.3
0.025	−0.025	30.8	33.5	43.1	69.6
0.025	−0.050	22.0	23.1	35.5	47.3
0.025	−0.100	14.9	12.6	11.9	29.2
0.050	−0.010	21.6	21.2	22.5	45.5
0.050	−0.025	21.7	21.6	23.9	53.3
0.050	−0.050	21.7	21.9	25.0	39.7
0.050	−0.100	20.9	21.3	24.6	27.0
h_Y		2.907	3.078	3.198	$h_X = 3.176$
h_S		3.565	4.205	5.012	$h_R^+ = 4.760$
h_W		2.159	1.990	1.697	$h_R^- = 0.001$

Table 2 also provides SSATS values for the combination of the X chart with the two-sided MR chart. The two-sided MR chart has incorporated the lower control limit $h_R^- \sigma_0$ in order to detect downward drifts in σ. As in the case of the EWMA charts, modifying the MR chart to detect downward drifts in σ results in slightly worse SSATS performance for drifts in μ and/or upward drifts in σ.

Rows 20–23 in Table 2 correspond to downward drifts in σ while μ is constant. The remaining 8 rows correspond to downward drifts in σ when accompanied by a drift in μ. A downward drift in σ will tend to mask a small change in μ and make this change more difficult to detect. For example, when $\lambda = 0.2$, a drift in μ of 0.05 per hour takes an average of 21.5 hours to detect when σ is constant. However, if this drift in μ is accompanied by a downward drift in σ at the rate of 0.05 per hour, then an average of 25.0 hours is required. A drift in μ will also make a downward drift in σ harder to detect. For example, if σ is decreasing at a rate of 0.10 per hour and μ remains constant, then an average of 10.2 hours is required to detect this downward drift in σ when $\lambda = 0.2$. However, if this drift in σ is accompanied by a drift in μ at a rate of 0.05 per hour, then an average of 24.6 hours is required. Choosing a small value of λ in the EWMA_Y and EWMA_S charts can reduce the masking effect of a decrease in σ.

Examining the SSATS values for downward drifts shows that the combination of the X chart with the two-sided MR chart is not effective at detecting downward drifts in σ. For this combination, the smallest SSATS value in the last 12 rows is 27.0 hours.

The overall conclusion that can be drawn from Table 2 is that decreases in σ can be detected by adding the EWMA_W chart to the pair of the EWMA_Y and EWMA_S charts. Assuming that a constant average false-alarm rate is maintained, adding the ability to detect decreases in σ comes at the expense of small increases in the time required to detect other process changes. The VSI feature and its effect on the performances of the combined X and MR charts and the combined EWMA_Y and EWMA_S charts for the problem of detecting drifts in μ and/or σ are investigated in the next two sections.

7. ADDING THE VSI FEATURE TO IMPROVE PERFORMANCE

Adding the VSI feature to a control chart generally results in significant reductions in the detection delay of small and moderate-size shifts in process parameters. Minimal work has been done on VSI charts for simultaneously monitoring μ and σ. Chengular, Arnold, and Reynolds [33] considered VSI Shewhart charts for monitoring μ and σ when samples of $n > 1$ observations are taken. Shamma and Amin [20] considered the performance of a single VSI EWMA chart that can be used to monitor μ and σ. Reynolds and Stoumbos [8] investigated the application of the VSI feature to the combined X and EWMA_Y charts and the combined EWMA_Y and EWMA_S charts, when the objective is to detect shifts in μ and/or σ. These two chart combinations will be considered here for the problem of detecting drifts in μ and/or σ. The VSI feature will be discussed first for each individual control chart, and then for combinations of charts.

The basic idea of the VSI feature is to use a short sampling interval if there is some indication of a possible process change, and a long sampling interval if there is no such indication. This means that a short sampling interval should be used next if the current value of the control statistic is close to a control limit, and a long sampling interval should be used otherwise. Previous work on VSI control charts for detecting shifts has shown that it is sufficient to use only two possible values for the sampling intervals (see Stoumbos, Mittenthal, and Runger [26] and Reynolds [34]). Let these possible values be denoted as d_1 and d_2, where $0 < d_1 \leq d_2$. For example, if the current FSI control chart is sampling every hour, a VSI chart for the same application might use a short sampling interval of $d_1 = 10$ minutes and a long one of $d_2 = 1.5$ hours.

Previous research has demonstrated that d_1 should be chosen as small as possible to detect shifts most effectively, so the choice of d_1 would usually depend on how soon it is feasible to sample again after the current observation is taken from the process. The long sampling interval d_2 could be chosen so that the resulting control chart will have an acceptable average sampling rate and an acceptable ability to detect changes in the process parameters being monitored.

To implement the VSI feature in the Shewhart X chart, an additional set of limits inside of the control limits can be used that will determine which sampling interval to use next. In particular, the long sampling interval d_2 would be used after observation k if X_k falls inside of limits constructed at

$$\mu_0 \pm g_X \sigma_0,$$

where $0 \leq g_X \leq h_X$. The short sampling interval d_1 would be used if X_k falls outside of these limits, but inside the control limits $\mu_0 \pm h_X \sigma_0$. If X_k falls outside of the control limits $\mu_0 \pm h_X \sigma_0$, then a signal would be given as in the case of a usual FSI X chart.

The VSI feature can be applied to the EWMA_Y chart by adding an extra set of limits inside of the control limits, as was done for the VSI X chart. In particular, the long sampling interval d_2 would be used after observation k when the EWMA statistic Y_k falls inside of the limits constructed at

$$\mu_0 \pm g_Y \sigma_0 \sqrt{\lambda/(2-\lambda)},$$

where $0 \leq g_Y \leq h_Y$. The short sampling interval d_1 is used if Y_k falls outside of these limits, but inside of the regular control limits.

The VSI feature can be added to the EWMA_S chart by introducing a limit

$$\sigma_0^2 + g_S\sigma_0^2\sqrt{2\lambda/(2-\lambda)},$$

where $0 \le g_S \le h_S$. The long sampling interval d_2 would be used next if S_k is below this limit, and the short d_1 would be used if S_k is between this limit and the chart control limit.

When two or more VSI control charts are used in combination, each control chart will specify a sampling interval to use next, but the individual charts may specify different sampling intervals. A reasonable decision rule to employ in this case is to use the short sampling interval d_1 if any of the control charts specifies d_1, and otherwise use the long sampling interval d_2.

8. THE PERFORMANCE OF VSI CONTROL CHARTS

For the evaluation of the performances of the VSI control charts considered here, the sampling intervals and limits for the VSI charts were chosen to give a joint in-control ANOS of approximately 370.4 and a joint in-control ATS of approximately 370.4 hours. This means that the average in-control sampling rate of the VSI charts is very close to one observation per hour, and their average false-alarm rate is very close to 1 per 370.4 hours. The limits of the VSI control charts that were paired to form chart combinations were chosen so that the two VSI charts would have the same individual in-control ANOS values and the same individual in-control ATS values.

Table 3 provides ATS and SSATS values for the combination of the VSI X and VSI EWMA_Y charts and the combination of the VSI EWMA_Y and VSI EWMA_S charts. In this table, $d_1 = 0.1$ hours, $\lambda = 0.1$, and $d_2 = 1.25$, 1.50, or 1.90 hours. The control limit constants h_X, h_Y, and h_S are the same as in Table 1 and yield an in-control ANOS of approximately 370.4 as discussed above. For each pair of sampling intervals (d_1, d_2), the limits g_X, g_Y, and g_S were selected to produce an in-control ATS of approximately 370.4 hours. The drifts in μ and σ considered in Table 3 are the same as those in Tables 1 and 2. The ATS and SSATS values for the four FSI schemes given in columns 3–6 serve as a basis of comparison. Table 4 is the same as Table 3, except that $\lambda = 0.2$. Tables 5 and 6 are the same as Tables 3 and 4, respectively, except that $d_1 = 0.25$ hours.

The results in Tables 3–6 show that the VSI charts have notably lower expected detection times for slow-rate and moderate-rate drifts in μ and/or σ compared to the corresponding FSI charts. The combined VSI X and VSI EWMA_Y charts have about the same SSATS values as the combined VSI EWMA_Y and EWMA_S charts for drifts in μ, but the two VSI EWMA charts are generally better for drifts in σ, particularly when μ remains near μ_0.

As expected from previous research on VSI control charts, using $d_1 = 0.1$ hours is better than $d_1 = 0.25$ hours. In general, d_1 should be as small as practically feasible. In some applications, it may not be feasible to have a very small value of d_1, but a VSI control chart with $d_1 = 0.25$ hours still offers a notable performance advantage over an FSI control chart. Also, the results in Tables 3–6 demonstrate that a relatively large value of d_2 gives slightly better performance for detecting slow drifts in μ and/or σ, while a relatively small value of d_2 gives slightly better performance for fast drifts.

9. CONCLUDING REMARKS

This paper has demonstrated that the traditional approach of using the Shewhart X chart together with the Shewhart MR chart to monitor μ and σ is ineffective for detecting slow-rate and moderate-rate drifts in these parameters. If a Shewhart chart is to be used, using the X chart alone gives better overall performance. These conclusions agree with previous research that considered the problem of detecting shifts in μ and σ.

Better alternatives for monitoring μ and σ are combinations of control charts that involve at least one EWMA chart. The combination of the X chart and EWMA_Y chart will offer faster detection of slow drifts in μ. This combination has the advantage that the two charts can be superimposed on the same plot.

A slightly better combination is that of the EWMA_Y and EWMA_S charts. This combination is just as fast as the combined X and EWMA_Y charts at detecting drifts in μ and is a little faster at detecting upward drifts in σ. When it is desirable to have a control chart for the detection of downward drifts in σ, the EWMA_W chart can be added to the EWMA_Y and EWMA_S charts. For a given average false-alarm rate, adding the EWMA_W chart for the detection of downward drifts in σ results in slight increases in the expected time required to detect other types of process drifts.

Control charts based on individual observations are very sensitive to the assumed normal distribution used to determine the control limits. When it is desirable to have charts that are more robust to non-normal distributions, the EWMA_S chart can be replaced with the EWMA_V chart (see also Stoumbos and Reynolds [9]).

Adding the VSI feature to the control charts considerably reduces the detection delay of slow-rate and moderate-rate drifts in either μ or σ. The VSI feature is especially effective when added to the combination of the EWMA_Y and EWMA_S charts. As a point of interest, Baxley [35], of Monsanto Fibers Business, recently presented an interesting application of a VSI EWMA chart, based on subgroups ($n > 1$), for monitoring the mean of a property of nylon fiber. Baxley,

Table 3. ATS ($r_\mu = r_\sigma = 0$) and SSATS ($r_\mu > 0$ and/or $r_\sigma > 0$) Values with Parameter Values for Matched FSI and VSI Shewhart Charts and FSI and VSI EWMA Charts with $\lambda = 0.1$

Drift Rates		FSI Control Charts				VSI Control Charts with $d_1 = 0.1$					
r_μ	r_σ	X Chart	X Chart & MR Chart	X Chart & $EWMA_Y$	$EWMA_Y$ & $EWMA_S$	X Chart & $EWMA_Y$			$EWMA_Y$ & $EWMA_S$		
0.000	0.000	372.0	370.8	370.8	370.7	370.5	371.1	370.0	370.7	370.4	370.7
0.010	0.000	89.6	98.8	53.2	52.9	47.1	45.9	45.0	47.0	45.7	44.9
0.025	0.000	49.2	54.0	30.4	30.4	25.8	25.0	24.5	25.7	25.0	24.6
0.050	0.000	30.5	33.1	20.0	20.0	16.4	15.9	15.5	16.4	15.9	15.7
0.100	0.000	18.4	19.8	13.3	13.2	10.5	10.2	10.0	10.6	10.3	10.2
0.250	0.000	9.3	9.9	7.8	7.7	6.0	5.8	5.7	6.0	5.8	5.8
0.500	0.000	5.5	5.8	5.1	5.0	3.9	3.7	3.7	3.9	3.8	3.7
1.000	0.000	3.3	3.4	3.3	3.2	2.5	2.4	2.4	2.5	2.4	2.4
2.000	0.000	2.0	2.0	2.0	2.1	1.6	1.6	1.7	1.6	1.6	1.7
4.000	0.000	1.2	1.3	1.3	1.4	1.1	1.2	1.4	1.1	1.2	1.3
0.000	0.010	44.0	42.8	45.3	39.2	42.0	41.3	40.7	34.3	33.1	32.3
0.000	0.025	25.7	25.1	26.6	23.2	24.3	23.8	23.5	19.7	19.0	18.6
0.000	0.050	17.0	16.6	17.5	15.6	15.8	15.5	15.4	13.0	12.5	12.3
0.000	0.100	11.2	11.0	11.6	10.6	10.3	10.1	10.1	8.6	8.3	8.2
0.000	0.250	6.5	6.4	6.8	6.4	5.9	5.8	5.8	5.1	5.0	4.9
0.000	0.500	4.4	4.3	4.5	4.4	4.0	3.9	4.0	3.5	3.4	3.4
0.000	1.000	3.0	3.0	3.1	3.1	2.7	2.7	2.8	2.5	2.4	2.4
0.000	2.000	2.1	2.1	2.2	2.2	1.9	1.9	2.0	1.8	1.8	1.8
0.000	4.000	1.5	1.5	1.6	1.6	1.4	1.5	1.6	1.4	1.4	1.4

0.050	0.050	15.4	15.4	15.0	13.9	13.1	12.8	12.6	11.5	11.1	11.0
0.050	0.250	6.5	6.4	6.7	6.3	5.9	5.8	5.8	5.1	4.9	4.9
0.050	1.000	3.0	3.0	3.1	3.1	2.7	2.7	2.8	2.5	2.4	2.4
0.250	0.050	8.2	8.6	7.5	7.4	5.9	5.8	5.7	5.8	5.7	5.6
0.250	0.250	5.7	5.7	5.8	5.6	5.0	4.9	4.9	4.5	4.4	4.4
0.250	1.000	3.0	3.0	3.1	3.0	2.7	2.7	2.7	2.5	2.4	2.4
1.000	0.050	3.2	3.4	3.3	3.2	2.5	2.5	2.5	2.5	2.5	2.5
1.000	0.250	3.1	3.2	3.2	3.2	2.6	2.5	2.6	2.5	2.5	2.5
1.000	1.000	2.6	2.6	2.7	2.7	2.3	2.3	2.4	2.2	2.2	2.2
d_2		—	—	—	—	1.25	1.5	1.9	1.25	1.5	1.9
h_X		3.000	3.161	3.190	—	3.190	3.190	3.190	—	—	—
h_R^+		—	4.445	—	—	—	—	—	—	—	—
h_Y		—	—	2.938	2.924	2.938	2.938	2.938	2.924	2.924	2.924
h_S		—	—	—	3.883	—	—	—	3.883	3.883	3.883
g_X		—	—	—	—	1.537	1.247	1.010	—	—	—
g_Y		—	—	—	—	1.493	1.216	0.986	1.477	1.192	0.960
g_S		—	—	—	—	—	—	—	1.370	0.971	0.665

Table 4. ATS ($r_\mu = r_\sigma = 0$) and SSATS ($r_\mu > 0$ and/or $r_\sigma > 0$) Values with Parameter Values for Matched FSI and VSI Shewhart Charts and FSI and VSI EWMA Charts with $\lambda = 0.1$

Drift Rates		FSI Control Charts				VSI Control Charts with $d_1 = 0.25$					
r_μ	r_σ	X Chart	X Chart & MR Chart	X Chart & EWMA$_Y$	EWMA$_Y$ &$_S$ EWMA$_S$	X Chart & EWMA$_Y$			EWMA$_Y$ & EWMA$_S$		
0.000	0.000	372.0	370.8	370.8	370.7	371.1	370.6	370.3	370.3	370.4	370.4
0.010	0.000	89.6	98.8	53.2	52.9	47.9	46.9	46.2	47.7	46.6	46.1
0.025	0.000	49.2	54.0	30.4	30.4	26.4	25.8	25.5	26.3	25.8	25.5
0.050	0.000	30.5	33.1	20.0	20.0	16.9	16.5	16.3	16.9	16.5	16.4
0.100	0.000	18.4	19.8	13.3	13.2	11.0	10.7	10.6	11.0	10.7	10.6
0.250	0.000	9.3	9.9	7.8	7.7	6.3	6.1	6.1	6.3	6.1	6.1
0.500	0.000	5.5	5.8	5.1	5.0	4.1	4.0	4.0	4.1	4.0	4.0
1.000	0.000	3.3	3.4	3.3	3.2	2.6	2.6	2.6	2.6	2.6	2.6
2.000	0.000	2.0	2.0	2.0	2.1	1.7	1.7	1.8	1.7	1.7	1.7
4.000	0.000	1.2	1.3	1.3	1.4	1.1	1.2	1.3	1.1	1.2	1.2
0.000	0.010	44.0	42.8	45.3	39.2	42.4	41.7	41.3	34.8	33.8	33.1
0.000	0.025	25.7	25.1	26.6	23.2	24.5	24.1	23.9	20.1	19.5	19.1
0.000	0.050	17.0	16.6	17.5	15.6	16.0	15.8	15.6	13.3	12.9	12.7
0.000	0.100	11.2	11.0	11.6	10.6	10.5	10.3	10.2	8.8	8.6	8.5
0.000	0.250	6.5	6.4	6.8	6.4	6.0	5.9	5.9	5.3	5.1	5.1
0.000	0.500	4.4	4.3	4.5	4.4	4.0	4.0	4.0	3.6	3.5	3.5
0.000	1.000	3.0	3.0	3.1	3.1	2.7	2.7	2.8	2.5	2.5	2.5
0.000	2.000	2.1	2.1	2.2	2.2	1.9	1.9	2.0	1.8	1.8	1.8
0.000	4.000	1.5	1.5	1.6	1.6	1.4	1.4	1.5	1.4	1.4	1.4

0.050	0.050	15.4	15.4	15.0	13.9	13.3	13.0	12.9	11.8	11.5	11.3
0.050	0.250	6.5	6.4	6.7	6.3	6.0	5.9	5.9	5.2	5.1	5.0
0.050	1.000	3.0	3.0	3.1	3.1	2.7	2.7	2.8	2.6	2.5	2.5
0.250	0.050	8.2	8.6	7.5	7.4	6.2	6.0	6.0	6.1	5.9	5.9
0.250	0.250	5.7	5.7	5.8	5.6	5.1	5.0	5.0	4.6	4.5	4.5
0.250	1.000	3.0	3.0	3.1	3.0	2.7	2.7	2.7	2.5	2.5	2.5
1.000	0.050	3.2	3.4	3.3	3.2	2.6	2.6	2.6	2.6	2.6	2.6
1.000	0.250	3.1	3.2	3.2	3.2	2.7	2.6	2.7	2.6	2.6	2.6
1.000	1.000	2.6	2.6	2.7	2.7	2.3	2.3	2.4	2.2	2.2	2.2
d_2		—	—	—	—	1.25	1.5	1.9	1.25	1.5	1.9
h_X		3.000	3.161	3.190	—	3.190	3.190	3.190	—	—	—
h_R^+		—	4.445	—	—	—	—	—	—	—	—
h_Y		—	—	2.938	2.924	2.938	2.938	2.938	2.924	2.924	2.924
h_S		—	—	—	3.883	—	—	—	3.883	3.883	3.883
g_X		—	—	—	—	1.463	1.171	0.942	—	—	—
g_Y		—	—	—	—	1.422	1.142	0.920	1.401	1.118	0.891
g_S		—	—	—	—	—	—	—	1.259	0.871	0.577

Table 5. ATS ($r_\mu = r_\sigma = 0$) and SSATS ($r_\mu > 0$ and/or $r_\sigma > 0$) Values with Parameter Values for Matched FSI and VSI Shewhart Charts and FSI and VSI EWMA Charts with $\lambda = 0.2$

Drift Rates		FSI Control Charts				VSI Control Charts with $d_1 = 0.1$					
						X Chart & EWMA$_Y$			EWMA$_Y$ & EWMA$_S$		
r_μ	r_σ	X Chart	X Chart & MR Chart	X Chart & EWMA$_Y$	EWMA$_Y$ &$_S$ EWMA$_S$						
0.000	0.000	372.0	370.8	371.2	370.3	371.5	371.1	370.0	370.4	370.4	370.2
0.010	0.000	89.6	98.8	58.4	58.2	53.3	52.1	51.2	53.0	51.8	51.0
0.025	0.000	49.2	54.0	32.3	32.2	28.4	27.7	27.1	28.3	27.5	27.1
0.050	0.000	30.5	33.1	20.6	20.6	17.5	17.0	16.7	17.4	16.9	16.7
0.100	0.000	18.4	19.8	13.2	13.2	10.8	10.5	10.3	10.8	10.5	10.4
0.250	0.000	9.3	9.9	7.4	7.4	5.8	5.7	5.6	5.9	5.7	5.7
0.500	0.000	5.5	5.8	4.9	4.8	3.7	3.6	3.6	3.8	3.7	3.7
1.000	0.000	3.3	3.4	3.2	3.1	2.4	2.4	2.4	2.4	2.4	2.4
2.000	0.000	2.0	2.0	2.0	2.0	1.6	1.6	1.7	1.6	1.6	1.7
4.000	0.000	1.2	1.3	1.3	1.3	1.1	1.2	1.4	1.1	1.2	1.3
0.000	0.010	44.0	42.8	44.3	40.5	41.1	40.4	39.9	36.3	35.2	34.3
0.000	0.025	25.7	25.1	25.9	23.8	23.8	23.3	22.9	20.7	20.0	19.5
0.000	0.050	17.0	16.6	17.2	15.8	15.5	15.2	15.0	13.5	13.0	12.8
0.000	0.100	11.2	11.0	11.3	10.5	10.1	9.9	9.8	8.8	8.6	8.4
0.000	0.250	6.5	6.4	6.6	6.3	5.8	5.7	5.7	5.2	5.0	5.0
0.000	0.500	4.4	4.3	4.5	4.3	3.9	3.8	3.9	3.5	3.5	3.5
0.000	1.000	3.0	3.0	3.1	3.0	2.7	2.7	2.8	2.5	2.4	2.5
0.000	2.000	2.1	2.1	2.2	2.1	1.9	1.9	2.1	1.8	1.8	1.8
0.000	4.000	1.5	1.5	1.6	1.6	1.4	1.5	1.6	1.3	1.4	1.4

0.050	0.050	15.4	15.4	14.7	14.0	13.0	12.7	12.6	11.9	11.5	11.3
0.050	0.250	6.5	6.4	6.6	6.2	5.8	5.7	5.7	5.1	5.0	5.0
0.050	1.000	3.0	3.0	3.1	3.0	2.7	2.7	2.8	2.5	2.4	2.5
0.250	0.050	8.2	8.6	7.2	7.1	5.8	5.7	5.7	5.8	5.6	5.6
0.250	0.250	5.7	5.7	5.7	5.5	4.9	4.8	4.8	4.5	4.4	4.4
0.250	1.000	3.0	3.0	3.0	3.0	2.6	2.6	2.7	2.4	2.4	2.5
1.000	0.050	3.2	3.4	3.1	3.1	2.4	2.4	2.4	2.4	2.4	2.5
1.000	0.250	3.1	3.2	3.1	3.1	2.5	2.5	2.5	2.5	2.5	2.5
1.000	1.000	2.6	2.6	2.6	2.6	2.3	2.3	2.4	2.2	2.2	2.2
d_2		—	—	—	—	1.25	1.5	1.9	1.25	1.5	1.9
h_X		3.000	3.161	3.183	—	3.183	3.183	3.183	—	—	—
h_R^+		—	4.445	—	—	—	—	—	—	—	—
h_Y		—	—	3.067	3.053	3.067	3.067	3.067	3.053	3.053	3.053
h_S		—	—	—	4.617	—	—	—	4.617	4.617	4.617
g_X		—	—	—	—	1.510	1.211	0.976	—	—	—
g_Y		—	—	—	—	1.487	1.196	0.964	1.473	1.179	0.940
g_S		—	—	—	—	—	—	—	1.285	0.838	0.504

Table 6. ATS ($r_\mu = r_\sigma = 0$) and SSATS ($r_\mu > 0$ and/or $r_\sigma > 0$) Values with Parameter Values for Matched FSI and VSI Shewhart Charts and FSI and VSI EWMA Charts with $\lambda = 0.2$

Drift Rates		FSI Control Charts				VSI Control Charts with $d_1 = 0.25$					
r_μ	r_σ	X Chart	X Chart & MR Chart	X Chart & EWMA_Y	EWMA_Y & EWMA_S	X Chart & EWMA_Y			EWMA_Y & EWMA_S		
0.000	0.000	372.0	370.8	371.2	370.3	371.1	371.2	371.5	370.3	370.4	370.5
0.010	0.000	89.6	98.8	58.4	58.2	53.9	53.0	52.3	53.6	52.6	51.9
0.025	0.000	49.2	54.0	32.3	32.2	28.9	28.3	27.9	28.8	28.2	27.9
0.050	0.000	30.5	33.1	20.6	20.6	17.9	17.5	17.3	17.8	17.5	17.3
0.100	0.000	18.4	19.8	13.2	13.2	11.2	10.9	10.8	11.1	10.9	10.8
0.250	0.000	9.3	9.9	7.4	7.4	6.1	6.0	6.0	6.1	6.0	6.0
0.500	0.000	5.5	5.8	4.9	4.8	3.9	3.9	3.9	3.9	3.9	3.9
1.000	0.000	3.3	3.4	3.2	3.1	2.5	2.5	2.5	2.5	2.5	2.5
2.000	0.000	2.0	2.0	2.0	2.0	1.6	1.7	1.7	1.7	1.6	1.7
4.000	0.000	1.2	1.3	1.3	1.3	1.1	1.2	1.3	1.1	1.2	1.2
0.000	0.010	44.0	42.8	44.3	40.5	41.4	40.8	40.4	36.7	35.8	35.1
0.000	0.025	25.7	25.1	25.9	23.8	24.0	23.6	23.3	21.1	20.5	20.1
0.000	0.050	17.0	16.6	17.2	15.8	15.7	15.4	15.3	13.8	13.4	13.1
0.000	0.100	11.2	11.0	11.3	10.5	10.3	10.1	10.0	9.0	8.8	8.7
0.000	0.250	6.5	6.4	6.6	6.3	5.9	5.8	5.8	5.3	5.2	5.1
0.000	0.500	4.4	4.3	4.5	4.3	3.9	3.9	3.9	3.6	3.5	3.5
0.000	1.000	3.0	3.0	3.1	3.0	2.7	2.7	2.7	2.5	2.5	2.5
0.000	2.000	2.1	2.1	2.2	2.1	1.9	1.9	2.0	1.8	1.8	1.8
0.000	4.000	1.5	1.5	1.6	1.6	1.4	1.4	1.5	1.3	1.3	1.4
0.050	0.050	15.4	15.4	14.7	14.0	13.2	13.0	12.9	12.1	11.8	11.6
0.050	0.250	6.5	6.4	6.6	6.2	5.9	5.8	5.8	5.3	5.1	5.1

0.050	1.000	3.0	3.0	3.1	3.0	2.7	2.7	2.7	2.5	2.5	2.5
0.250	0.050	8.2	8.6	7.2	7.1	6.0	5.9	5.9	6.0	5.9	5.9
0.250	0.250	5.7	5.7	5.7	5.5	5.0	4.9	4.9	4.6	4.5	4.5
0.250	1.000	3.0	3.0	3.0	3.0	2.7	2.7	2.7	2.5	2.5	2.5
1.000	0.050	3.2	3.4	3.1	3.1	2.5	2.5	2.6	2.6	2.5	2.5
1.000	0.250	3.1	3.2	3.1	3.1	2.6	2.6	2.6	2.6	2.5	2.6
1.000	1.000	2.6	2.6	2.6	2.6	2.3	2.3	2.4	2.2	2.2	2.2
d_2		—	—	—	—	1.25	1.5	1.9	1.25	1.5	1.9
h_X		3.000	3.161	3.183	—	3.183	3.183	3.183	—	—	—
h_R^+		—	4.445	—	—	—	—	—	—	—	—
h_Y		—	—	3.067	3.053	3.067	3.067	3.067	3.053	3.053	3.053
h_S		—	—	—	4.617	—	—	—	4.617	4.617	4.617
g_X		—	—	—	—	1.429	1.137	0.909	—	—	—
g_Y		—	—	—	—	1.407	1.122	0.898	1.395	1.103	0.871
g_S		—	—	—	—	—	—	—	1.162	0.729	0.413

however, did not address the issue of monitoring the variance of this property, although his analysis suggested that this issue is important.

In general, the conclusions reached here about the performance of control charts for the problem of detecting drifts in μ and/or σ are similar to those reached by Reynolds and Stoumbos [8] and Stoumbos and Reynolds [9] for the problem of detecting sustained shifts in μ and/or σ. Thus, the control schemes that they recommended for detecting parameter shifts can also be recommended for the problem of detecting parameter drifts.

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