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Fluctuation Tests for a Change in Persistence*

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Abstract

In this paper, we develop a set of new persistence change tests which are similar in spirit to those of Kim [*Journal of Econometrics* (2000) Vol. 95, pp. 97–116], Kim *et al.* [*Journal of Econometrics* (2002) Vol. 109, pp. 389–392] and Busetti and Taylor [*Journal of Econometrics* (2004) Vol. 123, pp. 33–66]. While the exisiting tests are based on ratios of sub-sample Kwiatkowski *et al.* [*Journal of Econometrics* (1992) Vol. 54, pp. 158–179]-type statistics, our proposed tests are based on the corresponding functions of sub-sample implementations of the well-known maximal recursive-estimates and re-scaled range fluctuation statistics. Our statistics are used to test the null hypothesis that a time series displays constant trend stationarity [$I(0)$] behaviour against the alternative of a change in persistence either from trend stationarity to difference stationarity [$I(1)$], or vice versa. Representations for the limiting null distributions of the new statistics are derived and both finite-sample and asymptotic critical values are provided. The consistency of the tests against persistence change processes is also demonstrated. Numerical evidence suggests that our proposed tests provide a useful complement to the extant persistence change tests. An application of the tests to US inflation rate data is provided.

I. Introduction

The ability to correctly decompose a time series into its separate difference stationary, $I(1)$, and trend stationary, $I(0)$, components, where they exist, has important implications for effective model building and forecasting in applied

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economics and finance. A number of recent testing procedures have been developed that aim to distinguish such behaviour. These include the ratio-based persistence change tests of, *inter alia*, Kim (2000), Kim, Belaire Franch and Badillo Amador (2002) and Busetti and Taylor (2004), and the sub-sample augmented Dickey–Fuller-type tests of Banerjee, Lumsdaine and Stock (1992) and Leybourne *et al.* (2003). The first three of these assume a null hypothesis of $I(0)$ throughout, while the last two assume a null of $I(1)$ throughout. For each, the alternative is a change from $I(0)$ to $I(1)$, or vice versa. Busetti and Taylor (2004) also propose locally best invariant (LBI) tests of the constant $I(0)$ null against a change in persistence and explore the behaviour of both full-sample and sub-sample Kwiatkowski, Phillips, Schmidt and Shin (1992) (KPSS) stationarity tests against persistence change processes.

In this paper, we focus attention on the ratio-based class of persistence change tests. The test statistics adopted in Kim (2000), Kim *et al.* (2002) and Busetti and Taylor (2004) are based on ratios of sub-sample implementations of KPSS-type stationarity test statistic (they differ from a sub-sample KPSS statistic only in that they need not be scaled by a long-run variance estimator). It is well known that the KPSS statistic diverges with the sample size against $I(1)$ data, but is of $O_p(1)$ against $I(0)$ data (see, e.g. Kwiatkowski *et al.*, 1992, pp. 165–9). The ratio-based testing approach exploits these facts, as the ratio statistic will be of $O_p(1)$ against either constant $I(0)$ or constant $I(1)$ processes (because the sub-sample KPSS-type statistics in the numerator and denominator of the ratio will be of the same order in probability), but will diverge where a persistence change occurs (because of the different orders, in probability, of the two sub-sample KPSS-type statistics).

The KPSS test belongs to the class of (generalized) fluctuation tests (see, *inter alia*, Kuan and Hornik, 1995 and Kuan, 1998). So, just as we can obtain consistent tests for a change in persistence through ratios of sub-sample KPSS-type statistics, consistent inference may also be obtained from the corresponding functions of other sub-sample fluctuation tests. Two further tests that have been widely considered in the fluctuations testing literature are the maximal recursive-estimates (or generalized Kolmogorov–Smirnov) test of Sen (1980) and Ploberger, Kramer and Kontrus (1989), and the re-scaled range test of Lo (1991), Kuan and Hornik (1995) and Kuan (1998), *inter alia*. Xiao (2001) has shown that the maximal recursive-estimates test, when used as a test of stationarity against a unit root, has very similar finite-sample size and power properties to the KPSS test, while Cavaliere and Taylor (2003) present results which show that the re-scaled range test is often more powerful against the unit-root alternative than the KPSS test. It therefore seems worth exploring the application of these two important fluctuation tests to the present problem of testing the null hypothesis of stationarity against a change in persistence in order to compare these with the extant tests based on KPSS-type statistics.

The paper is organized as follows. Section II outlines the model of persistence change which we focus on. In section III we provide a brief review of tests of Kim (2000), Kim *et al.* (2002) and Busetti and Taylor (2004), which are based on ratios of KPSS-type statistics. In section IV, we detail our new ratio test statistics, based on the maximal recursive-estimates and re-scaled range statistics, and derive their large sample properties. In section V, using Monte Carlo simulation, we provide critical values and compare the finite-sample size and power properties of the new tests with the corresponding KPSS-based tests. In section VI we apply the tests discussed in this paper to the US inflation rate. Section VII concludes.

II. The persistence change model

Kim (2000, p. 99), considers the null hypothesis, denoted H_0, that the scalar time-series process y_t is formed as the sum of a purely deterministic component, d_t, and a short-memory [$I(0)$] component; i.e.

$$y_t = d_t + v_t, \quad t = 1, \ldots, T \tag{1}$$

where v_t satisfies the familiar strong mixing conditions of, *inter alia*, Phillips and Perron (1988, p. 336) with strictly positive and bounded long-run variance $\omega^2 \equiv \lim_{T \to \infty} E(\sum_{t=1}^{T} v_t)^2$. In what follows, reference to a series as being $I(0)$ is taken to imply that such conditions hold. In equation (1), the deterministic kernel $d_t = \mathbf{x}_t'\beta$, where $\mathbf{x}_t$ is a $(k+1) \times 1$, $k < T - 1$, fixed sequence the first element of which is fixed at unity throughout [so that equation (1) always contains an intercept term], with associated parameter vector β. The vector $\mathbf{x}_t$ is assumed to satisfy the mild regularity conditions of Phillips and Xiao (1998): precisely, there exists a scaling matrix δ_T and a bounded piecewise continuous function $\mathbf{x}(\cdot)$ on [0, 1] such that $\delta_T \mathbf{x}_{\lfloor \cdot T \rfloor} \to \mathbf{x}(\cdot)$ uniformly on [0, 1], where $\lfloor \cdot \rfloor$ denotes the integer part of its argument, and $\int_0^1 \mathbf{x}(s)\mathbf{x}(s)' \mathrm{d}s$ is positive definite. A leading example satisfying these conditions is given by the kth-order polynomial trend, $\mathbf{x}_t = (1, t, \ldots, t^k)'$, within which the constant ($d_t = \beta_0$) and constant plus linear time trend ($d_t = \beta_0 + \beta_1 t$) are special cases. The broken intercept and broken intercept and trend functions of Busetti and Harvey (2001) are also permitted and are obtained by specifying

$$\mathbf{x}_t'\beta = \sum_{j=0}^{i} \beta_j t^j + \sum_{j=0}^{i} \beta_{m,j} t_m^j \text{ for } i = 0, 1, \text{ respectively,}$$

with

$$t_m^j := (t-m)^j \mathbb{I}(t \in \{m+1, \ldots, T\}),$$

where $\mathbb{I}(\cdot)$ is the usual indicator function, and where m satisfies $\lim_{T\to\infty}(m/T) = \mu \in (0, 1)$ (see Phillips and Xiao, 1998, p. 448).

Kim (2000) considers two alternative hypotheses: the first, denoted H_{01}, is that y_t displays a shift from $I(0)$ to $I(1)$ behaviour[1] at time $t = \lfloor \tau^* T \rfloor$, while the second, denoted H_{10}, is that there is a shift from $I(1)$ to $I(0)$ behaviour at time $t = \lfloor \tau^* T \rfloor$. These may be expressed conveniently within the persistence change data generating process (DGP) of Kim (2000, p. 100),

$$y_t = d_t + z_{t,1}, \quad t = 1, \ldots, \lfloor \tau^* T \rfloor, \ \tau^* \in (0,1) \tag{2}$$

$$y_t = d_t + z_{t,2}, \quad t = \lfloor \tau^* T \rfloor + 1, \ldots, T. \tag{3}$$

In the case of H_{01}, $z_{t,2} = z_{t-1,2} + v_t$, with v_t and $z_{t,1}$ both $I(0)$, while in the case of H_{10}, $z_{t,1} = z_{t-1,1} + v_t$ with v_t and $z_{t,2}$, both $I(0)$.

III. Kim's ratio-based tests

Kim (2000), and subsequent modifications proposed independently by Kim *et al.* (2002) and Busetti and Taylor (2004), develop tests for the constant $I(0)$ DGP (H_0) against the $I(0)$–$I(1)$ change DGP (H_{01}) which are based on the ratio statistic

$$K(\tau) = \frac{(T - \lfloor \tau T \rfloor)^{-2} \sum_{t=\lfloor \tau T \rfloor+1}^{T} (S_{t,n}(\tau))^2}{\lfloor \tau T \rfloor^{-2} \sum_{t=1}^{\lfloor \tau T \rfloor} (S_{t,d}(\tau))^2} \tag{4}$$

where

$$S_{t,n}(\tau) \equiv \sum_{i=\lfloor \tau T \rfloor+1}^{t} \breve{v}_{i,\tau}, \quad S_{t,d}(\tau) \equiv \sum_{i=1}^{t} \hat{v}_{i,\tau} \tag{5}$$

where, in order to obtain exact invariance to β (the vector of parameters characterizing d_t), $\hat{v}_{t,\tau}$ are the residuals from the ordinary least squares (OLS) regression of y_t on $\mathbf{x}_t$, for $t = 1, \ldots, \lfloor \tau T \rfloor$. In the constant case ($\mathbf{x}_t = 1$), e.g.

$$\hat{v}_{t,\tau} = y_t - \bar{y}(\tau) \quad \text{with } \bar{y}(\tau) = \lfloor \tau T \rfloor^{-1} \sum_{t=1}^{\lfloor \tau T \rfloor} y_t$$

that is, the data are de-meaned over $t = 1, \ldots, \lfloor \tau T \rfloor$. Similarly, $\breve{v}_t$ are the OLS residuals from the regression of y_t on $\mathbf{x}_t$ for $t = \lfloor \tau T \rfloor + 1, \ldots, T$.

[1]An $I(1)$ series is defined to be the one formed from the accumulation of an $I(0)$ series.

Remark 1. Recall that the standard stationarity test of KPSS rejects H_0 in favour of the alternative of constant $I(1)$ behaviour for large values of the statistic

$$K = T^{-2}\hat{\omega}^{-2}\sum_{t=1}^{T} S_t^2, \quad S_t \equiv \sum_{i=1}^{t} e_i \tag{6}$$

where e_t are the OLS residuals from the regression of y_t on $\mathbf{x}_t$, $t = 1, \ldots, T$, and $\hat{\omega}^2$ is an estimator of the long-run variance, ω^2, which is consistent under H_0 and has the form

$$\hat{\omega}^2 \equiv \sum_{j=-T+1}^{T-1} k\left(\frac{j}{m}\right)\hat{\gamma}(j), \quad \hat{\gamma}(j) \equiv \frac{1}{T}\sum_{t=|j|+1}^{T} e_t e_{t-|j|} \tag{7}$$

where m is a bandwidth parameter and $k(\cdot)$ is a weighting function. KPSS assume Bartlett weights; i.e. $1/m + T^{-1/2}m \to 0$ as $T \to \infty$ and $k(x) = 1 - |x|.\mathbb{I}(|x| \leq 1)$. The statistic K is of $O_p(T/m)$ under the constant $I(1)$ model (see Kwiatkowski *et al.*, 1992, p. 169). Notice, therefore, that the numerator (denominator) of the statistic $K(\tau)$ is nothing other than a standard KPSS stationarity test statistic applied to the second (first) sub-sample of the data, but without any long-run variance estimator used to scale the statistic.

As the true changepoint, τ^*, is assumed to be unknown, Kim (2000), Kim *et al.* (2002) and Busetti and Taylor (2004) consider three statistics based on the sequence of statistics $\{K(\tau), \tau \in \Lambda\}$, where $\Lambda = [\tau_l, \tau_u]$ is a compact subset of [0, 1] with $\tau_l \leq \tau^* \leq \tau_u$. These are:

$$K_1 = \max_{s\in\{\lfloor\tau_l T\rfloor,\ldots,\lfloor\tau_u T\rfloor\}} K(s/T)$$

$$K_2 = T_*^{-1}\sum_{s=\lfloor\tau_l T\rfloor}^{\lfloor\tau_u T\rfloor} K(s/T)$$

$$K_3 = \ln\left\{T_*^{-1}\sum_{s=\lfloor\tau_l T\rfloor}^{\lfloor\tau_u T\rfloor} \exp\left(\frac{1}{2}K(s/T)\right)\right\},$$

where $T_* \equiv \lfloor\tau_u T\rfloor - \lfloor\tau_l T\rfloor + 1$. The first of these, after Andrews (1993), takes the maximum over the sequence, the second uses Hansen's (1991) mean score statistic, and the third, Andrews and Ploberger's (1994) mean-exponential statistic.

Busetti and Taylor (2004) demonstrate that for $0 < \tau < 1$, and under the condition that both $\int_0^\tau \mathbf{x}(s)\mathbf{x}(s)'\mathrm{d}s$ and $\int_\tau^1 \mathbf{x}(s)\mathbf{x}(s)'\mathrm{d}s$ are positive definite,

$$\omega^{-1}T^{-1/2}(S_{\lfloor T\cdot\rfloor,n}(\cdot), S_{\lfloor T\cdot\rfloor,d}(\cdot)) \Rightarrow (N_v(\cdot,\cdot), D_v(\cdot,\cdot)) \tag{8}$$

where '$\Rightarrow$' is used to denote weak convergence as $T \to \infty$ and where

$$N_v(\tau, r) \equiv W_v(r) - W_v(\tau) - \int_\tau^1 \mathbf{x}(r)'\mathrm{d}W_v(r)\left(\int_\tau^1 \mathbf{x}(r)\mathbf{x}(r)'\mathrm{d}r\right)^{-1} \times \int_\tau^r \mathbf{x}(s)\mathrm{d}s, \quad r \in (\tau, 1], \tag{9}$$

and

$$D_v(\tau, r) \equiv W_v(r) - \int_0^\tau \mathbf{x}(r)'\mathrm{d}W_v(r)\left(\int_0^\tau \mathbf{x}(r)\mathbf{x}(r)'\mathrm{d}r\right)^{-1}\int_0^r \mathbf{x}(s)\mathrm{d}s, \quad r \in [0, \tau], \tag{10}$$

with $W_v(\cdot)$ a standard Brownian motion process on [0, 1]: here defined by

$$\omega^{-1}T^{-1/2}\sum_{t=1}^{\lfloor rT \rfloor} v_t \Rightarrow W_v(r), \quad r \in [0, 1].$$

In the de-meaned case ($d_t = \beta_0$), e.g., equations (9) and (10) reduce to

$$N_v(\tau, r) \equiv W_v(r) - W_v(\tau) - (r - \tau)(1 - \tau)^{-1}\{W_v(1) - W_v(\tau)\}$$
$$D_v(\tau, r) \equiv W_v(r) - r\tau^{-1}W_v(\tau).$$

The limiting distributions of the three statistics K_1, K_2 and K_3 under H_0 then follow directly from (8), using applications of the continuous mapping theorem (CMT); viz.,

$$K_1 \Rightarrow \sup_{\tau \in [\tau_l, \tau_u]} A(\tau) \tag{11}$$

$$K_2 \Rightarrow \int_{\tau_l}^{\tau_u} A(\tau)\,\mathrm{d}\tau \tag{12}$$

$$K_3 \Rightarrow \ln\left\{\int_{\tau_l}^{\tau_u} \exp\left(\frac{1}{2}A(\tau)\right)\mathrm{d}\tau\right\} \tag{13}$$

where

$$A(\tau) \equiv \frac{(1-\tau)^{-2}\int_\tau^1 N_v(\tau, r)^2\,\mathrm{d}r}{\tau^{-2}\int_0^\tau D_v(\tau, r)^2\,\mathrm{d}r}.$$

Notice that these limiting representations do not depend on the long-run variance, ω^2, although no long-run variance estimators are used in calculating the statistics.

In order to test H_0 against the $I(1)$–$I(0)$ change DGP (H_{10}), Busetti and Taylor (2004) propose further tests based on the sequence of *reciprocals* of $K(\tau)$, $\tau \in \Lambda$; precisely,

$$K_1' = \max_{s \in \{\lfloor \tau_l T \rfloor, \ldots, \lfloor \tau_u T \rfloor\}} K(s/T)^{-1}$$

$$K_2' = T_*^{-1} \sum_{s=\lfloor \tau_l T \rfloor}^{\lfloor \tau_u T \rfloor} K(s/T)^{-1}$$

$$K_3' = \ln\left\{ T_*^{-1} \sum_{s=\lfloor \tau_l T \rfloor}^{\lfloor \tau_u T \rfloor} \exp\left(\frac{1}{2} K(s/T)^{-1}\right) \right\},$$

and, in order to test against an unknown direction of change [i.e. either a change from $I(0)$ to $I(1)$ or vice versa], they also propose

$$K_4 = \max(K_1, K_1'), \quad K_5 = \max(K_2, K_2'), \quad K_6 = \max(K_3, K_3')$$

and again the limiting distributions of these six statistics under H_0 follow straightforwardly from equation (8) and applications of the CMT.

As regards test consistency, under H_{01} it is shown in Busetti and Taylor (2004) that K_1 to K_6 are each of $O_p(T^2)$, while K_1' to K_3' are each of $O_p(1)$. Similarly, under H_{10}, Busetti and Taylor (2004) show that K_1' to K_3' and K_4 to K_6 are each of $O_p(T^2)$, while K_1 to K_3 are of $O_p(1)$. Thus, tests which reject for large values of K_1 to K_3 can be used to detect H_{01}, while tests which reject for large values of K_1' to K_3' can be used to detect H_{10}, and, finally, tests which reject for large values of K_4 to K_6 can be used to detect either H_{01} or H_{10}.

It is also shown in Busetti and Taylor (2004) that all of the foregoing statistics share the important property that they do not diverge against constant $I(1)$ processes. Consequently, the ratio-based tests will not be consistent against constant $I(1)$ processes. This property is *not* shared by either the LBI-based persistence change tests of Busetti and Taylor (2004) or the full- and sub-sample KPSS tests, all of which are based on statistics which diverge against series which are $I(1)$ throughout. These tests are therefore not useful as persistence change tests and will not be discussed further in this paper.

IV. Ratio-based fluctuations tests

The KPSS statistic, K, essentially maps the sequence $\{S_t\}$ (defined in Remark 1) onto [0, 1] by properly averaging the squared values of the sequence. However, as Xiao (2001, p. 88) argues, '... fluctuation tests can provide another way to distinguish between stationary and unit root processes': such tests can clearly be obtained simply by taking different mappings of the sequence $\{S_t\}$. For example, if the supremum of the absolute value of $\{S_t\}$ is considered, then the maximal recursive-estimates stationarity test proposed in Xiao (2001) is obtained which rejects H_0 for large values of the statistic

$$KS = T^{-1/2}\hat{\omega}^{-1} \max_{t=1,\ldots,T} |S_t|.$$

By taking the range of $\{S_t\}$, the re-scaled range test which rejects H_0 for large values of statistic

$$RS = T^{-1/2}\hat{\omega}^{-1}(\max_{t=1,\ldots,T} S_t - \min_{t=1,\ldots,T} S_t),$$

is obtained.

Under the constant $I(1)$ model, Xiao (2001, p. 94) demonstrates that KS is of $O_p(T/m)$, while Cavaliere and Taylor (2003) show that the same rate applies in this case to the RS statistic. Summarizing the simulation evidence provided in his paper comparing the K and KS tests, Xiao (2001, p. 99) states that 'It is clear from the Monte Carlo evidence that in general these two tests have very similar finite sample behavior…'. Moreover, Cavaliere and Taylor (2003) compare the power properties of the K, RS and KS tests and conclude that the RS test is competitive on power, often outperforming the KPSS K test.

Given these encouraging results for applying fluctuation statistics to testing H_0 against the constant $I(1)$ alternative, it seems worthwhile exploring persistence change tests based on sub-sample implementations of (long-run variance uncorrected) KS and RS fluctuation statistics. Specifically, and by analogy to equation (4), we therefore consider tests based on the two ratio statistics:

$$KS(\tau) = \frac{(T - \lfloor \tau T \rfloor)^{-1/2} \max_{t=\lfloor \tau T \rfloor+1,\ldots,T} |S_{t,n}(\tau)|}{\lfloor \tau T \rfloor^{-1/2} \max_{t=1,\ldots,\lfloor \tau T \rfloor} |S_{t,d}(\tau)|} \tag{14}$$

$$RS(\tau) = \frac{(T - \lfloor \tau T \rfloor)^{-1/2} \left(\max_{t=\lfloor \tau T \rfloor+1,\ldots,T} S_{t,n}(\tau) - \min_{t=\lfloor \tau T \rfloor+1,\ldots,T} S_{t,n}(\tau) \right)}{\lfloor \tau T \rfloor^{-1/2} \left(\max_{t=1,\ldots,\lfloor \tau T \rfloor} S_{t,d}(\tau) - \min_{t=1,\ldots,\lfloor \tau T \rfloor} S_{t,d}(\tau) \right)} \tag{15}$$

where $S_{t,n}(\tau)$ and $S_{t,d}(\tau)$ are as defined in equation (5).

As the true breakpoint τ^* is unknown, as might be the direction of change, under the alternative of a change in persistence, we may, as with the tests of section III, consider tests based on the sequences of ratio statistics $\{KS(\tau), \tau \in \Lambda\}$, $\{RS(\tau), \tau \in \Lambda\}$, $\{KS(\tau)^{-1}, \tau \in \Lambda\}$ and $\{RS(\tau)^{-1}, \tau \in \Lambda\}$. We shall denote the resulting statistics as $KS_1, \ldots, KS_6$ and $KS'_1, \ldots, KS'_3$ for the tests based on the $\{KS(\tau)\}$ sequence, and $RS_1, \ldots, RS_6$ and $RS'_1, \ldots, RS'_3$ for the tests based on the $\{RS(\tau)\}$ sequence, with an entirely obvious notation.

We now establish the limiting distributions of our proposed statistics under the null, H_0. Using equation (8) and applications of the CMT, we obtain immediately that for the maximal recursive-estimates-based tests,

$$KS_1 \Rightarrow \sup_{\tau\in[\tau_l,\tau_u]} B(\tau), \quad KS_1' \Rightarrow \sup_{\tau\in[\tau_l,\tau_u]} B(\tau)^{-1} \tag{16}$$

$$KS_2 \Rightarrow \int_{\tau_l}^{\tau_u} B(\tau)\mathrm{d}\tau, \quad KS_2' \Rightarrow \int_{\tau_l}^{\tau_u} B(\tau)^{-1}\mathrm{d}\tau \tag{17}$$

$$KS_3 \Rightarrow \ln\left\{\int_{\tau_l}^{\tau_u} \exp\left(\frac{1}{2}B(\tau)\right)\mathrm{d}\tau\right\}, \quad KS_3' \Rightarrow \ln\left\{\int_{\tau_l}^{\tau_u} \exp\left(\frac{1}{2}B(\tau)^{-1}\right)\mathrm{d}\tau\right\} \tag{18}$$

where

$$B(\tau) \equiv \frac{(1-\tau)^{-1/2}\sup_{s\in(\tau,1]}|N_v(\tau,s)|}{\tau^{-1/2}\sup_{s\in[0,\tau]}|D_v(\tau,s)|},$$

while for the re-scaled range-based tests,

$$RS_1 \Rightarrow \sup_{\tau\in[\tau_l,\tau_u]} C(\tau), \quad RS_1' \Rightarrow \sup_{\tau\in[\tau_l,\tau_u]} C(\tau)^{-1} \tag{19}$$

$$RS_2 \Rightarrow \int_{\tau_l}^{\tau_u} C(\tau)\mathrm{d}\tau, \quad RS_2' \Rightarrow \int_{\tau_l}^{\tau_u} C(\tau)^{-1}\mathrm{d}\tau \tag{20}$$

$$RS_3 \Rightarrow \ln\left\{\int_{\tau_l}^{\tau_u} \exp\left(\frac{1}{2}C(\tau)\right)\mathrm{d}\tau\right\}, \quad RS_3' \Rightarrow \ln\left\{\int_{\tau_l}^{\tau_u} \exp\left(\frac{1}{2}C(\tau)^{-1}\right)\mathrm{d}\tau\right\} \tag{21}$$

where

$$C(\tau) \equiv \frac{(1-\tau)^{-1/2}\sup_{s,s'\in(\tau,1]}|N_v(\tau,s)-N_v(\tau,s')|}{\tau^{-1/2}\sup_{s,s'\in[0,\tau]}|D_v(\tau,s)-D_v(\tau,s')|}.$$

Notice that, as with the persistence change statistics of section III, these limiting representations do not depend on the long-run variance, ω^2, although no long-run variance estimators are used in calculating the statistics.

Finally, for the KS_4 statistic, for example, noting that the function max(x, y) is continuous in both arguments, it follows immediately from equation (16) and the CMT that

$$KS_4 \Rightarrow \max\left\{\sup_{\tau\in[\tau_l,\tau_u]} B(\tau),\ \sup_{\tau\in[\tau_l,\tau_u]} B(\tau)^{-1}\right\}.$$

The corresponding results for KS_5, KS_6, and $RS_4,\dots,RS_6$, follow from equations (17)–(21) and applications of the CMT in exactly the same way.

Under H_{01}, the first sub-sample residuals $\hat{v}_{t,\tau}$, $t = 1,\dots,\lfloor\tau T\rfloor$, are clearly of $O_p(1)$ provided $\tau \le \tau^*$, but are of $O_p(T^{1/2})$ otherwise. The second sub-sample residuals, $\breve{v}_{t,\tau}$, are seen to be of $O_p(T^{1/2})$, $t = \lfloor\tau T\rfloor + 1,\dots,T$, regardless of τ. Therefore, it follows immediately that for $\tau \le \tau^*$, $KS(\tau)$ and $RS(\tau)$ of (14) and (15) are both of $O_p(T)$, while for $\tau > \tau^*$, they are both of $O_p(1)$. Consequently, KS_1 to KS_6 and RS_1 to RS_6 are of $O_p(T)$ under H_{01}, while KS'_1 to KS'_3 and RS'_1 to RS'_3 are of $O_p(1)$ under H_{01}. Under H_{10}, y_t is $O_p(T^{1/2})$ for $t = 1,\dots,\lfloor\tau^* T\rfloor$ and, hence, the first sub-sample residuals, $\hat{v}_{t,\tau}$, $t = 1,\dots,\lfloor\tau T\rfloor$, are also of $O_p(T^{1/2})$, regardless of τ. For $\tau > \tau^*$, the second sub-sample residuals $\breve{v}_{t,\tau}$, are of $O_p(1)$, $t = \lfloor\tau T\rfloor + 1,\dots,T$, but are otherwise $O_p(T^{1/2})$. Consequently, $KS(\tau)$ and $RS(\tau)$ are both of $O_p(T^{-1})$ for all $\tau \ge \tau^*$, but $O_p(1)$ otherwise. Therefore, KS_4 to KS_6, RS_4 to RS_6, KS'_1 to KS'_3 and RS'_1 to RS'_3 are of $O_p(T)$ under H_{01}, while KS_1 to KS_3 and RS_1 to RS_3 are of $O_p(1)$ under H_{10}. Finally, all of the persistence change statistics considered above are trivially seen to be of $O_p(1)$ against constant $I(1)$ processes.

In section V we use Monte Carlo simulation methods to *investigate* the relative size and power properties of the tests proposed in this section *vis-à-vis* the corresponding tests of section III. We also provide both finite-sample and asymptotic critical values for the various tests discussed.

V. Numerical results

Critical values

Table 1 reports both finite-sample and asymptotic upper tail null critical values for the K_1 to K_6 and K'_1 to K'_3 persistence change tests of section III, while Tables 2 and 3 report the corresponding quantities for the KS_1 to KS_6 and KS'_1 to KS'_3, and the RS_1 to RS_6 and RS'_1 to RS'_3 tests, respectively, of section IV.[2] Precisely, the finite-sample critical values of Tables 1–3 were obtained by Monte Carlo simulation using pseudo-data generated according to the pure-noise DGP:

$$y_t = \varepsilon_t \sim \text{n.i.i.d.}(0,1), \quad t = 1,\dots,T.$$

[2]In what follows, we will, at times, refer to these generically as the KPSS-, KS- and RS-based tests, respectively.

TABLE 1

Critical values for KPSS-based tests of stationarity against a change in persistence

T	K_1	K_2	K_3	K_1'	K_2'	K_3'	K_4	K_5	K_6
(A) De-meaned case									
60									
10%	12.56	3.56	3.52	12.64	3.57	3.53	16.71	4.72	5.31
5%	16.90	4.72	5.39	16.78	4.73	5.34	21.46	5.95	7.50
1%	28.93	7.87	11.08	28.43	7.89	10.87	34.76	9.36	14.00
120									
10%	12.92	3.50	3.42	12.93	3.53	3.45	16.95	4.62	5.14
5%	16.97	4.56	5.14	17.16	4.67	5.23	21.70	5.85	7.28
1%	28.31	7.52	10.37	28.83	7.78	10.63	34.22	9.26	13.23
240									
10%	13.34	3.52	3.43	13.26	3.52	3.41	17.52	4.63	5.14
5%	17.69	4.61	5.19	17.61	4.66	5.18	22.20	5.82	7.24
1%	29.11	7.59	10.41	29.22	7.63	10.42	34.45	9.07	12.92
∞									
10%	13.87	3.55	3.45	13.65	3.50	3.39	18.07	4.63	5.12
5%	18.33	4.67	5.22	18.08	4.59	5.11	22.95	5.90	7.24
1%	30.26	7.74	10.51	29.91	7.72	10.41	35.98	9.35	13.22
(B) De-meaned and de-trended case									
60									
10%	6.74	2.43	1.60	6.78	2.44	1.61	8.44	2.99	2.12
5%	8.47	2.97	2.12	8.53	3.00	2.15	10.39	3.56	2.78
1%	13.23	4.40	3.93	13.11	4.41	3.88	15.73	5.06	4.88
120									
10%	6.65	2.36	1.51	6.72	2.37	1.53	8.27	2.90	1.99
5%	8.28	2.89	1.99	8.35	2.91	2.01	9.98	3.46	2.55
1%	12.60	4.23	3.47	12.37	4.22	3.44	14.48	4.86	4.25
240									
10%	6.79	2.35	1.50	6.74	2.34	1.49	8.32	2.86	1.94
5%	8.39	2.86	1.95	8.34	2.86	1.94	10.05	3.41	2.48
1%	12.58	4.18	3.37	12.35	4.18	3.28	14.49	4.79	4.09
∞									
10%	7.00	2.36	1.50	7.00	2.36	1.50	8.61	2.88	1.95
5%	8.68	2.89	1.97	8.64	2.88	1.96	10.38	3.42	2.49
1%	12.92	4.20	3.38	13.00	4.19	3.40	14.94	4.84	4.14

Results are reported for de-meaned ($\mathbf{x}_t = 1$) and de-meaned and de-trended [$\mathbf{x}_t = (1, t)'$] data in panels A and B of the tables, respectively. In each case, finite-sample critical values are given for $T = 60$, 120 and 240, while the rows labelled '∞' give asymptotic critical values for the tests, obtained by direct simulation of the appropriate limiting functionals of sections III and IV using discrete approximations for $T = 1{,}000$. For each test we used $\Lambda = [0.2, 0.8]$, as is typical in this literature; this choice is applied

TABLE 2

Critical values for maximal recursive-estimates-based tests of stationarity against a change in persistence

T	KS_1	KS_2	KS_3	KS'_1	KS'_2	KS'_3	KS_4	KS_5	KS_6
(A) De-meaned case									
60									
10%	3.06	1.62	0.86	3.06	1.63	0.86	3.51	1.85	0.98
5%	3.53	1.84	0.98	3.52	1.85	0.99	3.98	2.05	1.11
1%	4.57	2.32	1.27	4.60	2.33	1.27	5.08	2.53	1.39
120									
10%	2.94	1.58	0.83	2.94	1.58	0.83	3.32	1.78	0.94
5%	3.34	1.78	0.94	3.34	1.78	0.94	3.73	1.97	1.05
1%	4.23	2.22	1.19	4.23	2.24	1.20	4.62	2.42	1.31
240									
10%	2.88	1.55	0.81	2.87	1.56	0.81	3.23	1.74	0.92
5%	3.24	1.74	0.91	3.25	1.75	0.92	3.60	1.93	1.02
1%	4.05	2.16	1.14	4.06	2.17	1.15	4.38	2.33	1.24
∞									
10%	2.81	1.53	0.79	2.80	1.52	0.79	3.14	1.71	0.89
5%	3.16	1.71	0.89	3.15	1.71	0.89	3.48	1.89	0.99
1%	3.93	2.12	1.12	3.91	2.11	1.11	4.25	2.29	1.21
(B) De-meaned and de-trended case									
60									
10%	2.57	1.46	0.76	2.56	1.47	0.76	2.89	1.62	0.85
5%	2.90	1.62	0.85	2.91	1.62	0.85	3.23	1.77	0.93
1%	3.71	1.95	1.03	3.70	1.96	1.04	4.07	2.10	1.12
120									
10%	2.41	1.42	0.73	2.41	1.42	0.73	2.66	1.56	0.81
5%	2.67	1.56	0.80	2.68	1.56	0.81	2.92	1.70	0.88
1%	3.26	1.87	0.97	3.27	1.86	0.97	3.52	2.00	1.04
240									
10%	2.32	1.39	0.72	2.32	1.39	0.71	2.55	1.52	0.78
5%	2.57	1.52	0.79	2.56	1.52	0.78	2.79	1.65	0.85
1%	3.09	1.81	0.94	3.09	1.81	0.94	3.32	1.93	1.00
∞									
10%	2.26	1.37	0.70	2.25	1.37	0.70	2.46	1.49	0.77
5%	2.48	1.50	0.77	2.47	1.49	0.76	2.67	1.61	0.83
1%	2.94	1.76	0.91	2.94	1.76	0.90	3.14	1.88	0.97

throughout this section of the paper. The simulations were performed using 80,000 Monte Carlo replications and the RNDN function of Gauss 3.2.

Size properties

In this section, we use Monte Carlo simulation methods to investigate the behaviour of the KPSS-based tests of section III relative to the corresponding

TABLE 3

Critical values for re-scaled range-based tests of stationarity against a change in persistence

T	RS_1	RS_2	RS_3	RS_1'	RS_2'	RS_3'	RS_4	RS_5	RS_6
(A) De-meaned case									
60									
10%	2.44	1.48	0.76	2.45	1.48	0.76	2.75	1.64	0.85
5%	2.77	1.63	0.85	2.77	1.64	0.85	3.08	1.79	0.93
1%	3.53	1.97	1.03	3.52	1.99	1.04	3.86	2.12	1.12
120									
10%	2.27	1.43	0.73	2.28	1.43	0.73	2.53	1.57	0.81
5%	2.54	1.57	0.81	2.54	1.57	0.81	2.78	1.70	0.88
1%	3.11	1.87	0.97	3.12	1.87	0.97	3.36	2.00	1.04
240									
10%	2.19	1.40	0.72	2.19	1.40	0.72	2.42	1.54	0.79
5%	2.43	1.54	0.79	2.42	1.54	0.79	2.64	1.66	0.85
1%	2.93	1.82	0.93	2.92	1.82	0.93	3.12	1.93	0.99
∞									
10%	2.12	1.38	0.70	2.12	1.38	0.70	2.32	1.50	0.77
5%	2.32	1.50	0.77	2.33	1.50	0.77	2.51	1.62	0.83
1%	2.76	1.77	0.90	2.76	1.77	0.90	2.95	1.87	0.96
(B) De-meaned and de-trended case									
60									
10%	2.50	1.45	0.75	2.49	1.45	0.75	2.80	1.60	0.83
5%	2.81	1.59	0.83	2.82	1.60	0.83	3.12	1.73	0.91
1%	3.59	1.90	1.00	3.58	1.91	1.01	3.92	2.04	1.08
120									
10%	2.31	1.40	0.72	2.32	1.40	0.72	2.55	1.53	0.79
5%	2.56	1.53	0.79	2.56	1.53	0.79	2.79	1.65	0.85
1%	3.11	1.81	0.94	3.12	1.81	0.94	3.34	1.92	1.00
240									
10%	2.22	1.38	0.70	2.22	1.37	0.70	2.43	1.49	0.77
5%	2.44	1.49	0.77	2.44	1.49	0.77	2.65	1.61	0.83
1%	2.92	1.75	0.91	2.93	1.75	0.90	3.12	1.86	0.96
∞									
10%	2.15	1.35	0.69	2.14	1.35	0.69	2.33	1.46	0.75
5%	2.34	1.46	0.75	2.33	1.46	0.75	2.52	1.57	0.80
1%	2.76	1.70	0.87	2.77	1.70	0.87	2.94	1.80	0.92

KS- and RS-based tests proposed in section IV when applied to data generated by the following *constant parameter* stable and invertible autoregressive moving average (ARMA) process:

$$y_t = \phi y_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}, \quad t = -100, \ldots, T, \tag{22}$$

with $\varepsilon_t \sim$ n.i.i.d.(0, 1), and the design parameters $\phi \in \{0.0, 0.50, 0.90\}$ and $\theta \in \{0.0, \pm 0.6\}$.[3] Notice that, in all cases, y_t is an $I(0)$ process and, hence,

[3]Results are not presented for $\phi = \theta = 0$ since exact critical values were used.

H_0 holds. However, other things equal, y_t increasingly resembles an $I(1)(I(-1))$ process in finite samples as $\phi(\theta)$ tends towards unity. Results are reported for de-meaned ($\mathbf{x}_t = 1$) and de-meaned and de-trended [$\mathbf{x}_t = (1, t)'$] data in panels A and B of the tables, respectively. Notice that, because of invariance, we have set $d_t = 0$ with no loss of generality.

Table 4 reports the empirical rejection frequencies of the tests of sections III and IV for $T = 60$. Only results pertaining to the K_1 to K_6, KS_1 to KS_6 and RS_1 to RS_6 tests are reported: results for the K_1' to K_3', KS_1' to KS_3', and RS_1' to RS_3' tests were virtually identical to those given for K_1 to K_3, KS_1 to KS_3, and RS_1 to RS_3, respectively. All tests were run at the nominal 5% level using the relevant finite-sample critical values from Tables 1–3. In order to control for initial effects, the first 100 observations were discarded.

Consider first the results for the KPSS-based tests K_1 to K_6. As a general rule, size distortions are lowest, other things equal, for the tests based on Hansen's (1991) mean score (K_2 and K_5). The tests based on Andrews' (1993) maximum statistic and Andrews and Ploberger's (1994) mean-exponential statistic display similar distortions which are, in general, rather worse than for the mean score tests. In the case of the KS-based tests (KS_1 to KS_6) and the RS-based tests (RS_1 to RS_6), the pattern is different with the worst distortions tending to be seen in the maximum statistics and the smallest distortions seen with the mean score tests.

In terms of a relative comparison between the KPSS-, KS- and RS-based persistence change tests, the KPSS-based tests tend to fare best in the case of Andrews' (1993) maximum statistic in both the de-meaned and de-meaned and de-trended cases. The KPSS-, KS- and RS-based tests display broadly comparable distortions in the mean-exponential case when the data are de-meaned, but the KPSS-based tests display greater size distortions than the KS- and RS-based tests in the de-meaned and de-trended case. Finally, for the mean score case, the KS- and RS-based tests perform very similarly to one another and rather better overall than the corresponding KPSS-based tests in both de-meaned and de-meaned and de-trended environments.

Size distortions for all of tests tend to be worse, other things equal, for the de-meaned and de-trended case than for the de-meaned case, with the exception of $\phi = 0$, $\theta = 0.6$, where the reverse is true. Although not reported here, corresponding results for $T = 120$ and 240 are presented in the accompanying working paper (Taylor, 2004), where it is shown that in all cases size distortions are also ameliorated, other things equal, as the sample size is increased, as predicted by the limiting distribution theory. It is fair to say, however, that for all of the tests, finite-sample size distortions can be quite high, with large values of ϕ combined with large negative values of θ causing the tests the greatest problems. A comparison of the results in Table 4 with the results reported in Kwiatkowski *et al.* (1992; Table 3, p. 171) and Xiao (2001,

TABLE 4

Empirical rejection frequencies of nominal 5% tests against a change in persistence: DGP (5.1), $T = 60$

ϕ	θ	K_1	K_4	K_2	K_5	K_3	K_6	KS_1	KS_4	KS_2	KS_5	KS_3	KS_6	RS_1	RS_4	RS_2	RS_5	RS_3	RS_6
(A) De-meaned case																			
0.0	0.6	0.4	0.4	0.5	0.4	0.4	0.4	0.4	0.3	1.0	0.9	0.8	0.6	0.5	0.4	0.8	0.6	0.7	0.5
	−0.6	6.7	9.1	6.3	8.5	6.8	9.2	7.7	10.0	6.4	8.1	6.9	9.0	9.0	12.2	6.9	8.8	7.4	9.7
0.5	0.0	11.1	16.0	9.4	13.6	11.2	16.0	13.1	17.6	8.9	12.0	10.5	14.7	15.3	21.8	9.6	13.3	11.4	16.0
	0.6	2.3	2.6	2.7	3.0	2.3	2.6	2.2	2.1	3.1	3.3	2.8	2.9	1.9	2.1	2.9	3.0	2.7	2.7
	−0.6	13.6	19.8	11.3	16.6	13.7	19.9	16.8	23.2	10.6	15.1	13.2	18.9	20.5	29.9	12.1	17.3	15.0	22.0
0.9	0.0	36.9	55.1	29.6	47.7	37.2	55.5	42.8	60.1	24.4	39.6	33.6	52.6	51.4	69.9	27.0	44.6	36.7	57.9
	0.6	25.6	38.4	20.3	32.4	25.7	38.7	27.2	38.7	15.8	24.1	20.8	32.4	30.2	42.6	15.8	23.8	20.7	31.9
	−0.6	38.2	56.7	30.9	49.4	38.5	57.1	44.9	63.0	26.3	42.5	36.0	55.3	54.0	73.0	29.4	47.9	39.5	61.4
(B) De-meaned and de-trended case																			
0.0	0.6	1.0	0.9	0.6	0.4	0.7	0.7	0.9	0.7	1.8	1.8	1.5	1.4	0.4	0.3	1.2	1.1	1.0	0.8
	−0.6	9.4	13.2	7.6	10.6	9.2	13.3	10.8	14.2	6.8	8.9	7.8	10.4	11.1	14.6	7.2	9.6	8.2	11.2
0.5	0.0	18.5	26.6	13.4	19.8	18.2	26.6	19.7	27.0	10.3	14.3	13.1	18.8	20.5	28.0	11.0	15.7	13.8	20.2
	0.6	2.2	2.4	2.4	2.8	2.0	2.4	2.1	2.3	3.1	3.5	2.9	3.1	1.9	2.0	3.0	3.3	2.6	2.9
	−0.6	24.7	36.3	17.2	26.1	24.4	36.3	27.9	39.2	13.4	19.2	18.2	27.0	28.7	40.1	14.2	21.2	19.2	29.0
0.9	0.0	59.9	78.5	44.0	65.5	59.0	78.5	61.9	79.0	30.2	47.9	44.4	65.3	63.4	80.3	31.9	51.7	45.9	68.0
	0.6	30.2	42.4	21.9	32.9	30.2	42.9	28.8	38.7	13.7	19.5	19.2	27.9	30.4	40.9	14.9	22.1	20.6	30.8
	−0.6	63.4	82.4	47.8	70.3	62.7	82.5	66.9	84.3	34.2	54.4	49.9	72.2	68.2	85.3	35.9	58.0	51.1	74.4

Tables IV–V, pp. 97–98), however, shows that these distortions are far less serious than for the corresponding full-sample KPSS and KS tests when a small bandwidth is used in the long-run variance estimator, $\hat{\omega}^2$ of equation (7), and are roughly comparable where a sample-size-dependent bandwidth is used.

Although the original ratio-based tests of Kim (2000), Kim *et al.* (2002) and Busetti and Taylor (2004) were based on statistics where no variance estimator is employed, Leybourne and Taylor (2004) have recently discussed tests based on statistics where the numerator and denominator of equation (4) are scaled by appropriate sub-sample variance estimators. They consider replacing $K(\tau)$ of equation (4), for each $\tau \in \Lambda$, by the Studentized statistic $K(\tau, m) \equiv (\hat{\omega}^2_{2,\tau}/\hat{\omega}^2_{1,\tau})K(\tau)$, where, $\hat{\omega}^2_{1,\tau}$ and $\hat{\omega}^2_{2,\tau}$ are variance estimators of the form given in equation (7), but applied only to the first $\lfloor \tau T \rfloor$ and last $T - \lfloor \tau T \rfloor$ sample observations, respectively, and proceeding as in section III, replacing $K(\tau)$ by $K(\tau, m)$ throughout. It is clear that exactly the same modification can be applied to the numerator and denominator of both $KS(\tau)$ of equation (14) and $RS(\tau)$ of equation (15).

Leybourne and Taylor (2004) find that significant improvements are seen in the finite-sample size properties of the tests based on $K(\tau, m)$. Unlike with the full-sample KPSS test, the bandwidth, m, used in $K(\tau, m)$ does not need to be of $o(T^{1/2})$ (see Remark 1) to obtain pivotal limiting distributions. Indeed, Leybourne and Taylor (2004) find that setting $m = 1$ provides a useful pragmatic balance between redressing the size problems of the tests yet keeping power losses against persistence change processes relatively small. We found much the same to be true of the tests based on the corresponding modifications of the KS- and RS-based tests. Table 5 reports results for $m = 1$ and $T = 60$; results for other values of m and other sample sizes are available on request. Roughly the same relative comparisons between the different tests noted above still apply to the modified tests, but across the board size distortions are very much improved over the corresponding results in Table 4.

Power properties

In this section, we report the empirical rejection frequencies of the tests of sections III and IV when the data are generated according to the $I(0)$–$I(1)$ switch DGP

$$y_t = \rho_t y_{t-1} + v_t, \quad t = -100, \ldots, T, \tag{23}$$

with $\rho_t = \rho$, $t = -100, \ldots, \lfloor \tau^* T \rfloor$, and $\rho_t = 1$, $t = \lfloor \tau^* T \rfloor + 1, \ldots, T$. We consider the following values for breakpoint and first sub-sample autoregressive parameters: $\tau^* \in \{0.25, 0.50, 0.75\}$ and $\rho \in \{0.0, 0.5, 0.9\}$, respectively. Results are reported for the noise process $v_t \sim$ n.i.i.d.(0, 1).

TABLE 5

Empirical rejection frequencies of nominal 5% Studentized tests (m = 1) against a change in persistence: DGP (5.1), T = 60

ϕ	θ	K_1	K_4	K_2	K_5	K_3	K_6	KS_1	KS_4	KS_2	KS_5	KS_3	KS_6	RS_1	RS_4	RS_2	RS_5	RS_3	RS_6
(A) De-meaned case																			
0.0	0.6	2.8	2.9	1.6	1.5	2.5	2.5	2.2	1.9	2.3	1.8	2.2	1.8	3.2	2.8	2.0	1.6	2.0	1.6
	−0.6	4.0	4.3	4.5	5.0	4.1	4.5	4.7	4.8	5.0	4.8	5.0	4.9	5.6	5.8	4.1	3.8	4.2	4.0
0.5	0.0	5.0	5.9	5.1	5.8	5.1	6.1	6.9	7.8	5.1	5.2	5.5	5.8	8.9	10.8	4.2	4.1	4.6	4.7
	0.6	3.7	4.0	3.7	4.1	3.6	4.0	3.3	3.1	4.4	4.3	4.3	4.0	3.2	3.1	4.1	4.0	3.9	3.8
	−0.6	4.1	4.5	4.3	4.9	4.3	4.5	6.3	6.7	4.4	4.3	4.7	4.7	7.9	8.9	3.0	2.5	3.4	3.0
0.9	0.0	10.9	14.5	6.9	8.8	10.8	14.5	16.1	20.2	5.0	5.0	7.1	7.9	21.2	27.3	3.2	2.8	5.0	5.1
	0.6	16.6	23.7	12.6	18.2	16.7	24.0	19.8	26.5	10.0	12.4	12.9	16.7	25.6	33.4	9.0	11.0	11.5	14.6
	−0.6	7.9	10.0	5.1	6.2	7.9	9.9	13.3	16.1	3.9	3.7	5.3	5.5	16.9	20.7	2.0	1.4	3.1	2.4
(B) De-meaned and de-trended case																			
0.0	0.6	5.8	6.4	1.6	1.4	3.2	3.6	5.0	5.0	3.1	2.9	3.1	3.0	3.2	3.0	2.3	2.2	2.4	2.1
	−0.6	3.7	3.9	4.1	4.5	4.0	4.3	5.0	5.4	3.7	3.7	3.9	3.9	5.0	5.4	3.9	3.8	4.0	4.0
0.5	0.0	6.6	7.7	5.6	6.6	6.4	7.8	9.5	11.2	4.3	4.4	4.9	5.1	10.1	12.0	4.6	4.8	5.3	5.8
	0.6	4.0	4.6	3.6	4.0	3.6	4.2	3.7	3.7	4.3	4.6	4.1	4.4	3.2	3.1	4.1	4.4	3.9	4.2
	−0.6	5.0	5.4	4.0	4.2	4.7	5.1	8.4	8.9	2.8	2.5	3.3	3.0	8.8	9.7	3.2	2.8	3.7	3.4
0.9	0.0	22.3	29.8	10.5	13.4	19.5	26.3	25.9	32.7	3.5	3.4	6.1	6.5	29.8	37.9	4.8	5.0	8.1	9.2
	0.6	19.9	26.9	14.1	19.6	19.3	26.6	22.2	28.4	8.4	9.8	10.7	13.3	24.9	32.3	9.8	12.0	12.6	16.3
	−0.6	16.1	20.9	6.6	7.7	13.4	17.3	20.0	24.1	2.0	1.6	3.3	3.0	24.1	29.3	2.9	2.5	4.7	4.6

Qualitatively similar conclusions are drawn for other values of τ^* and ρ, and from size-adjusted results for other stable and invertible noise processes; these results are available on request. Again we have set $d_t = 0$ because of the invariance properties of the tests considered, and have discarded the first 100 observations to control for initial effects.

Table 6 reports results for $T = 60$ for tests run at the nominal 5% level, again using the relevant finite-sample critical values from Tables 1–3. Results are reported only for the K_1 to K_6, KS_1 to KS_6 and RS_1 to RS_6 tests, as the K_1' to K_3', KS_1' to KS_3', and RS_1' to RS_3' tests are all inconsistent against the $I(0)$–$I(1)$ switch DGP. Results pertaining to de-meaned, and de-meaned and de-trended data are again reported in panels A and B of the tables, respectively.

Summarizing the results in Table 6, in general, the KPSS-based tests obtained from Hansen's (1991) mean score display superior power to the corresponding KS- and RS-based tests, while for the tests derived from Andrews' (1993) maximum statistic, the RS-based tests tend to be more powerful than the corresponding KS-based tests which in turn tend to be more powerful than the corresponding KPSS-based tests. For the tests based on Andrews and Ploberger's (1994) mean-exponential statistic, the KPSS-, KS- and RS-based tests display broadly comparable power properties in the de-meaned case, while in the de-meaned and de-trended case, the KPSS-based tests tend to outperform the corresponding RS-based tests which in turn outperform the corresponding KS-based tests. Finally, it is worth noting that, other things equal, for all the tests power is not necessarily lower in the de-meaned and de-trended case than for the de-meaned case; this phenomenon is also apparent in the simulation results presented in, e.g. Kwiatkowski *et al.* (1992; Table 4, p. 172). Corresponding results for $T = 120$ and 240 are again reported in Taylor (2004), where, as expected, power increases with the sample size throughout, other things equal.

Table 7 reports results for the corresponding Studentized tests (cf. Table 5) based on $K(\tau, m)$ for $m = 1$ and $T = 60$; again results for other values of m and other sample sizes are available on request. Again, roughly the same relative comparisons between the different tests noted above still apply to the modified tests, but in all cases power is reduced relative to Table 4. Taken in tandem with the results reported in section V, this reflects the usual size–power trade-off decision.

Although not reported, we also considered the corresponding $I(1)$–$I(0)$ switch DGP. These experiments yielded very similar results to those observed in Tables 6–7 for the K_4 to K_6, KS_4 to KS_6, and RS_4 to RS_6 tests on switching τ^* for $(1 - \tau^*)$, noting that this model can also be viewed as a process with a switch from $I(0)$ to $I(1)$ at $(1 - \tau^*)$ when the data are taken in reverse order. Similarly, in this case, the K_j', $j = 1, \ldots, 3$, KS_j', $j = 1, \ldots, 3$, and RS_j', $j = 1, \ldots, 3$, tests displayed almost identical rejection frequencies for a given τ^* to the corresponding results for the K_j, $j = 1, \ldots, 3$, KS_j, $j = 1, \ldots, 3$,

TABLE 6

Empirical rejection frequencies of nominal 5% tests against a change in persistence: DGP (5.2), $T = 60$

ρ	τ^*	K_1	K_4	K_2	K_5	K_3	K_6	KS_1	KS_4	KS_2	KS_5	KS_3	KS_6	RS_1	RS_4	RS_2	RS_5	RS_3	RS_6
(A) De-meaned case																			
0.0	0.25	92.2	93.0	86.2	89.2	92.3	93.2	94.0	94.4	75.4	80.9	88.1	90.9	96.2	96.5	78.7	84.0	89.9	92.6
	0.50	93.2	91.8	94.2	92.5	93.6	92.1	93.6	92.1	93.5	90.9	94.5	92.7	94.4	93.1	94.3	92.1	95.4	93.7
	0.75	80.5	76.1	84.4	81.1	81.3	76.8	78.3	73.2	84.3	80.6	83.7	79.8	76.6	71.4	83.8	79.7	83.4	79.3
0.5	0.25	82.5	86.3	74.8	81.4	82.7	86.7	85.4	88.8	63.8	72.4	77.3	83.8	89.4	92.0	67.1	76.0	79.8	86.3
	0.50	82.0	81.3	82.7	80.7	82.7	81.9	83.1	82.4	80.4	76.7	83.6	81.7	85.6	85.0	81.7	78.3	84.8	83.0
	0.75	64.3	59.9	68.4	64.2	65.5	60.8	63.8	59.7	67.2	62.3	68.0	63.3	64.2	61.1	65.8	60.7	66.9	62.6
0.9	0.25	62.0	76.3	53.5	70.7	62.3	76.8	67.0	80.1	44.1	62.0	56.8	74.1	74.8	86.2	47.1	66.5	59.9	78.2
	0.50	60.7	71.0	55.6	65.8	61.4	71.6	65.4	75.3	48.9	57.1	58.5	69.1	72.8	82.5	51.1	61.3	60.7	73.1
	0.75	51.9	62.6	47.2	57.0	52.6	63.2	56.5	67.2	41.0	49.1	49.5	60.7	64.0	75.8	42.9	53.1	51.5	65.0
(B) De-meaned and de-trended case																			
0.0	0.25	88.3	90.8	78.8	82.9	88.1	90.8	87.4	90.0	62.0	65.2	76.7	81.1	88.6	90.9	63.9	69.1	78.1	83.1
	0.50	86.9	87.3	87.1	85.4	88.4	88.3	83.4	84.3	79.8	75.1	83.6	81.9	84.3	85.0	81.1	77.6	84.6	83.5
	0.75	72.4	66.7	77.0	71.9	75.5	69.5	65.6	60.3	71.3	64.9	71.8	65.8	65.9	60.0	72.4	66.3	73.0	67.0
0.5	0.25	79.2	86.6	67.7	76.9	79.0	86.8	78.8	86.3	51.4	58.8	66.0	75.6	80.2	87.4	53.1	62.8	67.5	77.9
	0.50	74.4	80.0	70.0	72.2	75.3	80.4	72.1	78.6	58.8	56.7	66.3	69.3	73.7	79.9	60.7	60.2	68.0	71.8
	0.75	60.6	60.3	57.9	54.9	62.1	61.2	58.1	58.9	49.4	44.5	53.8	51.5	59.2	60.1	51.0	46.9	55.8	53.6
0.9	0.25	70.7	85.8	56.0	75.0	69.7	85.7	72.0	86.3	40.1	57.6	55.5	74.8	73.4	87.5	41.8	61.2	56.8	76.9
	0.50	68.0	83.7	53.3	71.9	67.1	83.6	69.5	84.5	38.5	54.4	52.7	71.9	70.7	85.7	40.3	58.1	54.2	74.3
	0.75	66.0	81.9	50.3	69.5	64.9	81.7	67.3	82.6	35.5	51.5	50.1	69.4	69.0	84.1	37.2	55.4	51.5	71.9

TABLE 7

Empirical rejection frequencies of nominal 5% Studentized tests (m = 1) against a change in persistence: DGP (5.2), T = 60

ρ	τ^*	K_1	K_4	K_2	K_5	K_3	K_6	KS_1	KS_4	KS_2	KS_5	KS_3	KS_6	RS_1	RS_4	RS_2	RS_5	RS_3	RS_6
(A) De-meaned case																			
0.0	0.25	72.9	68.6	54.4	48.3	72.4	68.6	76.2	73.2	29.6	21.0	51.0	44.0	85.5	82.0	29.2	19.0	48.9	38.1
	0.50	82.1	75.8	84.3	78.2	83.5	77.5	82.8	77.4	80.7	71.1	84.4	77.0	87.6	82.2	84.0	74.4	87.2	79.9
	0.75	70.7	62.3	79.3	73.3	73.5	65.0	68.4	60.3	79.4	73.0	78.7	71.9	66.9	58.0	78.5	71.0	77.8	70.5
0.5	0.25	47.5	44.9	32.5	28.2	47.7	45.5	51.4	50.7	15.5	10.8	27.4	22.6	58.2	55.6	12.0	6.9	21.1	15.0
	0.50	53.2	46.1	56.5	47.0	56.0	48.3	55.3	49.7	46.1	32.7	52.9	41.1	58.8	51.2	40.6	26.9	47.2	34.4
	0.75	38.8	29.6	51.2	41.6	42.6	32.3	39.6	31.4	47.5	36.6	47.7	36.9	36.6	28.6	36.8	25.7	37.5	27.0
0.9	0.25	22.6	26.1	14.4	15.3	22.7	26.4	26.8	31.7	6.9	6.1	11.2	11.3	31.4	36.3	4.2	3.2	7.1	6.4
	0.50	23.2	23.5	18.0	15.2	23.7	23.9	27.4	28.8	10.0	6.5	14.5	11.4	31.0	33.0	5.9	3.5	9.2	6.5
	0.75	18.3	17.4	15.4	12.6	18.7	17.8	22.3	22.6	9.5	6.7	12.2	9.7	24.7	27.1	5.1	3.4	7.3	5.7
(B) De-meaned and de-trended case																			
0.0	0.25	73.3	69.3	52.7	44.5	70.2	66.6	73.5	69.7	26.0	15.8	40.6	29.4	78.5	75.1	31.5	21.3	46.3	36.1
	0.50	80.1	74.4	81.5	74.5	83.5	77.9	76.8	71.5	67.9	54.1	73.5	62.4	80.0	74.3	72.6	61.4	77.3	68.2
	0.75	68.9	59.5	77.3	70.2	75.4	67.2	63.1	54.0	70.7	60.6	71.0	61.3	64.5	54.1	72.6	63.8	72.7	64.1
0.5	0.25	49.1	48.5	30.2	24.8	45.7	45.0	49.5	49.2	11.3	6.3	19.0	12.6	55.8	56.2	14.7	9.3	23.3	17.2
	0.50	48.5	44.7	45.4	36.2	51.5	45.9	46.7	45.2	25.3	14.9	32.3	21.5	52.0	49.6	30.6	19.9	37.6	27.1
	0.75	37.7	29.4	40.8	31.7	42.1	33.1	38.2	30.8	29.4	19.2	32.1	21.9	41.1	33.0	32.2	22.3	34.9	25.0
0.9	0.25	32.5	38.8	16.1	16.9	28.6	34.6	33.5	40.1	5.0	3.6	8.7	7.7	39.8	46.8	6.7	5.6	11.4	11.0
	0.50	30.0	35.5	15.5	15.8	26.8	31.5	31.3	37.4	5.1	3.6	8.6	7.2	37.4	43.6	6.9	5.5	11.2	10.3
	0.75	27.7	32.1	13.8	14.4	24.2	28.7	29.7	34.8	4.8	3.5	8.0	6.8	35.1	40.7	6.5	5.5	10.4	9.7

and RS_j, $j = 1, \ldots, 3$, tests for $(1 - \tau^*)$ in Tables 6 and 7. Full details of these experiments are available on request.

VI. Empirical application

In this section, we apply the tests discussed in this paper to the quarterly US inflation rate series originally analysed in Busetti and Taylor (2004). The data are observed for the period 1960Q2–2000Q4. Busetti and Taylor (2004) apply the un-Studentized KPSS-based statistics of section III to these data. Panel A of Table 8 replicates their results for the K_1 to K_3 and K_1' to K_3' tests (first row), together with the results for the KS- and RS-based tests of section IV. Panel B reports results for the corresponding Studentized tests. As in Tables 5 and 7, the Studentized statistics are run with a bandwidth of $m = 1$.

Considering the un-Studentized statistics first, we see that the outcomes of the KS- and RS-based tests are in concert with those of the KPSS-based tests with each of the K_j', KS_j' and RS_j', $j = 1, 2, 3$ tests rejecting H_0 in favour of H_{10}, a change in persistence from $I(1)$ to $I(0)$, at the 1% level. Notice also from these results that each of the K_4 to K_6, KS_4 to KS_6 and RS_4 to RS_6 tests yield outcomes which are significant at the 1% level. Turning to the Studentized results, we see that for all tests the strength of the rejections is much reduced, as expected, given the simulation results in section V. The KPSS-based tests now yield no significant evidence against H_0 even at the 10% level. However, the outcome of KS_1' is significant at the 10% level, while the outcomes of the RS_1' and RS_4 tests are both significant at the 5% level, consistent with the simulation evidence noted in section V for the tests derived from Andrews' (1993) maximum statistic. As discussed in Busetti and Taylor (2004), the estimated breakpoint occurs at 1990Q4.

TABLE 8

Results of persistence change tests for US inflation rate

	$K_1/KS_1/RS_1$	$K_1'/KS_1'/RS_1'$	$K_2/KS_2/RS_2$	$K_2'/KS_2'/RS_2'$	$K_3/KS_3/RS_3$	$K_3'/KS_3'/RS_3'$
(A) Un-Studentized tests						
KPSS-based	16.84*	151.91***	1.80	21.38***	4.08*	72.37***
KS-based	4.70***	12.31***	1.05	3.16***	0.71	3.67***
RS-based	4.80***	9.32***	1.08	3.02***	0.74*	2.61***
(B) Studentized tests						
KPSS-based	2.57	8.79	0.80	2.53	0.45	2.09
KS-based	1.84	2.94*	0.90	1.37	0.47	0.74
RS-based	1.88	2.51**	0.91	1.35	0.48	0.71*

Note: The superscripts *, ** and *** denote rejection of H_0 at the 10%, 5% and 1% levels, respectively, using asymptotic critical values.

TABLE 9

Results of persistence change tests for US inflation rate with estimated level shift at 1982

	$K(0.55)$	$K(0.55)^{-1}$	$KS(0.55)$	$KS(0.55)^{-1}$	$RS(0.55)$	$RS(0.55)^{-1}$
Un-Studentized	0.02	43.54***	0.19	5.27***	0.23	4.30***
Studentized	0.15	6.84**	0.48	2.09**	0.59	1.71**

Note: See Note to Table 8.

The results in Table 8 assume that there are no breaks in the level of the inflation series. Busetti and Taylor (2004) discuss the problem of testing for changes in persistence at an unknown point in the sample under a simultaneous level break. They suggest a two-stage procedure, first estimating the breakpoint, after Bai (1997), according to

$$\hat{\tau} = \arg\min_{\tau \in \Lambda} \sum_{t=1}^{T} \hat{\epsilon}_t(\tau)$$

where $\{\hat{\epsilon}_t(\tau)\}_{t=1}^{T}$ are the OLS residuals from the regression of y_t on $\mathbf{x}_t = (1,\ h_t(\tau))'$, where $h_t(\tau) = \mathbb{I}(t > \lfloor T\tau \rfloor)$. The estimator $\hat{\tau}$ is then used as if it were the true breakpoint and, hence, one simply computes in step two the statistic $K(\hat{\tau})$. The critical values presented for $K(\tau)$ in Tables 2.1 and 8 of Busetti and Taylor (2004) are asymptotically valid for $K(\hat{\tau})$. Exactly the same procedure can be applied to the KS- and RS-based tests, using the statistics $KS(\hat{\tau})$ and $RS(\hat{\tau})$, respectively. All of these tests may be Studentized in the manner outlined in section V.

Applying step one of this procedure to the US inflation data yields $\hat{\tau} = 0.55$, corresponding to a break in level in 1982. The resulting two-stage statistics computed at this estimated break date are reported in Table 9. Each of the un-Studentized tests designed for detecting $I(1)$ to $I(0)$ changes rejects H_0 at the 1% level, while the tests designed to detect $I(0)$ to $I(1)$ changes again provide no evidence against the null. Again the evidence against H_0 is reduced when one considers the Studentized tests although each of the tests for $I(1)$ to $I(0)$ changes still reject H_0 at the 5% level.

VII. Conclusions

In this paper, we have proposed a new set of tests for a change in persistence based on statistics formed from certain functions (namely the maximum, the mean score and the mean-exponential) of ratios of sub-sample maximal recursive-estimates (*KS*) and re-scaled range (*RS*) fluctuation statistics. Asymptotic null distributions of the proposed statistics were derived and associated tables of critical values provided. The consistency of the proposed tests against persistence change processes was demonstrated. A Monte Carlo study comparing the finite sample size and power properties of the proposed tests with their counterparts formed from sub-sample KPSS-type fluctuation tests was conducted. The results suggested that the new tests proposed in this paper provide a very useful complement to the extant KPSS-based tests. In particular, the functions taken of the sequences of ratios behave differently for each of the three fluctuation measures, with an evident and useful size/power trade-off existing between the three. In particular, while the RS- and KS-based tests proposed in this paper display smaller (larger) size distortions against weakly dependent $I(0)$ shocks than the corresponding KPSS-based tests for the mean score (maximum) case, they display lower (higher) power than the KPSS-based tests when there is a change in persistence. Finally, we applied the tests to the US inflation rate. When a simultaneous level break was allowed, the outcomes were consistent with a change in persistence from $I(1)$ to $I(0)$ in the early 1980s. Somewhat weaker evidence of an $I(1)$ to $I(0)$ shift, coupled with a later estimated break date, was found when not allowing for a simultaneous level break.

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