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# Testing and Modeling Threshold Autoregressive Processes

RUEY S. TSAY\*

The threshold autoregressive model is one of the nonlinear time series models available in the literature. It was first proposed by Tong (1978) and discussed in detail by Tong and Lim (1980) and Tong (1983). The major features of this class of models are limit cycles, amplitude dependent frequencies, and jump phenomena. Much of the original motivation of the model is concerned with limit cycles of a cyclical time series, and indeed the model is capable of producing asymmetric limit cycles. The threshold autoregressive model, however, has not received much attention in application. This is due to (a) the lack of a suitable modeling procedure and (b) the inability to identify the threshold variable and estimate the threshold values. The primary goal of this article, therefore, is to suggest a simple yet widely applicable model-building procedure for threshold autoregressive models. Based on some predictive residuals, a simple statistic is proposed to test for threshold nonlinearity and specify the threshold variable. Some supplementary graphic devices are then used to identify the number and locations of the potential thresholds. Finally, these statistics are used to build a threshold model. The test statistic and its properties are derived by simple linear regression. Its performance in the finite-sample case is evaluated by simulation and real-world data analysis. The statistic performs well as compared with an alternative test available in the literature. Further applications of threshold autoregressive models are also suggested, including handling heterogeneous time series and modeling random processes with periodic variances whose periodicity is not fixed. The latter phenomenon is commonly encountered in practice, especially in econometrics and biological sciences.

KEY WORDS: Arranged autoregression; Nonlinear time series; Nonlinearity test; Predictive residual; Sunspot.

## 1. INTRODUCTION

A time series  $Y_t$  is a self-exciting threshold autoregressive (TAR) process if it follows the model

$$Y_t = \Phi_0^{(j)} + \sum_{i=1}^p \Phi_i^{(j)} Y_{t-i} + a_t^{(j)},$$
$$r_{j-1} \leq Y_{t-d} < r_j, \quad (1)$$

where  $j = 1, \dots, k$  and  $d$  is a positive integer. The thresholds are  $-\infty = r_0 < r_1 < \dots < r_k = \infty$ ; for each  $j$ ,  $\{a_t^{(j)}\}$  is a sequence of martingale differences satisfying

$$E(a_t^{(j)} | F_{t-1}) = 0,$$
$$\sup_t E(|a_t^{(j)}|^\delta | F_{t-1}) < \infty \text{ a.s. for some } \delta > 2, \quad (2)$$

with  $F_{t-1}$  the  $\sigma$  field generated by  $\{a_{t-i}^{(j)} | i = 1, 2, \dots, j = 1, \dots, k\}$ . Such a process partitions the one-dimensional Euclidean space into  $k$  regimes and follows a linear AR model in each regime. The overall process  $Y_t$  is nonlinear when there are at least two regimes with different linear models. This nonlinear time series model was proposed by Tong (1978, 1983) and Tong and Lim (1980) as an alternative model for describing periodic time series. The model has certain features, such as limit cycles, amplitude dependent frequencies, and jump phenomena, that cannot be captured by a linear time series model. For instance, Tong and Lim (1980) showed that the threshold model is capable of producing asymmetric, periodic behavior exhibited in the annual Wolf's sunspot and Canadian lynx data.

Nevertheless, the TAR model has not been widely used in applications, primarily because (a) it is hard in practice to identify the threshold variable and estimate the associated threshold values, and (b) there is no simple modeling procedure available. The procedure proposed by Tong and Lim (1980) is complex. It involves several computing-intensive stages, and there were no diagnostic statistics available to assess the need for a threshold model for a given data set. The goal of this article, therefore, is to propose a procedure for testing threshold nonlinearity and building, if necessary, a TAR model. The proposed test is simple because it uses only familiar linear regression techniques. The modeling procedure consists of four steps, each informative. The steps can also be used iteratively when the number of regimes  $k$  is large or the degree of nonlinearity is weak.

The article is organized as follows. In Section 2, I discuss some sampling properties of least squares estimates of a TAR model. Least squares estimates are used throughout. Section 3 deals with testing the threshold nonlinearity. An arranged autoregression provides predictive residuals that are used in nonlinearity testing and threshold specification. Asymptotic distribution of the proposed test statistic is given, and the finite-sample performance is evaluated by simulation and analysis of several real data sets. Section 4 suggests some graphics that are informative in locating the values of the thresholds and thus useful in specifying the threshold regimes. Section 5 gives the proposed modeling procedure, and three illustrative examples (including the sunspot data and the Canadian lynx series) are given in Section 6. It is hoped that the article will broaden the use and stimulate further investigation of the TAR model.

For convenience, I refer to Model (1) as a TAR( $k; p, d$ ) model, where  $k$  is the number of regimes separated by

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$k - 1$  nontrivial thresholds  $r_j$ ,  $p$  denotes the AR order, and  $d$  is the threshold lag (called the delay parameter by Tong). The interval  $r_{j-1} \leq Y_{t-d} < r_j$  is the  $j$ th regime of  $Y_t$ . Note that (a) the AR order  $p$  may differ from regime to regime, (b) the TAR model becomes a nonhomogeneous linear AR model if only the noise variances  $\sigma_j^2 = \text{var}(a_t^{(j)})$  are different for different regimes, and (c) the TAR model reduces to a random-level shift model if only the constant terms  $\Phi_0^{(j)}$  are different for different  $j$ . These last two features are of special interest in various applications and are related to outliers and model changes in linear time series analysis. The fact that the TAR models encompass these special cases suggests that the models may have further applications beyond nonlinear time series analysis.

## 2. CONSISTENCY OF LEAST SQUARES ESTIMATES

Since the TAR model is a locally linear model, ordinary least squares techniques are useful in studying the process. I give a brief discussion of some useful results. For a given TAR( $k; p, d$ ) model of (1), denote by  $n_j$  the number of observations of  $Y_t$  that are in the  $j$ th regime  $r_{j-1} \leq Y_{t-d} < r_j$ . Assume that

$$n_j/n \rightarrow c_j \quad \text{in probability,} \quad (3)$$

for all  $j = 1, \dots, k$ , where  $n$  is the total sample size and  $c_j$  is a positive fraction such that  $\sum_{j=1}^k c_j = 1$ . Next, for each regime  $j$ , form the ordinary least squares autoregression of order  $p$  and denote the estimate of  $\Phi_i^{(j)}$  by  $\hat{\Phi}_i^{(j)}$  and the associated  $X'X$  matrix by  $X'X^{(j)}$ . Furthermore, assume that for each  $j$

$$\lambda_{j,\min} \rightarrow \infty \quad \text{a.s.,} \quad \ln \lambda_{j,\max} = o(\lambda_{j,\min}) \quad \text{a.s.,} \quad (4)$$

as  $n \rightarrow \infty$ , where  $\lambda_{j,\min}$  and  $\lambda_{j,\max}$  are the minimum and maximum eigenvalues of  $X'X^{(j)}$  based on sample size  $n$ .

**Theorem 2.1.** Suppose that  $Y_t$  follows the TAR Model (1) with  $a_t^{(j)}$ ,  $n_j$ , and  $X'X^{(j)}$  satisfying (2)–(4), respectively. Then, for given  $k, d$ , and the threshold values  $r_j$ , the ordinary least squares estimates  $\hat{\Phi}_i^{(j)}$  converge to  $\Phi_i^{(j)}$  almost surely.

Under Conditions (1) and (3),  $Y_t$  is a linear autoregression in every regime, with increasing sample size as the total sample size  $n$  goes to infinity. Theorem 2.1 then follows directly from the results of Lai and Wei (1982), where the almost sure convergence of least squares estimates was shown under Conditions (2) and (4) for general linear stochastic regressions. Condition (4) is general, and is satisfied, for instance, when  $Y_t$  is ergodic.

## 3. A TEST FOR THRESHOLD NONLINEARITY

In this section I consider testing threshold nonlinearity. The proposed test is related to the portmanteau test of nonlinearity of Petrucci and Davies (1986), in that it also is based on arranged autoregression and predictive residuals. Nevertheless, the two tests are different in ways in which the special features of the predictive residuals are

exploited. Roughly speaking, the proposed test is a combined version of the nonlinearity tests of Keenan (1985), Tsay (1986), and Petrucci and Davies (1986). It is extremely simple and widely applicable. Its asymptotic distribution under the linear model assumption is nothing but the usual  $F$  distribution.

### 3.1 Arranged Autoregression and Predictive Residuals

Write an AR( $p$ ) regression with  $n$  observations as  $Y_t = (1, Y_{t-1}, \dots, Y_{t-p})\beta + a_t$  for  $t = p + 1, \dots, n$ , where  $\beta$  is the  $(p + 1)$ -dimensional vector of coefficients and  $a_t$  is the noise. I refer to  $(Y_t, 1, Y_{t-1}, \dots, Y_{t-p})$  as a case of data for the AR( $p$ ) model. Then, an arranged autoregression is an autoregression with cases rearranged, based on the values of a particular regressor. For the TAR model (1), arranged autoregression becomes useful if it is arranged according to the threshold variable. To see this, consider case  $k = 2$ . That is, consider the situation of a nontrivial threshold  $r_1$ . For a given TAR(2;  $p, d$ ) model with  $n$  observations, the threshold variable  $Y_{t-d}$  may assume values  $\{Y_h, \dots, Y_{n-d}\}$ , where  $h = \max\{1, p + 1 - d\}$ . Let  $\pi_i$  be the time index of the  $i$ th smallest observation of  $\{Y_h, \dots, Y_{n-d}\}$ . We rewrite the model as

$$\begin{aligned} Y_{\pi_i+d} &= \Phi_0^{(1)} + \sum_{v=1}^p \Phi_v^{(1)} Y_{\pi_i+d-v} + a_{\pi_i+d}^{(1)} \quad \text{if } i \leq s \\ &= \Phi_0^{(2)} + \sum_{v=1}^p \Phi_v^{(2)} Y_{\pi_i+d-v} + a_{\pi_i+d}^{(2)} \quad \text{if } i > s, \end{aligned} \quad (5)$$

where  $s$  satisfies  $Y_{\pi_s} < r_1 \leq Y_{\pi_{s+1}}$ . This is an arranged autoregression with the first  $s$  cases in the first regime and the rest in the second regime. It is useful for the TAR model because it effectively separates the two regimes. More specifically, the arranged autoregression provides a means by which the data points are grouped so that all of the observations in a group follow the same linear AR model. Note that the separation does not require knowing the precise value of  $r_1$ . Only the number of observations in each group depends on  $r_1$ .

To illustrate the potential use of arranged autoregression in studying TAR models, I give the motivation of the proposed test. Consider Model (5). If one knew the threshold value  $r_1$ , then consistent estimates of the parameters could easily be obtained. Since the threshold value is unknown, however, one must proceed sequentially. The least squares estimates  $\hat{\Phi}_v^{(1)}$  are consistent for  $\Phi_v^{(1)}$  if there are sufficiently large numbers of observations in the first regime, that is, many  $i \leq s$ . In this case, the predictive residuals are white noise asymptotically and orthogonal to the regressors  $\{Y_{\pi_i+d-v} \mid v = 1, \dots, p\}$ . On the other hand, when  $i$  arrives at or exceeds  $s$  the predictive residual for the observation with time index  $\pi_{s+1} + d$  is biased because of the model change at time  $\pi_{s+1} + d$ . Here, it is easy to see that the predictive residual is a function of the regressors  $\{Y_{\pi_i+d-v} \mid v = 1, \dots, p\}$ . Consequently, the orthogonality between the predictive residuals and the regressors is destroyed once the recursive autoregression

goes on to the observations whose threshold value exceeds  $r_1$ . Notice that here the actual value of  $r_1$  is not required; all that is needed is the existence of a nontrivial threshold. Based on the aforementioned consideration, one way to test for threshold nonlinearity is to regress the predictive residuals of the arranged autoregression (5) on the regressors  $\{Y_{\pi_i+d-v} \mid v = 1, \dots, p\}$  and use the  $F$  statistic of the resulting regression.

For the arranged autoregression (5), let  $\hat{\beta}_m$  be the vector of least squares estimates based on the first  $m$  cases,  $P_m$  the associated  $X'X$  inverse matrix, and  $x_{m+1}$  the vector of regressors of the next observation to enter the autoregression, namely  $Y_{d+\pi_{m+1}}$ . (Note that the positions of  $d$  and  $\pi_{m+1}$  in the subscript of  $Y$  are interchanged to clarify that  $m + 1$  is a subscript of  $\pi$ .) Then, recursive least squares estimates can be computed efficiently by

$$\begin{aligned} \hat{\beta}_{m+1} &= \hat{\beta}_m + K_{m+1}[Y_{d+\pi_{m+1}} - x'_{m+1}\hat{\beta}_m], \\ D_{m+1} &= 1.0 + x'_{m+1}P_mx_{m+1}, \\ K_{m+1} &= P_mx_{m+1}/D_{m+1}, \end{aligned}$$

and

$$P_{m+1} = \left( I - P_m \frac{x_{m+1}x'_{m+1}}{D_{m+1}} \right) P_m$$

(see Ertel and Fowlkes 1976; Goodwin and Payne 1977), and the predictive and standardized predictive residuals by

$$\hat{a}_{d+\pi_{m+1}} = Y_{d+\pi_{m+1}} - x'_{m+1}\hat{\beta}_m \tag{6}$$

and

$$\hat{e}_{d+\pi_{m+1}} = \hat{a}_{d+\pi_{m+1}}/\sqrt{D_{m+1}}. \tag{7}$$

The predictive residuals can also be used to locate the threshold values once the need for a TAR model is detected, by using various scatterplots designed to show specific features of the TAR model. Details are given in Section 4. Note that the problem considered here is related to the change-point or switching-regression problem, for which voluminous references are available in the literature. For example, see Quandt (1960), Shaban (1980), Pole and Smith (1985), and Siegmund (1988). A key difference, however, is that here the data are serially correlated.

### 3.2 A Nonlinearity Test

I now give details of the proposed nonlinearity test. For fixed  $p$  and  $d$ , the effective number of observations in arranged autoregressions is  $n - d - h + 1$ , with  $h$  defined just before (5). Assume that the recursive autoregressions begin with  $b$  observations so that there are  $n - d - b - h + 1$  predictive residuals available. Do the least squares regression

$$\hat{e}_{\pi_i+d} = \omega_0 + \sum_{v=1}^p \omega_v Y_{\pi_i+d-v} + \varepsilon_{\pi_i+d}, \tag{8}$$

for  $i = b + 1, \dots, n - d - h + 1$ , and compute the

associated  $F$  statistic

$$\hat{F}(p, d) = \frac{(\sum \hat{e}_i^2 - \sum \hat{e}_i^2)/(p + 1)}{\sum \hat{e}_i^2/(n - d - b - p - h)}, \tag{9}$$

where the summations are over all of the observations in (8) and  $\hat{e}_i$  is the least squares residual of (8). The argument  $(p, d)$  of  $\hat{F}$  is used to signify the dependence of the  $F$  ratio on  $p$  and  $d$ .

**Theorem 3.1.** Suppose that  $Y_t$  is a linear stationary AR process of order  $p$ . That is,  $Y_t$  follows Model (1) with  $k = 1$ . Then, for large  $n$  the statistic  $\hat{F}(p, d)$  defined in (9) follows approximately an  $F$  distribution with  $p + 1$  and  $n - d - b - p - h$  df. Furthermore,  $(p + 1)\hat{F}(p, d)$  is asymptotically a chi-squared random variable with  $p + 1$  df.

This theorem can be proved by using the same techniques as Tsay (1986, theorem 1) and Keenan (1985, lemma 3.1). It uses the consistency property of least squares estimates of a linear AR model and a martingale central limit theorem of Billingsley (1961). (Details are omitted.) Note that the asymptotic distribution of the  $\hat{F}(p, d)$  statistic continues to hold if one replaces the standardized predictive residuals  $\hat{e}_i$  by the ordinary predictive residuals  $\hat{a}_i$  of (6). Nevertheless, the standardized predictive residuals appear to be preferable when the sample size is small. For a large sample, ordinary predictive residuals may save some computation.

Since the number and locations of the thresholds are unknown, there exists no (global) most powerful test for threshold nonlinearity. Relative power, feasibility, and simplicity are the major considerations in proposing the  $\hat{F}(p, d)$  statistic. The test can easily be implemented because it requires only a sorting routine and the linear regression method.

### 3.3 Power of the Test and Comparison

I study (via simulation and analysis of some well-known data sets) the power of the  $\hat{F}(p, d)$  statistic in detecting the threshold nonlinearity. I also compare it with the Petruccielli and Davies (1986) portmanteau test. Various threshold lags are used for each real data set. I choose  $b = (n/10) + p$ , with  $n$  the sample size and  $p$  the fitted AR order. To compute the portmanteau test, the standardized residual of (7) is further normalized by

$$z_{d+\pi_{m+1}} = \hat{e}_{d+\pi_{m+1}}/\sqrt{s_m^2}, \tag{10}$$

where  $s_m^2$  is an estimate of the residual variance  $\sigma^2$  computed recursively by

$$\begin{aligned} s_m^2 &= \text{RSS}_m/(m - p - 1), \\ \text{RSS}_{m+1} &= \text{RSS}_m + \hat{e}_{d+\pi_{m+1}}^2, \end{aligned} \tag{11}$$

where RSS denotes residual sum of squares. Asymptotically,  $z_t$  is standard Gaussian, so the  $P$  value of the portmanteau test can be evaluated by using an invariant principle (see Petruccielli and Davies 1986).

Table 1 summarizes the test results for some real data

Table 1. Nonlinearity Tests of Some Real Data

Test	Threshold lags, <i>d</i>										
	1	2	3	4	5	6	7	8	9	10	11
Series C: $p = 2, n = 226$											
$F_{3,197}$	.053	.209									
$P$	.869	.976									
Series A: $p = 7, n = 197$											
$F_{8,156}$	.978	1.100	.637	1.177	.800	.926	.599				
$P$	.441	.597	.835	.157	.188	.591	.868				
Logged lynx data: $p = 9, n = 114$											
$F_{10,75}$	2.426	2.484	2.038	1.579	.532	.677	1.460	1.775	2.171		
$P$	.1070	.0000	.0001	.3641	.3256	.2323	.2640	.7891	.0612		
Original lynx data: $p = 11, n = 114$											
$F_{12,69}$	1.41	1.46	3.59	2.08	2.90	1.58	1.68	2.22	1.54	1.17	1.71
$P$	.000	.000	.004	.270	.267	.910	.950	.257	.100	.049	.451
Sunspot series, 1700–1979: $p = 11, n = 280$											
$F_{12,218}$	3.10	10.55	3.89	1.85	1.98	2.73	2.20	2.08	1.93	.91	1.84
$P$	.000	.607	.573	.399	.195	.031	.790	.069	.217	.316	.171

NOTE:  $F_{u,v}$  denotes the proposed  $F$  statistic with  $u$  and  $v$  df, and  $P$  is the  $P$  value of the Petrucci and Davies (1986) portmanteau test. Series A and C are from Box and Jenkins (1976).

sets consisting of Series A and C of Box and Jenkins (1976), and the Canadian lynx and Wolf's sunspot data. Series A and C are known to be linear, whereas the others are believed to be nonlinear. The AR orders used are those commonly employed in the literature. From the table, I make the following observations. (a) Both the proposed  $\hat{F}(p, d)$  statistic and the portmanteau test clearly declare Series A and C to be linear. (b) The  $\hat{F}(p, d)$  statistic suggests that the lynx series (both the original and logged) and the sunspot data are nonlinear. (c) On the other hand, the results of the portmanteau test are mixed, especially for the sunspot series. That result depends heavily on the threshold lag (or the delay parameter) used. This observation agrees with the simulation results of Petrucci and Davies (1986).

Table 2 gives the empirical frequencies of rejecting a linear process based on 1,000 realizations and 1% and 5% critical values. The model used in the simulation is a TAR(2; 1, 1), with parameters  $(\Phi_0^{(1)}, \Phi_1^{(1)}, \Phi_0^{(2)}, r_1, \sigma_1^2, \sigma_2^2) = (1.0, .5, 1.0, 1.0, 1.0, 1.0)$  and  $\Phi_1^{(2)}$  given in the table. The sample sizes used are 50 and 100. For each realization of sample size  $n$  in the simulation,  $n + 200$  observations were generated and the first 200 values were

discarded, to reduce any effect of the starting value (0) in generating a TAR model. In the test,  $p = 1$  and  $d = 1$  were used. Again,  $b = (n/10) + p$ . From the table, it is clear that the proposed  $F$  statistic is more powerful than the portmanteau test in detecting threshold nonlinearity, except for the case  $\Phi_1^{(2)} = .0$ . For the linear models, that is,  $\Phi_1^{(2)} = .5$ , the  $F$  statistic does not result in large Type I error.

Table 3 shows results corresponding to those of Table 2 but with threshold value  $r_1 = .0$  and constant terms  $\Phi_0^{(j)} = .0$  for  $j = 1, 2$  both in the data-generating and testing. Since the constant term is related to the level of a process, it is important to see its effect on the testing. The  $F$  statistic again seems to be more powerful than the portmanteau test except for  $\Phi_1^{(2)} = -2.0$ . Based on the results of real examples and simulations, the proposed  $F$  statistic generally outperforms the portmanteau test in detecting threshold nonlinearity. It is relatively insensitive to the change in the threshold lag and is often more powerful than the portmanteau test. The portmanteau test, however, is not universally dominated by the  $F$  statistic. This is in agreement with the nonexistence of a global optimal test.

Table 2. The Empirical Frequencies of Rejecting a Linear Model Based on 1,000 Replications of a TAR(2; 1, 1) Model

$\Phi_1^{(2)}$	1%				5%				
	$F$	$P$	$F$	$P$	$F$	$P$	$F$	$P$	
	$n = 100, p = 1, d = 1$				$n = 50, p = 1, d = 1$				
-2.0	1,000	1,000	1,000	1,000	-2.0	1,000	907	1,000	977
-1.0	1,000	928	1,000	978	-1.0	917	584	988	784
-.5	836	589	943	790	-.5	416	254	672	478
.0	87	121	237	250	.0	35	74	120	170
.5	6	10	26	39	.5	9	20	38	57

NOTE: The parameters used are  $(\Phi_0^{(1)}, \Phi_1^{(1)}, \Phi_0^{(2)}, r_1, \sigma_1^2, \sigma_2^2) = (1.0, .5, 1.0, 1.0, 1.0, 1.0)$ , and  $F$  and  $P$  denote the  $F$  statistic and the portmanteau test.

Table 3. The Empirical Frequencies of Rejecting a Linear Model Based on 1,000 Replications of a TAR(2; 1, 1) Model

$\Phi_1^{(2)}$	1%				5%				
	$F$	$P$	$F$	$P$	$F$	$P$	$F$	$P$	
	$n = 100, p = 1, d = 1$				$n = 50, p = 1, d = 1$				
-2.0	740	945	856	995	-2.0	418	456	612	760
-1.0	672	539	804	820	-1.0	357	167	539	414
-.5	535	246	701	519	-.5	251	71	426	232
.0	229	56	374	195	.0	93	27	207	95
.5	19	23	59	72	.5	18	16	59	78

NOTE: The parameters used are  $(\Phi_0^{(1)}, \Phi_1^{(1)}, \Phi_0^{(2)}, r_1, \sigma_1^2, \sigma_2^2) = (.0, .5, .0, .0, 1.0, 1.0)$ . The constant term is omitted.  $F$  and  $P$  denote the  $F$  statistic and the portmanteau test.

#### 4. SPECIFYING THE THRESHOLD VARIABLE

##### 4.1 Selecting the Delay Parameter $d$

A major difficulty in modeling TAR models is the specification of the threshold variable, which plays a key role in the nonlinear nature of the model. For Model (1), the specification amounts to selection of the delay parameter  $d$ . Tong and Lim (1980) used the Akaike information criterion (Akaike 1974) to select  $d$  after choosing all of the other parameters. I propose a different procedure that selects  $d$  before locating the threshold values. The proposed method is motivated by the performance of the  $F$  statistic in analyzing real data. It assumes that the AR order  $p$  is given. For a given TAR process and an AR order  $p$ , one selects an estimate of the delay parameter, say  $d_p$ , such that

$$\hat{F}(p, d_p) = \max_{v \in S} \{\hat{F}(p, v)\}, \quad (12)$$

where  $\hat{F}(p, v)$  is the  $F$  statistic of (9), the subscript  $p$  signifies that the estimate of  $d$  may depend on  $p$ , and  $S$  is a set of prespecified positive integers, that is, a collection of possible values of  $d$ . For simplicity, assume that all of the test statistics  $\hat{F}(p, v)$  of (12) have the same degrees of freedom. This can be achieved by a proper selection of the starting point  $b$  of the recursion [see (8)]. When the degrees of freedom are different, one may compute the  $P$  values of the  $F$  statistics and select  $d_p$  based on the minimum of the resulting  $P$  values.

Note that the choice of  $d_p$  in (12) is somewhat heuristic. It is based on the idea that if TAR models are needed, then one might start with a delay parameter that gives the most significant result in testing for threshold nonlinearity. A more cautious data analyst may wish to try several values of  $d$ , such as those corresponding to the maximum and the second maximum of  $\hat{F}(p, d)$  in (12).

Table 1 provides some examples for the proposed method with  $S = \{1, \dots, p\}$ . For the sunspot series  $d_{11} = 2$ , and for the Canadian lynx data  $d_9 = 2$ . It is interesting to note that for the sunspot series  $d_p = 2$  for each  $p$  from 2 to 15, suggesting that the selection could be stable with respect to the AR order  $p$ . In general,  $d_p$  might vary with  $p$ , which is usually unknown. In this case, one may start with a reasonable AR order  $p$ , as suggested by some identification statistics such as the partial autocorrelation function of  $Y_t$ , and refine the order later if necessary. Details are given in the next section.

##### 4.2 Locating the Values of Thresholds

For a TAR model, special care is needed in estimating the threshold  $r_j$ 's. To see this, assume that  $k = 2$  and the true value of  $r_1$  satisfies  $Y_{\pi_j} < r_1 < Y_{\pi_{j+1}}$ . Then, any value in the interval  $[Y_{\pi_j}, Y_{\pi_{j+1}}]$  is as good as the other in providing an estimate of  $r_1$ , because they all give the same fitting results for a specified TAR model. Therefore, how to select an estimate of  $r_1$  with nice finite-sample properties from infinitely many possible values remains an open problem. In general, one may provide an interval estimate for each of the threshold values or use sample percentiles as

point estimates. I use the latter. That is, I adopt the approach of Tong and Lim (1980) by considering the empirical percentiles as candidates for the threshold values. But instead of prespecifying a set of finite numbers of sample percentiles to work with, I search through the percentiles to locate the threshold values. The only limitation is that a threshold is not too close to the 0th or 100th percentile. For these extreme points there are not enough observations to provide an efficient estimate.

The methods proposed to locate the thresholds, hence the partitions of the Euclidean space, are scatterplots of various statistics versus the specified threshold variable. The statistics used show the special features of the TAR model. Although the graphics are not formal testing and estimating statistics, they do provide useful information in locating the thresholds. The plots used are (a) the scatterplot of the standardized predictive residuals of (7) or the ordinary predictive residuals of (6) versus  $Y_{t-d_p}$ , and (b) the scatterplot of  $t$  ratios of recursive estimates of an AR coefficient versus  $Y_{t-d_p}$ . The rationale of each of the plots is discussed in the following, whereas illustrative examples are deferred to the applications section.

In the framework of arranged autoregression, the TAR model consists of various model changes that occur at each threshold value  $r_j$ . Therefore, the predictive residuals are biased at the threshold values. A scatterplot of the standardized predictive residuals versus the threshold variable thus may reveal the locations of the threshold values of a TAR model. On the other hand, for a linear time series the plot is random, except for the beginning of the recursion. This scatterplot is closely related to the traditional on-line residual plot for quality control. I use the scatterplot because it tells the locations of the threshold values directly. In practice, I have found the plot informative in TAR modeling, especially for TAR models in which the only differences between different regimes are the variances  $\sigma_j^2$ .

To motivate the use of a scatterplot of recursive  $t$  ratios of an AR coefficient versus the threshold variable, it is best to begin with a linear time series. In this case, the  $t$  ratios have two functions: (a) they show the significance of that particular AR coefficient, and (b) when the coefficient is significant the  $t$  ratios gradually and smoothly converge to a fixed value as the recursion continues. Next, consider the simple TAR model

$$\begin{aligned} Y_t &= \Phi_1^{(1)} Y_{t-1} + a_t^{(1)} & \text{if } Y_{t-d} \leq r_1 \\ &= \Phi_1^{(2)} Y_{t-1} + a_t^{(2)} & \text{if } Y_{t-d} > r_1, \end{aligned} \quad (13)$$

where  $\Phi_1^{(1)}$  and  $\Phi_1^{(2)}$  are different. Let  $\hat{\Phi}_1$  be the recursive estimate of the lag-1 AR coefficient in an arranged autoregression as in (5). By Theorem 2.1, the  $t$  ratios of  $\hat{\Phi}_1$  behave exactly as those of a linear time series before the recursion reaches the threshold value  $r_1$ . Once  $r_1$  is reached, the estimate  $\hat{\Phi}_1$  starts to change and the  $t$  ratio begins to deviate. The pattern of gradual convergence of  $t$  ratios is destroyed. In effect, the  $t$  ratio starts to turn and, perhaps, changes direction at the threshold value. For Model (13),  $\hat{\Phi}_1$  begins to change when  $Y_{t-d}$  reaches

$r_1$  and eventually is a compromise between  $\Phi_1^{(1)}$  and  $\Phi_1^{(2)}$ . This behavior also appears in the associated  $t$  ratios showing information on the value of  $r_1$ . In general, it is easy to see that the change in  $t$  ratio is substantial when the two AR coefficients are substantially different.

To gain insight into the  $t$  ratio plots under the condition of no model changes in a series, a simulation study was conducted. One thousand realizations of a Gaussian AR(1) model  $Y_t = .7Y_{t-1} + a_t$  were generated. As before, 300 points of  $Y_t$  were generated with  $Y_0 = 0$  for each realization, but only the last 100 observations were used as data points. The arranged autoregression was then fitted with  $p = 1$  and  $d = 1$ . Again, use  $b = (n/10) + p = 11$  data points to initiate a recursion. Denote the  $t$  ratio of  $\hat{\Phi}_1$  at the time point  $s$  by  $T_s$ , and define the percentage change in the  $t$  ratio as  $c_s = |T_{s+1} - T_s|/T_s \times 100\%$ . Let  $C(\alpha)$  be the empirical percentage that  $c_s \geq \alpha\%$ .  $C(15)$  and  $C(10)$  were then counted. Since the  $t$  ratio is relatively unstable at the beginning of the recursion, the percentage changes were tabulated in three different time intervals, namely (13, 100), (36, 100), and (51, 100). The following results were obtained: (a) for the time interval (13, 100),  $C(15) = 14.0\%$  and  $C(10) = 21.9\%$ ; (b) for the interval (36, 100),  $C(15) = 2.7\%$  and  $C(10) = 6.5\%$ ; (c) for the interval (51, 100),  $C(15) = .7\%$  and  $C(10) = 2.4\%$ . From these results, it is seen that the  $t$  ratios are stable except for the beginning of the recursion. Next, to check the possibility that the  $t$  ratio may change its direction, define that there is an *up-turn* at time  $s$  if  $T_{s-1} > T_s$  and  $T_s < T_{s+1} \leq T_{s+2}$ , and there is a *down-turn* at time  $s$  if  $T_{s-1} < T_s$  and  $T_s > T_{s+1} \geq T_{s+2}$ . Based on these definitions, a 5.9% down-turn rate was obtained, but only a single up-turn occurred. The results again indicate that the  $t$  ratios behave smoothly when there is no model change.

Some remarks on the  $t$ -ratio plot are of interest. First, the  $t$ -ratio plot of the constant term  $\hat{\Phi}_0$  is important because it signifies level changes. Second, the usefulness of the plot is not limited to the case of a single threshold value. In fact, the previous discussion applies to each threshold as long as the sample size in every regime is reasonable. Third, the technique can be used iteratively. For instance, one may drop the observations in the first regime to detect the second threshold. (See Sec. 6, Ex. 3, for an illustration.) Fourth, Haggan, Heravi, and Priestley (1984) considered scatterplots of recursive AR estimates in studying nonlinear time series. But their estimates are not based on arranged autoregression; their model is the state-dependent model of Priestley (1980), which, as mentioned by those authors, is not particularly useful in handling the TAR model. Finally, since the ordered  $Y_{t-d}$  are not equally spaced, it is often helpful in a scatterplot to omit some data points that have relatively large values in  $Y_{t-d}$ . In this article the last  $b = (n/10) + p$  points in all of the scatterplots are omitted.

## 5. MODELING TAR MODELS

In this section I use the results of previous sections to propose a procedure for modeling TAR models. The procedure consists of several steps, but each step is relatively

simple as compared with those outlined by Tong and Lim (1980). It is hoped that this procedure may help exploit the potential of TAR models in application. The procedure is as follows.

Step 1. Select the AR order  $p$  and the set of possible threshold lags  $S$ .

Step 2. Fit arranged autoregressions for a given  $p$  and every element  $d$  of  $S$ , and perform the threshold nonlinearity test  $\hat{F}(p, d)$ . If the nonlinearity of the process is detected, select the delay parameter  $d_p$  by the method of Section 4.1.

Step 3. For given  $p$  and  $d_p$ , locate the threshold values by using the scatterplots of Section 4.2.

Step 4. Refine the AR order and threshold values, if necessary, in each regime by using linear autoregression techniques.

In Step 1, the AR order  $p$  may be selected by considering the partial autocorrelation function (PACF) of  $Y_t$  or some information criteria such as the Akaike information criterion (AIC). I prefer PACF over information criteria because (a) PACF often provides guidance for a reasonable value of  $p$ , (b) the information criteria could be misleading when the process is indeed nonlinear, and (c) the AR order can be refined, if desired, at Step 4. Furthermore, since high-order AR models could provide reasonable approximations to a nonlinear model, it is hoped that the selection of  $p$  in Step 1 can reflect such approximations. I believe that PACF can show this property better because it imposes no penalty on high-order terms. On the other hand, information criteria are designed to find the best linear model for a time series and tend to penalize high-order terms. The set  $S$  of possible threshold lags may be  $\{1, \dots, p\}$  for a given  $p$ . It can also include seasonal lags when there is seasonality in the process. In Step 3,  $t$  ratios of various AR coefficients can be examined as long as the AR coefficients are significant. Scatterplots of insignificant AR coefficients are usually not informative. The model refinement at Step 4 may rely on information criteria such as AIC because of the linear nature of the TAR model. For details of using AIC in modeling TAR models, see Tong and Lim (1980).

## 6. APPLICATIONS

I now apply the proposed procedure and statistics to some real examples. The sunspot and logged lynx series are included because the procedure suggests different threshold models from those now available. Residual analysis and AIC are used to demonstrate that the models selected by the procedure are indeed better than those previously specified. A third data set that had previously been regarded as a periodic AR process is used to illustrate the potential use of TAR models in handling heterogeneous time series. It also demonstrates iterative use of the procedure.

*Example 1.* I begin with the annual sunspot data from 1700 to 1979, given by Tong (1983, p. 280). The series consists of 280 observations and is known to exhibit asym-

metric cyclic behavior. Various linear and nonlinear models have been proposed for this process, and some of them appear to be reasonable. For instance, the TAR models suggested by Tong and Lim (1980) can produce asymmetric limit cycles similar to those actually observed. In general, for this series it seems that different data spans would suggest different models. I use all of the 280 observations in model building.

As shown in Sections 3 and 4, based on  $p = 11$  the proposed  $F$  statistic confirms that the process is nonlinear and selects  $Y_{t-2}$  as the threshold variable. Therefore, I start with Step 3. Figure 1 gives the scatterplot of the  $t$  ratios of the lag-2 AR coefficient versus ordered  $Y_{t-2}$ . From the plot, it is clear that the  $t$  ratio is significant and changes its direction twice: once near  $Y_{t-2} = 35$  and again near  $Y_{t-2} = 72$ , suggesting that there are two nontrivial thresholds. That is, there are three regimes for the process. An examination of the actual values suggests that the possible estimates of  $r_1$  are  $\{34.0, 34.5, 34.8, 35.0, 35.4, 35.6, 36.0\}$  and those of  $r_2$  are  $\{70.0, 70.9, 73.0, 74.0\}$ . This step substantially simplifies the complexity in modeling the TAR model because it effectively identifies the number and locations of the thresholds. Notice that the possibility of more than one threshold in this series was noted by Hong (1983, pp. 256–257). Finally, in Step 4 I use AIC to refine the threshold values and AR orders. The final threshold values are  $r_1 = 34.8$  and  $r_2 = 70.9$ . The AR orders are 11, 10, and 10, and the numbers of observations are 116, 91, and 62. Details of the model are given in Table 4. The table also gives the ACF of the standardized residuals of the model, as well as the PACF of the squared standardized residuals. Both ACF and PACF fail to indicate any model inadequacy. In Table 4, many of the AR coefficients are small as compared with the corresponding standard errors, especially for those in the first regime. Nevertheless, the small coefficients are not 0, based on AIC. This shows that the major difficulty in analyzing the sunspot data comes from the first regime. Further analysis of this regime might be useful.

Using the data from 1700 to 1920, Tong (1983, p. 241) specified a TAR model for the sunspot series. Tong's

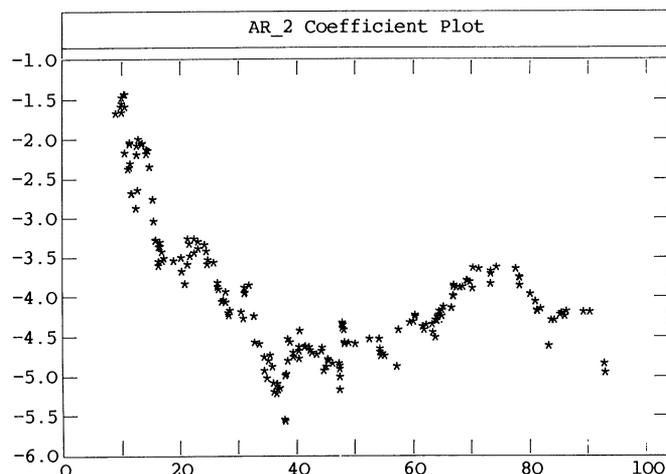


Figure 1. Scatterplot of Recursive  $t$  Ratios of the Lag-2 AR Coefficient Versus Ordered  $Y_{t-2}$  for the Sunspot Data. The X axis is  $Y_{t-2}$ .

model uses  $Y_{t-3}$  as the threshold variable with a threshold 36.6. As a rough comparison, I refit both Tong's and the previously specified model to this shorter data span. For Tong's model, I obtained close results, with an overall AIC 1,083.8. For my model, the overall AIC is 1,064.1, which is substantially smaller. In addition, the PACF of the squared standardized residuals of Tong's model has a value .16 at lag 2, which is significant. But the two TAR models have a similar first threshold that (from Fig. 1) is the most significant. This demonstrates that the proposed procedure can handle multiple thresholds in a direct manner. There is no need to assume knowledge of the number of thresholds.

*Example 2.* In this example I give a TAR model for the logged Canadian lynx data. Again, this data set has been extensively analyzed. See Lim (1987) for a summary and discussion. Since there are only 114 observations, I start with  $p = 3$  and  $S = \{1, 2, 3\}$  at Step 1 of the proposed modeling procedure. The  $F$  statistics of the nonlinearity test are 4.70, 6.13, and 4.46, respectively. Thus  $p = 3$  and  $d = 2$  were tentatively entertained. To specify the thresholds at Step 3, the recursive  $t$  ratios of the lag-2 AR coefficient are not helpful because the  $t$  ratios stay between  $-2$  and  $2$  throughout the range of  $Y_{t-2}$ , except for a few points at the end. Two other scatterplots are useful, however. Figure 2 shows the  $t$ -ratio plot of the lag-1 AR coefficient, from which a threshold with  $r_1 = 2.4$  is clearly seen. The plot also shows a large jump around  $Y_{t-2} = 2.6$ . But this point is not treated as a threshold for two reasons. First, the jump is basically due to three points. Second, there are only few observations with  $Y_{t-2}$  between 2.4 and 2.6, making a separation difficult to estimate. Figure 3 gives the scatterplot of the predictive residuals of (6) versus the ordered  $Y_{t-2}$ . From the plot, it can be seen that the predictive residuals start to deviate around  $Y_{t-2} = 3.1$ , suggesting yet another threshold value. [The deviation becomes much more clear when a horizontal line of .0 is drawn across the plot. Also, Tsay's (1988) technique of detecting a step change in residual variances could be used to help read the plot.] Consequently, there are two possible thresholds for the process: One is about 2.4 and the other about 3.1. Again, I use AIC in Step 4 to refine the model and obtain two thresholds ( $r_1 = 2.373$  and  $r_2 = 3.154$ ). The AR orders are 1, 7, and 2, whereas the numbers of observations are 21, 42, and 45. The final TAR model is

$$\begin{aligned}
 Y_t &= .083 + 1.096Y_{t-1} + a_t^{(1)}, & Y_{t-2} \leq 2.373 \\
 &= .63 + .96Y_{t-1} - .11Y_{t-2} + .23Y_{t-3} - .61Y_{t-4} \\
 &\quad + .48Y_{t-5} - .39Y_{t-6} + .28Y_{t-7} + a_t^{(2)}, & 2.373 < Y_{t-2} \leq 3.154 \\
 &= 2.323 + 1.530Y_{t-1} - 1.266Y_{t-2} + a_t^{(3)}, & 3.154 < Y_{t-2}.
 \end{aligned}$$

The residual variances are .015, .025, and .053, respectively. In model checking, the ACF and PACF of the standardized residuals and squared standardized residuals

Table 4. A TAR Model for the Annual Sunspot Series 1700-1979

Regimes	Lags												
	0	1	2	3	4	5	6	7	8	9	10	11	12
AR Coefficients													
1	3.14	1.86	-1.36	.06	.04	.06	-.04	.11	.04	.06	-.01	-.03	
2	11.4	1.06	-.03	-.70	.35	-.12	-.02	.14	-.19	.02	.27		
3	1.01	.63	.14	-.12	-.01	-.14	.10	.31	-.46	.21	.19		
ACF of standardized residuals													
		-.05	.00	.04	-.04	-.02	-.02	.06	.03	.03	-.03	-.05	.03
PACF of squared standardized residuals													
	.06	.05	.02	-.04	-.02	.00	.02	.04	-.04	.14	-.02	.06	

NOTE: The threshold variable is  $Y_{t-2}$  with thresholds  $r_1 = 34.8$  and  $r_2 = 70.9$ . The AR orders are 11, 10, and 10. The numbers of observations are 116, 91, and 62, and the residual variances are 173.3, 123.7, and 84.5. The overall AIC = 1,379.4.

of the model all fail to suggest any model inadequacy. Nevertheless, there is a slight deviation from symmetry in the histogram of the standardized residuals.

A TAR model for the lynx data was given by Tong (1983, p. 190) that has  $AIC = -337.6$ , whereas the AIC of the aforementioned model is  $-347.7$ . In comparing the two models, it is interesting to note that the single threshold obtained by Tong is 3.116, which is very close to the second threshold obtained here. In fact, there is only one observation, 3.142, between 3.116 and 3.154. Thus the two approaches arrive at a similar conclusion. On the other hand, the analysis of Haggan et al. (1984) of the logged lynx data indicates that there is a possible threshold around 2.2 (of  $Y_{t-1}$  rather than  $Y_{t-2}$ ). This appears to agree with my results. Therefore, for the lynx data the proposed approach is able to capture features previously noticed in the literature.

*Example 3.* Finally, I analyze a process of hourly attic temperatures, consisting of 251 observations. The data were obtained from the Twin Rivers Project conducted by the Princeton University Center for Environmental Studies, beginning May 26, 1976. A preliminary analysis shows that there are several potential outlying observations in the process. For simplicity, I have adjusted three obser-

vations to reduce the sizes of three relatively large residuals. The adjustments are from 102.5, 84.1, and 91.5 to 97.9, 81.0, and 88.0 at  $t = 20, 110,$  and  $112,$  respectively. A linear AR(9) was fitted to the data, and the residual ACF and PACF appear to be clean. Nevertheless, the PACF of the squared residuals assumes values .22 and .15 at lags 1 and 3, and the residual plot shows some heteroscedasticity. The latter feature is understandable in light of the periodic nature of outside temperature that apparently influenced the attic temperature. In view of this, a periodic autoregression or a seasonal time series model may be useful. But since the periodicity is not fixed throughout the process, further analysis is needed.

Following the procedure of Section 5, select  $p = 9$  and  $S = \{1, \dots, 9\}$  and perform a nonlinearity test. Using the predictive residuals of (6), the  $F$  statistics are 5.92, 5.59, 5.18, and 4.69, respectively, for  $d = 1, 2, 3,$  and  $4.$  As compared with an  $F$  distribution with 10 and 198 df, these results are highly significant. Thus I tentatively specify a TAR model with  $p = 9$  and  $d = 1.$  Figure 4 shows the  $t$ -ratio plot of the constant term in an arranged AR(9) regression with  $d = 1,$  that is, the  $t$  ratio of constant term versus  $Y_{t-1}.$  From the plot, the  $t$  ratios are significant and contain two major changes occurring approximately at  $Y_{t-1} = 70$  and  $82.$  The second change appears to be rel-

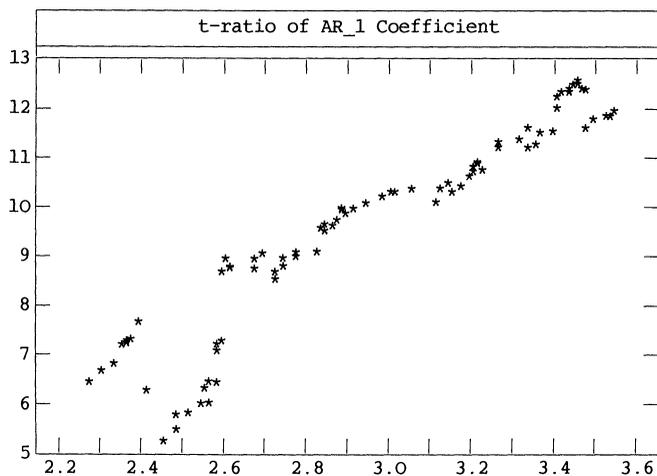


Figure 2. Scatterplot of Recursive  $t$  Ratios of the Lag-1 AR Coefficient Versus Ordered  $Y_{t-2}$  for the Logged Canadian Lynx Data. The X axis is  $Y_{t-2}.$

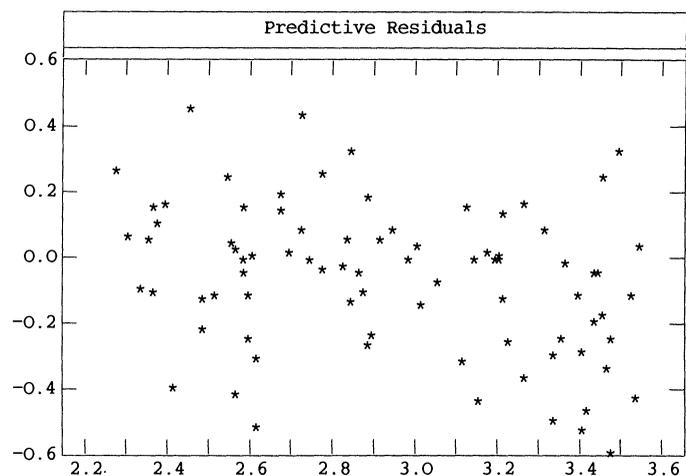


Figure 3. Scatterplot of Predictive Residuals Versus Ordered  $Y_{t-2}$  for the Logged Canadian Lynx Data. The X axis is  $Y_{t-2}.$

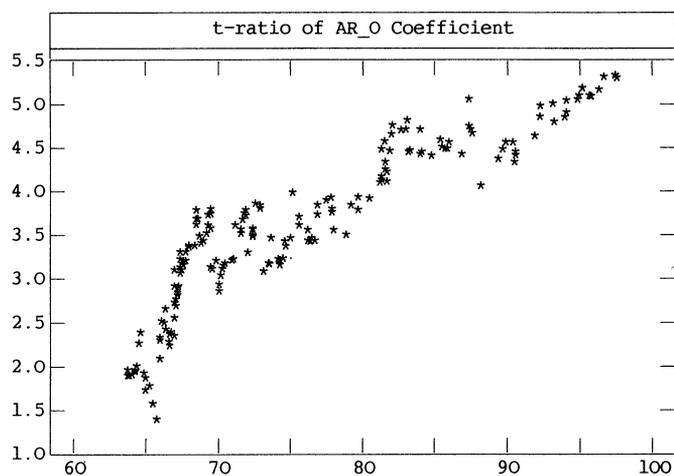


Figure 4. Scatterplot of Recursive  $t$  Ratios of the Constant Term Versus Ordered  $Y_{t-1}$  for Example 3. The  $X$  axis is  $Y_{t-1}$ .

atively small and I come back to it later. Finally, after some refinement, we arrive at a TAR model with two threshold values ( $r_1 = 69.3$  and  $r_2 = 83.0$ ). The AR orders are 7, 4, 6; the numbers of observations are 95, 75, 74; and the residual variances are .913, 2.386, 2.984. Details of the TAR model are

$$\begin{aligned}
 Y_t &= 7.53 + 2.51Y_{t-1} - 1.86Y_{t-2} + .02Y_{t-3} + .18Y_{t-4} \\
 &\quad + .06Y_{t-5} + .17Y_{t-6} - .17Y_{t-7} + a_t^{(1)}, \quad Y_{t-1} \leq 69.3 \\
 &= 8.40 + 1.69Y_{t-1} - .51Y_{t-2} \\
 &\quad - .57Y_{t-3} + .29Y_{t-4} + a_t^{(2)}, \quad 69.3 < Y_{t-1} \leq 83.0 \\
 &= 11.91 + 1.50Y_{t-1} - .68Y_{t-2} + .11Y_{t-3} \\
 &\quad + .43Y_{t-4} - .84Y_{t-5} + .34Y_{t-6} + a_t^{(3)}, \quad 83.0 < Y_{t-1}.
 \end{aligned}$$

The overall AIC of the model is 177.5. In model checking, the problems that appear in a linear AR(9) fit are no longer apparent. The ACF and PACF of the standardized resid-

uals are all clean. The first four PACF's of the squared standardized residuals are .10, .2,  $-.1$ , and .05, respectively. Note that the residual variance of regime 1 is much smaller than those of the other two regimes. This partially explains the heteroscedasticity observed in the linear model.

As mentioned earlier, the second threshold is less pronounced in Figure 4; this sometimes happens in application. An iterative procedure might be useful. I use this last example to illustrate the iteration. After locating the first threshold value, one may drop those cases of data in the first regime and carry out the recursive estimation of arranged autoregression, using the remaining cases to test for the need of a second threshold. For the attic temperature,  $\hat{F}(9, 1) = 2.32$  after removing those cases of data with  $Y_{t-1} \leq 69.3$ . Compared with an  $F$  distribution with 10 and 135 df, the test is significant at the 5% level but not at the 1% level. Similarly, one may also confirm the threshold location by using the reduced data set. Figure 5 gives the scatterplot of the standardized predictive residuals of the reduced data set of Example 3, from which  $r_2 = 82$  seems reasonable because there is an apparent difference in variance.

### 7. CONCLUDING REMARKS

I proposed a procedure for testing and building TAR models. The procedure is simple and requires no pre-specification of the number of regimes of a threshold model. I applied the procedure to three real data sets and obtained adequate models. In particular, for the Canadian lynx data of Example 2 the specified model gives rise to an asymmetric limit cycle similar to that of the data.

Note that the TAR model (1) has a striking feature of discontinuity at the threshold  $Y_{t-d} = r_j$  when the coefficients  $\Phi_i^{(j)}$  depend on  $j$ . This feature can capture jump phenomena observed in practice such as in the vibration study. Nevertheless, it appears to be somewhat counter-intuitive. To check this feature, Chan and Tong (1986) considered a class of smooth TAR models that put certain continuity constraints on the model. Much remains to be investigated for this continuity problem, however.

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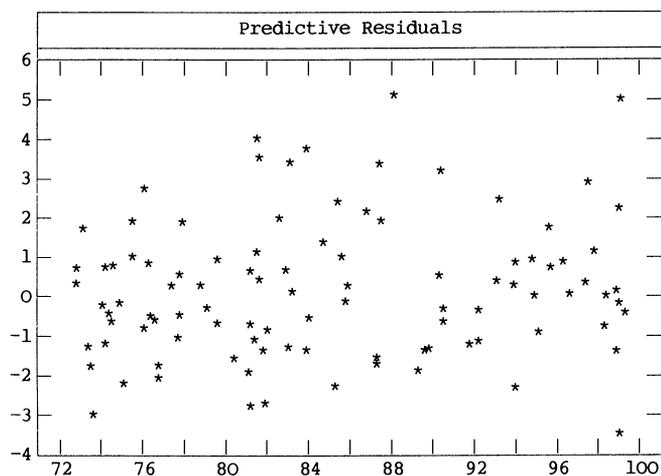


Figure 5. Scatterplot of Standardized Predictive Residuals Versus Ordered  $Y_{t-1}$  for Example 3 After Omitting Observations With  $Y_{t-1} \leq 69.3$ . The  $X$  axis is  $Y_{t-1}$ .

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