Biometrika (1982), **69**, 1, *pp.* 274–6
Printed in Great Britain

Discontinuous decision processes and threshold autoregressive time series modelling

BY H. TONG

Department of Mathematics, University of Manchester Institute of Science and Technology

SUMMARY

Setting the problem of approximating an underlying nonlinear time series model within the framework of Bayesian decision theory, we demonstrate how the general analysis of discontinuous decision processes developed by Smith, Harrison & Zeeman (1981) leads naturally to a threshold autoregression.

Some key words: Bayesian decision theory; Bounded loss function; Multimodality of expected loss; Nonlinear time series; Threshold autoregression.

1. INTRODUCTION

Let $\{X_n: n = 0, 1, 2, \ldots\}$ denote a discrete parameter time series. Given a realization $\{x_n\}$, we consider the modelling of the time series. Typically, a nonlinear model is called for. For concreteness and simplicity of discussion, we address ourselves to the simplest case, namely

$$E(X_n \mid X_{n-1} = x) = \mu(x)\, x, \tag{1·1}$$

where μ is a 'smooth' function. Suppose that we approximate $\mu(x)$ by θ, a constant, for all x. In subsequent discussion we suppress the argument x whenever this may be done without obscuring the context. Clearly this approximation will incur some errors. On the other hand, as argued by Tong & Lim (1980), we could approximate μ arbitrarily closely by a step function. At first sight, it seems that this could pose a horrendous computational problem. Indeed, this is the case from a purely deterministic point of view. However, we usually approximate the 'true' model with some purpose in mind, e.g. forecasting, control, filtering. Thus, we should really specify what we mean by approximating μ arbitrarily closely. Bayesian decision theory seems a natural approach to this problem. In particular, the recent general results of Smith *et al.* (1981) provide the necessary framework.

2. BAYES DECISION

Since a linear model has been found to be a generally acceptable first approximation, we start with the Bayesian linear model,

$$E(X_n \mid X_{n-1} = x) = \theta x, \tag{2·1}$$

$$\theta \sim N(c, V). \tag{2·2}$$

In order to quantify the closeness of the approximation model (2·1) to the 'true' model (1·1), we introduce the loss function, L, which is conjugate to the Gaussian belief given by (2·2)

$$L(\theta) = h[1 - \exp\{-\tfrac{1}{2}k^{-1}(\theta - \mu)^2\}]. \tag{2·3}$$

Here μ is the most desirable value of θ, k represents the relative tolerance to differences between μ and θ, and h quantifies the maximum loss.

With the explicit quantification of closeness by (2·3), we may now decide whether or not (2·1) is an acceptable approximation of (1·1) by evaluating the expected loss of making the decision. For this, we let D denote the class of possible decisions. The expected loss function, E_V, is defined by

$$E_V(\delta) = \int_{-\infty}^{\infty} L(\theta)\, dF_V(\theta \mid \delta), \quad \delta \in D, \tag{2·4}$$

where $F_V(\theta \mid \delta)$ denotes the distribution of θ given that the decision δ is employed. As a consequence of Gaussian belief and conjugate Gaussian loss, $F_V(\theta \mid \delta)$ is $N(c+\delta, V)$, from which we may write

$$E_V(\delta) = h[1-\{k/(k+V)\}^{\frac{1}{2}} \exp\{-\tfrac{1}{2}(k+V)^{-1}(\delta-d)^2\}], \tag{2·5}$$

where $d = \mu - c$ represents the distance of the desired value of θ from the expected value of δ. The minimizer of $E_V(\delta)$ with respect to $\delta \in D$ is called the Bayes decision. We denote it by δ^*. In Bayesian decision theory, a decision is acceptable if and only if it is a Bayes decision. Following are two results pertinent to nonlinear time series modelling.

(A) If both k and V are unaffected by the decision δ, then δ^* equals d. This implies that no approximation, linear or not, is acceptable, i.e. a Bayes decision. However, Smith *et al.* (1981) have discussed at length the unreasonableness of this attitude in many practical situations.

(B) They have also argued that, in many practical situations, it is not unreasonable to suppose that the uncertainty, V, of belief of the value of θ is an increasing function of δ, that is a bold decision increases the uncertainty of belief. In our context, if we accept that $\mu(x)$ is *a priori* 'smooth' in x, then we would not expect to have to make drastic adjustment in the value of θ. A mathematical expression of this supposition is that V depends on δ via

$$V(\delta) = \alpha + \beta \mid \delta \mid, \tag{2·6}$$

where $\alpha, \beta > 0$. In this case, for $d > 0, \delta^* \in [0, d]$, and by measuring δ in units of $\surd\{k+V(0)\}$, that is $\surd(k+d)$, and on replacing $(k+\delta)^{-\frac{1}{2}}$ by γ we obtain

$$\delta^* = \begin{cases} 0 & \text{for } 0 < d < \{(1+\gamma^2)^{\frac{1}{2}}-1\}\gamma^{-1}, \\ \tfrac{1}{2}[\{4(1+\gamma d)^2+\gamma^4\}^{\frac{1}{2}}-(2+\gamma^2)]/\gamma & \text{otherwise.} \end{cases} \tag{2·7}$$

The most significant implication of the Bayes decision given by (2·7) is that a discontinuous model, i.e. a threshold time series model, is the natural outcome. Next, so long as $\mu(x) - c < \{(1+\gamma^2)^{\frac{1}{2}}-1\}\gamma^{-1}$, the linear model (2·1), (2·2) is as true as the 'true' model (1·1)! We may therefore conclude that the class of threshold time series models introduced by us in a series of papers (Tong & Lim, 1980), is a natural approach to nonlinear time series modelling.

3. Discussion

Smith *et al.* (1981) have considered many other interesting cases which include, for example, a skew belief distribution, a δ-dependent k, a loss in the form of a step function, etc. We leave further exploration of this rich area for the future. Nevertheless, on the basis of our results here, it seems that we are not unjustified in believing that threshold time series modelling is founded on a sound principle.

I am most grateful to Professor Jeff Harrison for introducing me to the key reference before its publication.

References

SMITH, J. Q., HARRISON, P. J. & ZEEMAN, E. C. (1981). The analysis of some discontinuous decision processes. *Europ. J. Oper. Res.* **7**, 30–43.

TONG, H. & LIM, K. S. (1980). Threshold autoregression, limit cycles and cyclical data (with discussion) *J. R. Statist. Soc.* B **42**, 245–92.

[*Received April* 1981. *Revised July* 1981]